Limiting Absorption Principle for Singular Solutions to Maxwell’s Equations and Plasma Heating

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References


Related results

For the numerical implementation these results see, L.-M. Imbert-Gérard, Analyse mathématique et numérique de problèmes d’ondes apparaissant dans les plasmas magnétiques, PhD thesis, University Paris VI, 2013.

M. C. Pintos, B. Després, Constructive formulations of resonant Maxwell’s equations, 2016, hal-01278860.

For a local analysis around the resonance for analytic coefficients see, B. Després, L-M Inbert-Gérard, O. Laffite, Singular solutions for the plasma at resonance, 2015, hal-01097364.
I will consider the heating of plasmas with electromagnetic waves for the cool plasma model, in the case of hybrid resonances. This is a classical problem that has been extensively studied in the plasma physics literature. However, in context of the ITER project, there is currently interest in revisiting it with the purpose of obtaining precise mathematical methods that give a rigorous formula for the energy absorbed by the plasma, as well as efficient numerical methods.
The mathematical difficulty lies on the fact that the solutions that heat the plasma are singular, and that it is necessary to consider Maxwell’s equations in non-homogeneous and anisotropic media that do not belong to the classes that can be analyzed with the standard methods.
Let us consider Maxwell’s equations with a linear current and bulk magnetic field $B_0$.

\[
\begin{align*}
-\frac{1}{c^2} \partial_t E + \nabla \wedge B &= \mu_0 J, \\
\partial_t B + \nabla \wedge E &= 0, \\
m_e \partial_t u_e &= -e (E + u_e \wedge B_0) - m_e \nu u_e.
\end{align*}
\]

The velocity of the electrons is $u_e$. The electronic density is $N_e$. There is a friction between the electrons and the ions with collision frequency $\nu$. 
The loss of energy in a domain $\Omega$ is given by,

$$\frac{d}{dt} \int_{\Omega} \left( \frac{\varepsilon_0 |E|^2}{2} + \frac{|B|^2}{2\mu_0} + \frac{m_e N_e |u_e|^2}{2} \right) = - \int_{\Omega} \nu m_e N_e |u_e|^2$$

plus boundary terms. Therefore

$$Q(\nu) = \int_{\Omega} \nu m_e N_e |u_e|^2$$

represents the total loss of energy of the electromagnetic field plus the electrons in function of the collision frequency $\nu$. Since the energy loss is necessarily equal to what is gained by the ions, it will be referred to as the heating.
In typical applications in fusion plasmas the collision frequency \( \nu \) is much smaller that the frequency of the electromagnetic waves, and it is of interest to compute the heating in the limit when \( \nu \to 0 \), \( Q(0^+) \).

We will show that in certain conditions characteristic of the hybrid resonance in frequency domain, the heating does not vanish for vanishing collision friction, and moreover a large fraction of the energy of the incident wave, up to 95\% for normal incidence, can be transferred to the ions.
Shifting to the frequency domain: $\partial_t = -i\omega$, taking $\mathbf{B}_0 = (|\mathbf{B}_0|, 0, 0)$ and eliminating the variables of the electron we obtain Maxwell’s equations,

$$\nabla \wedge \nabla \wedge \mathbf{E} - \left( \frac{\omega}{c} \right)^2 \varepsilon \mathbf{E} = 0,$$

with the dielectric tensor

$$\varepsilon = \begin{pmatrix} 1 - \frac{\tilde{\omega}\omega_p^2}{\omega(\tilde{\omega}^2 - \omega_c^2)} & i \frac{\omega_c \omega_p^2}{\omega(\tilde{\omega}^2 - \omega_c^2)} & 0 \\ -i \frac{\omega_c \omega_p^2}{\omega(\tilde{\omega}^2 - \omega_c^2)} & 1 - \frac{\tilde{\omega}\omega_p^2}{\omega(\tilde{\omega}^2 - \omega_c^2)} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\tilde{\omega}\omega} \end{pmatrix},$$

where the cyclotron frequency is $\omega_c = \frac{e|\mathbf{B}_0|}{m_e}$, the plasma frequency $\omega_p = \sqrt{\frac{e^2 N_e}{\varepsilon_0 m_e}}$ depends on the electronic density $N_e$, and $\tilde{\omega} = \omega + i\nu$ is the complex pulsation.
We consider in this work $\omega \neq \omega_c$ that is the frequency is away of the cyclotron frequency.

As already mentioned, physical values in fusion plasmas are such that we can assume that $\nu = 0$ (However we will later come back to this point) and take

$$
\varepsilon(x) = \begin{pmatrix}
1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & i \frac{\omega_c \omega_p^2}{\omega (\omega^2 - \omega_c^2)} & 0 \\
-i \frac{\omega_c \omega_p^2}{\omega (\omega^2 - \omega_c^2)} & 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\
0 & 0 & 1 - \frac{\omega_p^2}{\omega^2}
\end{pmatrix}.
$$
The hybrid resonance concerns the upper-left block in the dielectric tensor $\varepsilon$, that corresponds to the mode $E = (E_x, E_y, 0)$, and $E_x, E_y$, independent of $z$. In this case we get the system,

$$
\begin{align*}
W &= \partial_x E_y - \partial_y E_x, \\
\partial_y W - \alpha \frac{\omega^2}{c^2} E_x - i \delta \frac{\omega^2}{c^2} E_y &= 0, \\
-\partial_x W + i \delta \frac{\omega^2}{c^2} E_x - \alpha \frac{\omega^2}{c^2} E_y &= 0,
\end{align*}
$$

where $W = i\omega B_z$ is the vorticity, and the coefficients are

$$
\alpha = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}, \quad \delta = \frac{\omega_c \omega_p^2}{\omega (\omega^2 - \omega_c^2)},
$$
$x = -L$

$B_0$

$N_e$

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To be more specific we consider the simplified 2D domain

\[ \Omega = \left\{ (x, y) \in \mathbb{R}^2, \quad -L \leq x, \quad y \in \mathbb{R}, \quad L > 0 \right\}. \]

We supplement the X-mode equations with a non-homogeneous boundary condition

\[ W + i\lambda E_y = g \text{ on the left boundary } x = -L, \quad \lambda > 0, \]

which models a given source, typically an radiating antenna. In real Tokamaks this antenna is used to heat or to probe the plasma. We will consider slab geometry also for the sake of simplicity, that is all coefficients \( \alpha \) and \( \delta \) are functions only of the variable \( x \)

\[ \partial_y \alpha = \partial_y \delta = 0. \]
Other assumptions which correspond to the physical context of idealized reflectometry or heating devices are the following. We assume that

\[ \delta \in C^1[-L, \infty[, \quad \delta(0) \neq 0. \]

The coefficient \( \alpha \) vanishes at a finite number of points and \( \alpha(x) = 0 \) implies that \( \alpha'(x) \neq 0 \). For the sake of simplicity we will now suppose that \( \alpha \) only vanishes at \( x = 0 \). More precisely

\[ \alpha \in C^2[-L, \infty[, \quad \alpha(0) = 0, \quad \alpha'(0) < 0, \quad (H1) \]

and

\[ \alpha_- \leq \alpha(x) \leq \alpha_+, \quad \forall x \in [-L, \infty[, \quad 0 < r \leq \left| \frac{\alpha(x)}{x} \right|, \quad \forall x \in [-L, H], (H2), \]

where \( H > 0 \). We will also assume that the coefficients are constant at large scale: there exists \( \delta_\infty \) and \( \alpha_\infty \) so that

\[ \delta(x) = \delta_\infty \text{ and } \alpha(x) = \alpha_\infty \quad H \leq x < \infty, \quad (H3) \]
We assume that the problem is coercive at infinity (this corresponds to the absorption of the waves at infinity)

\[ \alpha_\infty^2 - \delta_\infty^2 > 0, \quad (H4) \]

An additional condition is defined by

\[ 4 \| \delta \|_\infty^2 H < r. \quad (H5). \]

It expresses the fact that the length of the transition zone between \( x = 0 \) and \( x = H \) is small with respect to the other parameters of the problem.
\[ x = -L \quad x = H \quad \text{slope} \ -r \quad \delta \quad \alpha \]
Let us consider first that $\alpha$ and $\delta$ are constant. Consider a plane wave

$$(E_x, E_y) = \text{Re}e^{i(k_1 x + k_2 y)}, \quad R \in \mathbb{C}^2.$$ 

We assume that $c = 1$ for simplicity. We set $k = |k|d$ with $d = (\cos \theta, \sin \theta)$ the direction of the wave. The phase velocity $v_\varphi = \frac{\omega}{|k|}$ is given by

$$v_\varphi^2 = \frac{\alpha}{\alpha^2 - \delta^2}.$$
When the phase velocity is real we are in a propagating region, and when the phase velocity is pure imaginary we are in a non-propagating region. One distinguishes two cutoffs where the local phase velocity is infinite

\[ \text{Cutoff : } \alpha(x) = \pm \delta(x) \]

and one resonance where the phase velocity is null

\[ \text{Resonance : } \alpha(x) = 0. \]

This structure is characteristic of the hybrid resonance.
Sign of the square of the phase velocity. In this example \( \alpha = -x \) and \( \delta = 1 \), so that \( v^2_\varphi = \frac{x}{1-x^2} \).
The Budden problem

Consider the case where the solution is independent of \( y \), what for the plane waves corresponds to normal incidence. In this situation the Maxwell equations can be solved analytically in some cases which helps a lot to understand the singularity of the general problem. Let us consider that \( \alpha = -x \) and
\[
\delta(x) = \sqrt{x^2 - \frac{x}{4} + 1} > 0.
\]
Denote
\[
\omega(x) = -e^{x/2} + x e^{-x/2} E_i(x),
\]
where \( E_i(x) \) is the Exponential-integral function,
\[
E_i(x) = \ln |x| + \sum_{j=1}^{\infty} \frac{x^j}{j \cdot j!}.
\]
It follows that,
\[
\omega(x) = -1 + x \ln |x| + O(|x|), \quad |x| \to 0.
\]
The singular solution is given by,

\[ E_y = \omega \Rightarrow E_y \in L^\infty[ -\varepsilon, \varepsilon[. \]

\[ E_x = i \frac{\sqrt{x^2 - \frac{x}{4} + 1}}{x} \omega(x) \approx -i \frac{1}{x}, \quad x \to 0. \]

\[ W(x) = \omega'(x). \]

The problem now is that \( \frac{1}{x} \) is not a distribution, and then the singular solution is not really a solution in mathematical sense at \( x = 0 \). It has to be regularized. This can be done in many different ways, in general \( \frac{1}{x} \) can be replaced by

\[ \text{P.V.} \frac{1}{x} + \sum_{j=1}^{N} a_j \delta^{(j)}_D(x), \]

for some arbitrary coefficients \( a_j \). So, which possible realization we choose? The answer to this question has to come from the physics.
We will solve this problem, for general profiles that are not necessarily solvable, by considering first the case where there is a small friction between the electrons and the ions.

For this purpose we replace \( \alpha(x) \) by \( \alpha(x) + i\nu, \nu \neq 0 \). In this case the solution is regular and uniquely defined, and we take the limit as the friction goes to zero, \( \pm \nu \downarrow 0 \). This also gives us a explicit formula to compute the heating of the plasma. The answer for the Budden profile is quite simple.

Denote,

\[
E_{i,\pm}(x) = \ln_{\pm}(x) + \sum_{j=1}^{\infty} \frac{x^j}{j^j j!},
\]

where \( \ln_{\pm} \) are the branches of the logarithm, so that,

\[
\ln_{\pm}(x) = \ln |x| \pm i\pi, \quad x < 0.
\]
We designate,\
\[ \omega_{\pm}(x) := -e^{-x/2} + xe^{x/2} E_{i,\mp}(x). \]

We define, \((UB,\pm = (E_{x,\pm}, E_{y,\pm}, W^{\pm}))\)

\[ UB,\pm = \left( -\text{P.V.} \frac{1}{x} \mp i\pi\delta_D(x) + u^{B,\pm}(x), v^{B,\pm}(x), w^{B,\pm}(x) \right), \]

where,

\[ u^{B,\pm}(x) := \frac{1}{x} \sqrt{x^2 - \frac{x}{4} + 1} \omega_{\pm}(x) + \frac{1}{x}, \]

\[ v^{B,\pm}(x) := -i\omega_{\pm}(x), \]

\[ w^{B,\pm}(x) := \frac{d}{dx} v^{B,\pm}(x). \]

The \( UB,^+ \) solution heats the plasma and \( UB,- \) cools the plasma.
We replace $\alpha(x)$ by $\alpha(x) + i\nu, \nu \neq 0$.

We denote by $\hat{g}$ the Fourier transform of $g$,

$$
\hat{g}(\theta) := \int_{\mathbb{R}} g(y) e^{-i\theta y} dy.
$$

We apply the Fourier transform and we obtain the system:

$$
\begin{align*}
W^{\theta,\nu} + i\theta U^{\theta,\nu} - \frac{d}{dx} V^{\theta,\nu} &= 0, \\
i\theta W^{\theta,\nu} - (\alpha(x) + i\nu) U^{\theta,\nu} - i\delta(x) V^{\theta,\nu} &= 0, \\
- \frac{d}{dx} W^{\theta,\nu} + i\delta(x) U^{\theta,\nu} - (\alpha(x) + i\nu) V^{\theta,\nu} &= 0.
\end{align*}
$$
By elimination $U^{\theta,\nu}$ one gets a system of two coupled ordinary differential equations

$$\frac{d}{dx} \begin{pmatrix} V^{\theta,\nu} \\ W^{\theta,\nu} \end{pmatrix} = A^{\theta,\nu}(x) \begin{pmatrix} V^{\theta,\nu} \\ W^{\theta,\nu} \end{pmatrix}$$

with

$$A^{\theta,\nu}(x) = \begin{pmatrix} \frac{\theta \delta(x)}{\alpha(x) + iv} & 1 - \frac{\theta^2}{\alpha(x) + iv} \\ \frac{\delta(x)^2}{\alpha(x) + iv} - \alpha(x) - iv & -\frac{\theta \delta(x)}{\alpha(x) + iv} \end{pmatrix}$$

In the case $\nu \neq 0$ the matrix is non singular for all $x$, which gives a meaning to the regularized problem. One notices the matrix is singular for $\nu = 0$. 
Let us denote by \((A_\nu, B_\nu)\) the two fundamental solutions of the modified equation

\[-u'' - (\alpha(x) + i\nu)u = 0,\]

with the usual normalization

\[A_\nu(0) = 1, \quad A_\nu'(0) = 0 \quad \text{and} \quad B_\nu(0) = 0, \quad B_\nu'(0) = 1.\]

Let us denote \(D_\theta^z\) the operator

\[D_\theta^zh(z) = i\theta \partial_z h(z) - i\delta(z)h(z).\]

Let us define the kernel

\[k_\nu(x, z) = B_\nu(z)A_\nu(x) - B_\nu(x)A_\nu(z).\]
Proposition

Any triplet \((U, V, W) = (E_x, E_y, W)\) solution of the regularized system of differential equations admits the following integral representation. One first chooses an arbitrary reference point \(G \in [–L, \infty[\).

- The \(x\) component of the electric field is solution of the integral equation

\[
U(x) - \int_G^x \frac{D_x^\theta D_z^\theta k^\nu(x, z)}{\alpha(x) + i\nu} U(z)dz = \frac{F^{\theta,\nu}(x)}{\alpha(x) + i\nu},
\]

where the right hand side is

\[
F^{\theta,\nu}(x) = a_G D_x^\theta A_\nu(x) + b_G D_x^\theta B_\nu(x).
\]
The \( y \) component of the electric field is recovered as

\[
V(x) = a_G A_\nu(x) + b_G B_\nu(x) + \int_G^x D_z^\theta k^\nu(x, z) U(z) dz,
\]

and the vorticity is recovered as

\[
W(x) = a_G A'_\nu(x) + b_G B'_\nu(x) + \int_G^x \partial_x D_z^\theta k^\nu(x, z) U(z) dz.
\]
The two complex numbers \((a_G, b_G)\) solve the linear system

\[
\begin{align*}
\begin{cases}
    a_G A_\nu(G) + b_G B_\nu(G) &= V(G), \\
    a_G A_\nu'(G) + b_G B_\nu'(G) &= W(G).
\end{cases}
\end{align*}
\]
So, we have to solve the integral equation,

$$U(x) - \int_G^x \frac{D^\theta_x D^\theta_z k^\nu(x, z)}{\alpha(x) + i\nu} U(z)dz = \frac{F^{\theta,\nu}(x)}{\alpha(x) + i\nu},$$

When $\nu \neq 0$ this is a standard integral equation of the second-class.
However, when $\nu = 0$ this is a singular integral equation of the third class that has been seldom studied. Some important contributions are E. Picard, 1911, D. Hilbert 1912, G. R. Bart and R. L. Warnock 1973.
The integral kernel
\[ \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^0(x, z)}{\alpha(x)} \]
and the right-hand side
\[ \frac{F^{\theta,0}(x)}{\alpha(x)} \]
are singular because \( \alpha(0) = 0 \). If \( G \neq 0 \), in the integration domain \( \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\mu(x, z)}{\alpha(x)} \) is not even bounded, and if \( G = 0 \) it is bounded but not Hölder continuous.
A singular Limiting Absorption Principle

The solution to this problem is rather involved. The main ideas are as follows.
Let us denote by $U_{2}^{\theta,\nu} = \left( E_{x}^{\theta,\nu}, E_{y}^{\theta,\nu}, W^{\theta,\nu} \right)$, the, properly normalized, solution that goes to zero as $x \to \infty$.

$$U_{2}^{\theta,\nu}(0) = \frac{1}{i\nu}.$$ 

Then,

**Theorem**

*For some $C_{p}^{\theta}$ and $C^{\theta}$ which are continuous with respect to $\theta$*

$$\left\| U_{2}^{\theta,\nu} - \frac{1}{\alpha(\cdot) + i\nu} \right\|_{L^{p}(-L,H)} \leq C_{p}^{\theta}, \quad 1 \leq p < \infty,$$

$$\left\| U_{2}^{\theta,\nu} \right\|_{H_{1,loc}^{1}[-L,0) \cup (0,H]} \leq C^{\theta}.$$
This allows us to pass to the limit $\nu \to 0$

**Theorem**

The solution $U^{\theta,\nu}_2$ admits a limit in the sense of distributions for $\nu = 0^+$ as follows:

$$U^{\theta,\nu}_2 \to U^{\theta,+}_2 = \left( P.V. \frac{1}{\alpha(x)} + \frac{i\pi}{\alpha'(0)} \delta_D + u^{\theta,+}_2, v^{\theta,+}_2, w^{\theta,+}_2 \right)$$

where $u^{\theta,+}_2, v^{\theta,+}_2, w^{\theta,+}_2 \in L^2(-L, \infty)$ and $\delta_D$ is the Dirac mass at the origin.
Theorem

The singular solution $U_2^{\theta,+}$ is the unique solution of the following system.

Find a triplet $(u_2^{\theta,+}, v_2^{\theta,+}, w_2^{\theta,+}) \in L^2(-L, \infty) \oplus L^2(-L, \infty) \oplus L^2(-L, \infty)$ which satisfies the system,

$$w_2^{\theta,+} - \frac{d}{dx} v_2^{\theta,+} + i\theta u_2^{\theta,+} = -i\theta P.V. \frac{1}{\alpha(x)} + \frac{\theta\pi}{\alpha'(0)} \delta_D,$$

$$i\theta w_2^{\theta,+} - \alpha u_2^{\theta,+} - i\delta(x) v_2^{\theta,+} = 1,$$

$$-\frac{d}{dx} w_2^{\theta,+} + i\delta(x) u_2^{\theta,+} - \alpha(x) v_2^{\theta,+} = -i P.V. \frac{\delta(x)}{\alpha(x)} + \frac{\delta(0)\pi}{\alpha'(0)} \delta_D.$$
Remark
We observe the similarity with the standard limiting absorption principle in scattering theory. In scattering theory the solutions obtained by the limiting absorption principle are characterized as the unique solutions that satisfy the radiation condition, i.e., they are uniquely determined by the behavior at infinity. Here, the singular solutions are uniquely determined by their behavior at $+\infty$ and by their singular part $P.V.\frac{1}{\alpha(x)} \pm \frac{i\pi}{\alpha'(0)} \delta_D$ Note that it is natural that we have to specify the singularity at $x = 0$ because our equations are degenerate at $x = 0$. 

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Representation of the Solution

Recall the boundary condition

$$W + i\lambda E_y = g \text{ on the left boundary } x = -L, \quad \lambda > 0,$$

We have the following representation of the singular solution

$$\begin{pmatrix} E_x^+ \\ E_y^+ \\ W^+ \end{pmatrix}(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\hat{g}(\theta)}{\tau^{\theta,+}} \left( P.V. \frac{1}{\alpha(x)} + \frac{i\pi}{\alpha'(0)} \delta_D + u_{\theta,+}^{\theta} \right) e^{i\theta y} d\theta.$$
Where we denote,

$$\tau^{\theta,+} := \lim_{\nu \downarrow 0} \left( W_2^{\theta,\nu}(-L) + i \lambda V_2^{\theta,\nu}(-L) \right).$$

Moreover, the heating coefficient is given by,

$$Q = \omega \varepsilon_0 \lim_{\nu \to 0^+} \nu \int |E_{\nu}^\nu(x, y)|^2 dxdy = \frac{\omega \varepsilon_0}{2} \int_{\mathbb{R}} \frac{|\hat{g}(\theta)|^2}{|\alpha'(0)| |\tau^{\theta,+}|^2} d\theta.$$
We consider the evolution of wave packets. We denote,

\[ P_\varphi(t) := \int_{\mathbb{R}} e^{-i\omega t} A(\omega) \left( U_{2,\omega}^{\theta,+}, \varphi \right) d\omega, \]

where \( A(\omega) \) is an integrable function with compact support. The function \( \varphi \) is continuously differentiable on \( \mathbb{R} \), with values in \( \mathbb{C}^3 \), and has compact support. We made explicit the dependence in \( \omega \) of the singular solution \( U_{2,\omega}^{\theta,+} \).

We have that,

\[ \lim_{t \to \infty} P_\varphi(t) = 0. \]

This shows that our singular solution properly describes the physics of our problem: for large times, in weak sense, the incoming solution tends to zero in any compact region of space, hence part of the solution is reflected to minus infinite and the rest is absorbed by the plasma in the region \((-L, \infty)\).
We construct a basis of solutions to the Maxwell equations with $\nu \neq 0$. We denote by

$$U := (U, V, W) = (E_x, E_y, W)$$

a general solution to Maxwell’s equations. The regular solution

$$U_{1,\nu}^{\theta} = \left( U_{1,\nu}^{\theta}, V_{1,\nu}^{\theta}, W_{1,\nu}^{\theta} \right), \quad U_{1,\nu}^{\theta}(0) = 0,$$

$$V_{1,\nu}^{\theta}(0) = i\theta, \quad \text{and} \quad W_{1,\nu}^{\theta}(0) = i\delta(0) \quad (\neq 0).$$
The regular solution $U_{1}^{\theta,\nu}$ is uniformly bounded with respect to $\nu$: for any interval $\theta \in [\theta_{-}, \theta_{+}]$ and any $H \in ]L, \infty[$, there exists a constant independent of $\nu$ such that

$$\left\| U_{1}^{\theta,\nu} \right\|_{L^{\infty}(-L,H)} + \left\| V_{1}^{\theta,\nu} \right\|_{L^{\infty}(-L,H)} + \left\| W_{1}^{\theta,\nu} \right\|_{L^{\infty}(-L,H)} \leq C.$$
This is true because as in this case $F^{\theta,\nu}(0) = 0$, the integral equation

$$U(x) - \int_0^x \frac{D_x^\theta D_z^\theta k^\nu(x, z)}{\alpha(x) + i\nu} U(z) dz = \frac{F^{\theta,\nu}(x)}{\alpha(x) + i\nu},$$

is a standard integral equation of the second kind. Furthermore, $U_1^{\theta,\nu}$ extends by continuity to a regular solution for $\nu = 0$

However, for $\nu \neq 0$ the regular solution increases exponentially and this is also true for $\nu = 0$, at least for $|\theta|$ small enough.
The Singular Solution

$U_{2,\nu} = (U_{2,\nu}, V_{2,\nu}, W_{2,\nu})$, is built with two requirements: It is exponentially decreasing at infinity, and

$$U_{2,\nu}^{\theta,\nu}(0) = \frac{1}{i \nu}.$$
To ensure that these conditions can be satisfied, we observe that since for \( x \geq H \), \( \alpha = \alpha_\infty \), \( \delta = \delta_\infty \) are constant, any solution satisfies,

\[
U = c_+ R_+ e^{\lambda^{\theta,\nu} x} + c_- R_- e^{-\lambda^{\theta,\nu} x}, \quad x \geq H,
\]

where \( c_\pm \in \mathbb{C} \), \( R_\pm \in \mathbb{C}^3 \), and \( \lambda^{\theta,\nu} \) has a positive real part because of the coercivity assumption.

Consider a third solution

\[
U_{3}^{\theta,\nu} = (U_{3}^{\theta,\nu}, V_{3}^{\theta,\nu}, W_{3}^{\theta,\nu})(x) = R_- e^{-\lambda^{\theta,\nu} x}, \quad x \geq H,
\]

and it is smoothly extended to \(-L \leq x \leq H\) so that \( U_{3}^{\theta,\nu} \) is a solution.
It turns out that since $\delta(0) \neq 0$, we have that $U^{\theta,\nu}_3(0) \neq 0$ and we can take,

$$U^{\theta,\nu}_2 = c_- U^{\theta,\nu}_3,$$

$$c_- = \frac{1}{i\nu U^{\theta,\nu}_3(0)}.$$
Let us consider a general solution \( \mathbf{U} = (U, V, W) \) of the integral equation with prescribed data in \( H \) under the form

\[
V(H) = a_H \quad \text{and} \quad W(H) = b_H.
\]

Let us introduce the compact notation

\[
\|H\| = |a_H| + |b_H|.
\]

**PROPOSITION**

There exists a constant \( C_\theta \) with continuous dependence with respect to \( \theta \) such that

\[
|U(x)| \leq \frac{C_\theta}{\sqrt{r^2 x^2 + \nu^2}} \|H\|, \quad 0 < x \leq H.
\]
Let us define

\[ Q(U) = V_{1}^{\theta, \nu}(H) W(H) - W_{1}^{\theta, \nu}(H) V(H). \]

This quantity is the Wronskian of the current solution \( U \) against the first basis function. It is therefore independent of the position \( H \) which is used to evaluate \( Q(U) \). Note that

\[ |Q(U)| \leq C^{\theta} ||H||. \]

To pursue the analysis, we begin by rewriting the integral equation for \( U \) in a way that shows that the various singularities of the equation can be recombined under a more convenient form.
\[(\alpha(x) + i\nu)U(x) = \tilde{a}m^{\theta,\nu}(x)x + \tilde{b}n^{\theta,\nu}(x)x + Q(U)\]

\[-\int_x^{H} D_x^\theta D_z^\theta k^\nu(x,0)U(z)dz\]

\[+\int_0^x \left(D_x^\theta D_z^\theta k^\nu(x,z) - D_x^\theta D_z^\theta k^\nu(x,0)\right)U(z)dz,\quad \forall x \in [-L, \infty[.\]

Where, \(|\tilde{a}| \leq C_5^\theta \|H\|\), \(|\tilde{b}| \leq C_6^\theta \|H\|\) are bounded uniformly with respect to \(\nu\) and \(m^{\theta,\nu}(x), n^{\theta,\nu}(x)\) are bounded functions.
Setting \( u(x) = U(x) - \frac{Q(U)}{\alpha(x) + i\nu} \), this equation can be written as:

\[
\begin{align*}
    u(x) + \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\nu(x,0)}{\alpha(x) + i\nu} \int_x^H u(z) dz & \\
    = \frac{x}{\alpha(x) + i\nu} p^{\theta,\nu}(x) - Q(U) \frac{\mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\nu(x,0)}{\alpha(x) + i\nu} \int_x^H \frac{1}{\alpha(z) + i\nu} dz
\end{align*}
\]

where

\[
p^{\theta,\nu}(x) = \tilde{a}m^{\theta,\nu}(x) + \tilde{b}n^{\theta,\nu}(x) + \frac{1}{x} \int_0^x \left( \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\nu(x, z) - \mathcal{D}_x^\theta \mathcal{D}_z^\theta k^\nu(x, 0) \right) U(z) dz
\]

\[
\|p^{\theta,\nu}\|_{L^\infty(0,H)} \leq C^\theta \|H\|, \quad \forall \nu \in [0,1].
\]
As a consequence one has

**PROPOSITION**

For all $1 \leq p < \infty$, there exists a constant $C_p^\theta$ independent of $\nu$ and which depends continuously on $\theta$ such that

$$
\left\| U - \frac{Q(U)}{\alpha(\cdot) + i\nu} \right\|_{L^p(0,H)} \leq C_p^\theta \| H \|.
$$

Now we have to cross the singularity at $x = 0$. 
\[ D^{\theta,\nu}(x) := -\frac{D_{\chi}D_{\zeta}k^{\nu}(x, 0)}{\alpha(x) + i\nu} \int_{0}^{H} \frac{1}{\alpha(z) + i\nu} dz \]

\[ + \int_{0}^{x} \frac{D_{\chi}D_{\zeta}k^{\nu}(x, z)}{\alpha(x) + i\nu} \frac{1}{\alpha(z) + i\nu} dz \]

which is nothing than the integral part of our reorganized integral equation where \( U \) is replaced by the function \( \frac{1}{\alpha(\cdot) + i\nu} \).

**PROPOSITION**

Let \( 1 \leq p < \infty \). One has

\[ \left\| D^{\theta,\nu} \right\|_{L^p(-L, H)} \leq C_p^\theta \]

where the constant depends continuously on \( \theta \) and does not depend on \( \nu \).
Setting $u(x) = U(x) - \frac{Q(U)}{\alpha(x) + i\nu}$ we write our integral equation as

$$u(x) = \frac{x}{\alpha(x) + i\nu} \tilde{p}^{\theta,\nu}(x) - Q(U)D^{\theta,\nu}(x) - \int_0^H \frac{D^\theta_x D^\theta_z k^\nu(x, z)}{\alpha(x) + i\nu} u(z) dz.$$
PROPOSITION

For all $1 \leq p < \infty$, there exists a constant $C_p^\theta$ independent of $\nu$ such that

$$\left\| U - \frac{Q(U)}{\alpha(\cdot) + i\nu} \right\|_{L^p(-L,H)} \leq C_p^\theta \| H \|.$$
We apply the results above to the second basis function for which

\[ Q(U_{\theta,\nu}^2) = V_{1,\nu}^{\theta}(x)W_{2,\nu}^{\theta}(x) - W_{1,\nu}^{\theta}(x)V_{2,\nu}^{\theta}(x) = 1 \quad \forall x. \]

In this case we have that,

\[ \left\| U_{\theta,\nu}^{\theta,\nu} - \frac{1}{\alpha(\cdot) + i\nu} \right\|_{L^p(-L,H)} \leq C_\theta^p \left( \left| V_{2,\nu}^{\theta}(H) \right| + \left| W_{2,\nu}^{\theta}(H) \right| \right), \]

for \( 1 \leq p < \infty \).
PROPOSITION

There exists a maximal value $\theta_{\text{thresh}} > 0$ such that: If 
$4 \|\delta\|_2^2 < r$ and $|\theta| < \theta_{\text{thresh}},$ then $U^\theta_1$ increases exponentially
at infinity.

We define:

$$\sigma(\theta, \nu) = V^{\theta, \nu}_1(H)W^{\theta, \nu}_3(H) - W^{\theta, \nu}_1(H)V^{\theta, \nu}_3(H).$$

The transversality condition is defined:

$$\sigma(\theta, 0) \neq 0.$$
PROPOSITION

Assume the transversality condition. There exists a constant $C^\theta$ independent of $\nu$ and continuous with respect to $\theta$ such that

$$\left| V_2^{\theta,\nu}(H) \right| + \left| W_2^{\theta,\nu}(H) \right| \leq C^\theta.$$

Proof: The pair $(v, w) = (V_2^{\theta,\nu}(H), W_2^{\theta,\nu}(H))$ is solution of the linear system

$$\begin{cases} 
-vW_1^{\theta,\nu}(H) + wV_1^{\theta,\nu}(H) = 1, \\
vW_3^{\theta,\nu}(H) - wV_3^{\theta,\nu}(H) = 0.
\end{cases}$$
The determinant of this linear system is equal to the value of the function \(-\sigma(\theta, \nu)\). So the transversality condition establishes that

\[
\det \begin{pmatrix}
-W_1^{\theta, \nu}(H) & V_1^{\theta, \nu}(H) \\
W_3^{\theta, \nu}(H) & -V_3^{\theta, \nu}(H)
\end{pmatrix} = -\sigma(\theta, \nu) \neq 0.
\]

Therefore the solution of the linear system is given by,

\[
v = -\frac{V_3^{\theta, \nu}(H)}{\sigma(\theta, \nu)}, \quad w = -\frac{W_3^{\theta, \nu}(H)}{\sigma(\theta, \nu)}
\]

and then, it is bounded uniformly with respect to \(\nu\).
Theorem

Assume the same transversality condition. The second basis function satisfies the following estimates for some $C_p^\theta$ and $C^\theta$ which are continuous with respect to $\theta$

$$\left\| U^{\theta,\nu}_2 - \frac{1}{\alpha(\cdot) + i\nu} \right\|_{L^p(-L,H)} \leq C_p^\theta, \quad 1 \leq p < \infty,$$

$$\left\| U^{\theta,\nu}_2 \right\|_{H^1_{\text{loc}}([-L,0) \cup (0,H])} \leq C^\theta.$$
The space $\mathbb{X}^{\theta,+}$

The basis of solutions on the limit $\nu \to 0^+$, is given by: $\mathbb{X}^{\theta,+}$ where

$$\mathbb{X}^{\theta,+} = \text{Span} \left\{ U_1^{\theta}, U_2^{\theta,+} \right\} \subset H^1_{\text{loc}}\left((−L, \infty) \setminus \{0\}\right).$$
The space $X^{\theta,-}$

It is of course possible to do all the analysis with negative $\nu < 0$ and to study the limit $\nu \rightarrow 0^-$. The first basis function is exactly the same. The second basis function is chosen exponentially decreasing at infinity and such that

$$i\nu U_2^{\theta,\nu} = 1 \quad \nu < 0.$$ 

We also have that,

$$\left\| U_2^{\theta,\nu} - \frac{1}{\alpha(\cdot) + i\nu}\right\|_{L^p(-L,H)} \leq C_{\theta}^\nu, \quad -1 \leq \nu < 0.$$
We have that,

\[ U_{2}^{\theta,\nu} \rightarrow U_{2}^{\theta,-} = \left( P.V. \frac{1}{\alpha(x)} - \frac{i\pi}{\alpha'(0)} \delta_{D} + u_{2}^{\theta,-}, v_{2}^{\theta,-}, w_{2}^{\theta,-} \right) \]

where \( u_{2}^{\theta,-}, v_{2}^{\theta,-}, w_{2}^{\theta,-} \in L^{2}(-L, \infty) \) and \( \delta_{D} \) is the Dirac mass at the origin.
As in the case of \( \nu \downarrow 0 \), the singular solution \( U^\theta_2^- \) is the unique solution of the following system.

Find a triplet \((u^\theta_2^-, v^\theta_2^-, w^\theta_2^-) \in L^2(-L, \infty)^3\) which satisfies the system,

\[
w_2^\theta,- - \frac{d}{dx} v_2^\theta,- + i\theta u_2^\theta,- = -i\theta P.V. \frac{1}{\alpha(x)} - \frac{\theta\pi}{\alpha'(0)} \delta_D,
\]

\[
i\theta w_2^\theta,- - \alpha u_2^\theta,- - i\delta(x) v_2^\theta,- = 1,
\]

\[-\frac{d}{dx} w_2^\theta,- + i\delta(x) u_2^\theta,- - \alpha(x) v_2^\theta,- = -iP.V. \frac{\delta(x)}{\alpha(x)} - \frac{\delta(0)\pi}{\alpha'(0)} \delta_D.
\]
Passing to the limit $\nu \to 0^-$, it defines the limit space $\mathfrak{X}^\theta, -$ 

$$\mathfrak{X}^\theta, - = \text{Span}\left\{ U_1^\theta, U_2^\theta, - \right\} \subset H^1_{\text{loc}} \left( (-L, \infty) \setminus \{0\} \right).$$
The first basis function $U_1^\theta$ is independent of the sign and belongs to $X^{\theta,+} \cap X^{\theta,-}$. Since the limit equation and the normalization at $x = H$ are the same, we readily observe that the limits of the second basis functions are identical for $0 < x$.

$$U_{2,+}^\theta(x) = U_{2,-}^\theta(x) \quad 0 < x.$$ 

**PROPOSITION**

One has

$$U_{2,+}^\theta(x) - U_{2,-}^\theta(x) = \frac{-2i\pi}{\alpha'(0)} U_1^\theta(x) \quad x < 0. \quad (1)$$
Lise-Marie Imbert-Gérard.
Analyse mathématique et numérique de problèmes d’ondes apparaissant dans les plasmas magnétiques,

The numerical tests show a fast pointwise convergence of the numerical solution to the exact one, except at the origin of course. Moreover our numerical tests show that a large part of the incoming energy of the wave may be absorbed by the heating, around 90 % in some cases.
This is for example the case for the Fourier mode $\theta = 0$, with $L = 2$, $c = \omega = 2$ and $\omega_c = 1$, so that $\alpha = 1 - 2\delta$. We consider the profiles

$$\alpha(x) = \begin{cases} 1, & -L \leq x \leq -1, \\ -x, & -1 \leq x \leq 3, \\ -3, & 3 \leq x < \infty, \end{cases} \quad \delta(x) = \begin{cases} 0, & -L \leq x \leq -1, \\ \frac{x+1}{2}, & -1 \leq x \leq 3, \\ 2, & 3 \leq x < \infty, \end{cases}$$

which satisfy additionally $|\alpha_\infty| > |\delta_\infty|$ and the fact that the electronic density is increasing from the left to the right.

We compute numerically the singular solution $U_{2,\nu}^0$ taking $\nu = 10^{-3}$ as a small regularization parameter.
The efficiency of the heating is defined as the ratio of the heating $Q$ over the incoming energy. In this case our calculations show an efficiency of around 95%. Another calculation in oblique incidence $\theta = \cos \frac{\pi}{4}$ shows an efficiency still around 76.7%. These values indicate a high efficiency.

The numerical results below were obtained with $H = 2$, $\alpha(x) = -x$, $x \leq 2$, $\alpha(x) = -2$, $x \geq 2$, $\delta(x) = 0.25$, $\theta = 1.5$ The real part of the singular solution $(U^{\theta,\nu}_2, V^{\theta,\nu}_2, W^{\theta,\nu}_2)$ is plotted. The real part of $(-x + i\nu)^{-1}$ is also represented, and evidences the blow up of the solution at the resonance as $\nu$ goes to zero.
$\mu = 10^{-2}$

Diagram showing the real parts of $v_2$, $w_2$, $u_2$, $1/(\mu - x + \text{i} \mu)$ for different values of $\mu$. The graph indicates the behavior and stability of these functions as $\mu$ changes.
$\mu = 10^{-3}$

- $Re \ v_2$
- $Re \ w_2$
- $Re \ u_2$
- $Re \ 1/(\mu - x + i\mu)$
Below the real parts of the three components of the singular solution \( (U_2^{\theta,\nu}, V_2^{\theta,\nu}, W_2^{\theta,\nu}) \) are represented. The scale is now fixed to show the strong convergence observed on \( ] - 10, 5 [ \{0\} \): the convergence observed is strong everywhere but at the resonance point \( x = 0 \).
\( \mu = 10^{-2} \)
\( \mu = 10^{-3} \)
THANKS FOR YOUR ATTENTION