On q-polynomials and some of their applications

Renato Álvarez-Nodarse

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Selected Topics in Mathematical Physics
In honor of Professor Natig Atakishiyev
Instituto de Matemáticas, Cuernavaca, UNAM, 28–30 November 2016
Why to talk about $q$-polynomials today?

This Conference was organized in honor to Natig Atakishiyev. Natig was born in Azerbaijan and in 1991 moved to Mexico where he is living now. He has 109 papers reviewed in MathSciNet and more than 100 in Journals of the Citation Index. He is very well known for his mathematical works on SF and OP and specially for his important contributions to the theory of $q$-polynomials but also for his works related with different kind of harmonic quantum oscillators.
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Special Functions (SF) appear in (almost) all context of Mathematics and other Sciences.

As Alberto Grunbaum one time said: “Special functions are to mathematics what pipes are to a house: nobody wants to exhibit them openly but nothing works without them”.

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**Definition 1** Given a sequence of normal pol. \((P_n)_n\) we said that \((P_n)_n\) is an OPS w.r.t. \(\mu\) if \(\forall n \neq m \in \mathbb{N},\)

\[
\int_{\mathbb{R}} P_n(x)P_m(x)d\mu(x) = \delta_{n,m}d_n, \quad d_n \neq 0
\]

If \(\mu\) is a positive measure \(\Rightarrow d_n > 0 \forall n, \Rightarrow \) we said that SOP is positive definite.
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\Rightarrow \text{If } d\mu(x) = \rho(x)dx \Rightarrow \rho \text{ is a continuous weight function}
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- If \( d\mu(x) = \rho(x)dx \) \( \Rightarrow \) \( \rho \) is a continuous weight function
- If \( d\mu(x) = \sum_k \delta(x - x_k)\rho(x_k)dx \) \( \Rightarrow \) \( \rho \) is a discrete weight function
TTRR: A characterization of an OPS

\[ \int_{\mathbb{R}} p_n(x) p_m(x) \, d\mu(x) = \delta_{n,m} \Rightarrow \exists (a_n) \text{ such that} \]

\[ x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_n(x), \quad n \geq 0. \]

¿There exists a converse result? ¿There are any other characterizations?
If \( \int_{\mathbb{R}} P_n(x)P_m(x)d\mu(x) = \delta_{n,m} \Rightarrow \exists (a_n)_n \text{ y } (b_n)_n \text{ such that}

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¿There exists a converse result?

¿There are any other characterizations?
The classical OP.

Sonin (1887): The only OPS \((P_n)_n\) such that their derivatives \((P'_n)_n\) also constitute and OPS are the Jacobi, Laguerre, and Hermite polynomials.

W. Hahn (1935) rediscovered it and found a new family: Bessel pol.

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\text{Jacobi, Laguerre y Hermite + Bessel = classical OP}
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Hahn (1937) also proved and extension of the above characterization: Given and OPS \((P_n)_n\) it is classical iff the sequence \((P^{(k)}_n)_n\) is orthogonal for some \(k \in \mathbb{N}\)

¿What else we can said about classical families?
Bochner (1929): They are the only solution of

\[ \sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0, \quad \deg \sigma \leq 2, \quad \deg \tau = 1. \]

This is the hypergeometric equation!
Some characterizations of classical OP

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Tricomi (1955): They satisfy the Rodrigues Eq.

\[ P_n(x) = \frac{B_n}{\rho(x)} \frac{d^n}{dx^n} [\rho(x)\sigma^n(x)], \quad n = 0, 1, 2, \cdots \quad \rho(x) \geq 0 \quad (\text{FR}) \]
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Hildebrandt (1931): \( \rho \) satisfy the Pearson Eq.: 

\[ [\rho(x)\sigma(x)]' = \tau(x)\rho(x), \quad \text{deg}(\sigma) \leq 2, \quad \text{deg}(\tau) = 1 \]
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etc.
The $q$–polynomials are extensions of the classical ones

- The first families: Markov 1884
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- Stieltjes 1898 in connection with the moment problem: Stieltjes-Wiegert OP

\[ \text{Find, if there exist, the OPS such that:} \]

1. \( (\Theta_w q^P_n(x))^n \) is an OPS
2. \( \sigma(x) \Theta_w q^P_n(x) - 1 P_n(x) + \tau(x) \Theta_w P_n(x) + \lambda P_n(x) = 0 \) (DE)
3. \( \exists \pi \in \mathbb{P} \text{ s.t. } \rho(x) P_n(x) = [\Theta_w q^P_n] \pi(x) \rho(x) \) (RF)

If \( w = 0 \) and \( q \to 1 \Rightarrow \Theta_w f(x) \to d/dx: \text{Classical case!} \)
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- The Hahn problem (1949): Let be the OPS $(P_n)_n$ and define the operator $\Theta^w_q : \mathbb{R} \mapsto \mathbb{R}$, $\Theta^w_q p(x) = \frac{p(qx + w) - p(x)}{(q - 1)x + w}$
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Find, if there exist, the OPS such that:

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\begin{align*}
1. & \quad (\Theta^w_q P_n(x))_n \text{ is and OPS} \\
2. & \quad \sigma(x)\Theta^w_q \Theta^w_{q^{-1}} P_n(x) + \tau(x)\Theta^w_q P_n(x) + \lambda P_n(x) = 0 \quad \text{(DE)}
\end{align*}
\]

\[
\begin{align*}
3. & \quad \exists \pi \in \mathbb{P} \text{ y } \rho \text{ t.q. } \rho(x)P_n(x) = [\Theta^w_q]^n[\pi(x)\rho(x)] \quad \text{(RF)}
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1. \quad (\Theta^w_q P_n(x))_n & \text{ is and OPS} \\
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3. \quad \exists \pi \in \mathbb{P} \text{ y } \rho \text{ t.q.} \quad \rho(x)P_n(x) = [\Theta^w_q]^n[\pi(x)\rho(x)] \quad \text{(RF)}
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- If \(w = 0\) and \(q \to 1 \Rightarrow \Theta^w_q f(x) \to \frac{d}{dx}: \text{Classical case!}\)
“Discrete” polynomials and $q-$polynomials

- Case $q = 1$ and $w = 1 \Rightarrow$ “discrete” (Lesky, 1962)

$$\Theta^w_q f(x) = \Delta f(x) := f(x + 1) - f(x), \hspace{1cm} \nabla f(x) = \Delta f(x - 1)$$

$$\sigma(x) \Delta \nabla P_n(x) + \tau(x) \Delta P_n(x) + \lambda P_n(x) = 0$$
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- Case $q \in (0, 1)$ y $w = 0 \Rightarrow q’s \ (Hahn 1949, \ldots)$

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  \[ \sigma(x) \Theta_q \Theta_q^{-1} P_n(x) + \tau(x) \Theta_q P_n(x) + \lambda P_n(x) = 0 \]

In the next years the $q$-polynomials appeared in several contexts.
**q–Polynomials: In the 1980’s there were two approaches**

$q$-Polynomials

Askey

Nikiforov & Uvarov

\[ \phi_p \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_p \end{array} \left| q; z \right. \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_p; q)_k} \frac{z^k}{(q; q)_k} \left[ (-1)^k q^k \frac{k(k-1)}{2} \right]^{p-r+1} \]

\[
\sigma(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \nabla y(s) + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda_n y(s) = 0
\]
In 1998 Koekoek and Swarttouw compiled a report all known families of \( q \)-polynomials that was called the \( q \)-Askey Tableu.

All \( q \)-classical polynomials can be obtained from the Askey-Wilson:

\[
p_n(x, a, b, c, d) = 4\phi_3 \left( \begin{array}{c} q^{-n}, q^{n-1}abcd, ae^{-i\theta}, ae^{i\theta} \\ ab, ac, ad \end{array} \left| \begin{array}{c} q, q \end{array} \right. \right), \quad x = \cos \theta
\]
The $q$-Hahn Tableau (Koornwinder, 1993)

Big $q-$ Jacobi polynomials (if $c = q^{-N-1} \rightarrow q$-Hahn)

$$p_n(x; a, b, c; q) = \frac{(aq; q)_n(cq; q)_n}{(abq^{n+1}; q)_n} \begin{pmatrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{pmatrix}_{q; q}.$$
The 1983 Nikiforov & Uvarov approach

Discretize $\tilde{\sigma}y''(x) + \tilde{\tau}y'(x) + \lambda y(x) = 0$ in a nonuniform lattice

\[ y'(x) \approx \frac{1}{2} \left[ \frac{y(x(s + h)) - y(x(s))}{x(s + h) - x(s)} + \frac{y(x(s)) - y(x(s - h))}{x(s) - x(s - h)} \right] \]

\[ y''(x) \approx \frac{1}{x(s + \frac{h}{2}) - x(s - \frac{h}{2})} \left[ \frac{y(x(s + h)) - y(x(s))}{x(s + h) - x(s)} - \frac{y(x(s)) - y(x(s - h))}{x(s) - x(s - h)} \right] \]
The $q$-hypergeometric Eq. of NU

\[ \tilde{\sigma} y''(x) + \tilde{\tau} y'(x) + \lambda y(x) = 0 \]

\[ \Rightarrow \]

\[ \sigma(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \frac{\nabla y(s)}{\nabla x(s)} + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda n y(s) = 0 \]

\[ \nabla f(s) = f(s) - f(s - 1), \quad \Delta f(s) = f(s + 1) - f(s) \]

\[ \sigma(s) = \tilde{\sigma}(x(s)) - \frac{1}{2} \tilde{\tau}(x(s)) \Delta x(s - \frac{1}{2}), \quad \tau(s) = \tilde{\tau}(x(s)). \]
The $q$-hypergeometric Eq. of NU

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There are nice polynomial solution for any function $x(s)$?
The $q$-hypergemetric Eq. of NU

\[ \tilde{\sigma} y''(x) + \tilde{\tau} y'(x) + \lambda y(x) = 0 \]

\[ \Downarrow \]

\[ \sigma(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \frac{\nabla y(s)}{\nabla x(s)} + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda_n y(s) = 0 \]

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There are nice polynomial solution for any function $x(s)$?

\[ x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_3(q) = c_1(q)[q^s + q^{-s-\mu}] + c_3(q) \]
The \( q \)-hypergemetric Eq. of NU

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\tilde{\sigma} y''(x) + \tilde{\tau} y'(x) + \lambda y(x) = 0
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\sigma(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \frac{\nabla y(s)}{\nabla x(s)} + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda_n y(s) = 0
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\[
\nabla f(s) = f(s) - f(s - 1), \quad \Delta f(s) = f(s + 1) - f(s)
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\sigma(s) = \tilde{\sigma}(x(s)) - \frac{1}{2} \tilde{\tau}(x(s)) \Delta x(s - \frac{1}{2}), \quad \tau(s) = \tilde{\tau}(x(s)).
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There are nice polynomial solution for any function \( x(s) \)?

\[
x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_3(q) = c_1(q)[q^s + q^{-s-\mu}] + c_3(q)
\]

Sufficient cond. NU (1983).

Some basic properties

- The q-analogue of the Rodrigues formula

\[ P_n(s) = \frac{B_n}{\rho(s)} \frac{\nabla}{\nabla x(s + \frac{1}{2})} \cdots \frac{\nabla}{\nabla x(s + \frac{n}{2})} \left[ \rho(s + n) \prod_{m=1}^{n} \sigma(s + m) \right] \]

\[ \nabla^{(n)} \rho_n(s) \]

where \( \rho(s) \) if the sol. of \( \Delta [\sigma(x)\rho(s)] = \tau(s)\rho(s)\Delta x(s - \frac{1}{2}) \),
Some basic properties

- The $q$-analogue of the Rodrigues formula

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P_n(s) = \frac{B_n}{\rho(s)} \frac{\nabla}{\nabla x(s + \frac{1}{2})} \cdots \frac{\nabla}{\nabla x(s + \frac{n}{2})} \nabla^{(n)} \rho_n(s)\]

where $\rho(s)$ if the sol. of \[
\Delta [\sigma(x)\rho(s)] = \tau(s)\rho(s)\Delta x(s - \frac{1}{2}),
\]

- Differentiation or ladder-type formulas \[
\tau_n(s) = \tau_n'x(s + \frac{n}{2}) + \tau_n(0)
\]

\[
\sigma(s) \frac{\nabla P_n(x(s))_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau_n'} P_n(x(s))_q - \frac{\alpha_n \lambda_{2n}}{[2n]_q} P_{n+1}(x(s))_q
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Some basic properties

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\[ P_n(s) = \frac{B_n}{\rho(s)} \frac{\nabla}{\nabla x(s + \frac{1}{2})} \cdots \frac{\nabla}{\nabla x(s + \frac{n}{2})} \nabla^{(n)} \left[ \rho(s + n) \prod_{m=1}^{n} \sigma(s + m) \right] \rho_n(s) \]

where \( \rho(s) \) if the sol. of \( \Delta [\sigma(x)\rho(s)] = \tau(s)\rho(s)\Delta x(s - \frac{1}{2}) \),

- Differentiation or ladder-type formulas \( \tau_n(s) = \tau'_n x(s + \frac{n}{2}) + \tau_n(0) \)

\[ \sigma(s) \frac{\nabla P_n(x(s))_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau'_n} P_n(x(s))_q - \frac{\alpha_n \lambda_{2n}}{[2n]_q} P_{n+1}(x(s))_q \]
Some basic properties

- **The q-analogue of the Rodrigues formula**

\[
P_n(s) = \frac{B_n}{\rho(s)} \nabla \left[ \nabla x(s + \frac{1}{2}) \cdots \nabla x(s + \frac{n}{2}) \right]^{(n)} \left[ \rho(s + n) \prod_{m=1}^{n} \sigma(s + m) \right] \rho_n(s)
\]

where \( \rho(s) \) if the sol. of \( \Delta [\sigma(x)\rho(s)] = \tau(s)\rho(s)\Delta x(s - \frac{1}{2}) \),

- **Differentiation or ladder-type formulas**

\[
\sigma(s) \frac{\nabla P_n(x(s))}{\nabla x(s)}_q = \frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau_n'} P_n(x(s))_q - \frac{\alpha_n\lambda_2n}{[2n]_q} P_{n+1}(x(s))_q
\]

For linear-type lattices \( x(s + \alpha) = A(\alpha)x(s) + B(\alpha) \) (q-Hahn Tableau) there is a complete study in Medem, et. al. *JCAM* (2001) and RAN, *JCAM* (2006). For the general case see Foupouagnigni et al. *Integral Transforms Spec. Funct.* (2011).
A corollary of the NU Eq. Let \( x(s) = c_1(q)[q^s + q^{-s-\mu}] + c_3(q) \).

The most general case of the NU Eq. corresponds to the choice:

\[
\sigma(s) = q^{-2s}(q^s - q^{s_1})(q^s - q^{s_2})(q^s - q^{s_3})(q^s - q^{s_4}).
\]

and the corresponding general polynomial solution can be expressed in term of basic hypergeometric series

\[
P_n(s) = 4\phi_3 \left( \begin{array}{c} q^{-n}, q^{2\mu+n-1+s_1+s_2+s_3+s_4}, q^{s_1-s}, q^{s_1+s+\mu} \\ q^{s_1+s_2+\mu}, q^{s_1+s_3+\mu}, q^{s_1+s_4+\mu} \end{array} \right | q, q \right)
\]

From the above solution we can obtain Askey-Wilson, \( q \)-Racah, \( q \)-duales de Hahn, \( q \)-Hahn, \ldots NU Integral Transforms Spec. Funct. (1993); Atakishiyev, Rahman y Suslov Const. Appr. (1993).

On q-polynomials and some of their applications

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The $q$-hypergeometric Eq. of NU: A final remark

There are a series of interesting papers by Natig (some of them with other people) that further developed the theory initiated by NU:

- The study of the orthogonality of Askey-Wilson polynomials
- The moments of the weight functions of $q$-polynomials
- The study of the continuous orthogonality of the solutions of the NU Eq. including the discrete case.
- etc.
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- The study of the orthogonality of Askey-Wilson polynomials
- The moments of the weight functions of $q$-polynomials
- The study of the continuous orthogonality of the solutions of the NU Eq. including the discrete case.
- etc.

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In RAN, Medem *JCAM* (2001) we found two new families within the $q$-Hahn tableau. One of them is a positive definite case that has been recently studied by Area et. al. (2016).
Some applications

Discrete oscillators

There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.

N.I. Lobachevsky
Given a Hamiltonian, that is a $2^\circ$ order diff. operator

$$\hat{H}\varphi_n = \lambda_n \varphi_n, \quad \hat{H} \text{ de orden } 2$$
Given a Hamiltonian, that is a 2\textsuperscript{nd} order diff. operator

\[ \mathcal{H}\varphi_n = \lambda_n \varphi_n, \quad \mathcal{H} \text{ de orden } 2 \]

To find 1\textsuperscript{st} order diff operators \(a\) and \(a^+\) such that

\[ \mathcal{H} = a^+ a, \quad a^+ \varphi_n = \alpha_n \varphi_{n+1}, \quad a \varphi_n = \beta_n \varphi_{n-1}, \quad (a^+)^* = a, \quad a^* = a^+. \]

**Interest:** Solving \(a \varphi_0 = 0\), one gets \(\varphi_0\), and \(a^+ \varphi_n\) generate the others
The typical example is the quantum harmonic oscillator

\[ \mathfrak{H} \psi_n(x) := -\psi_n''(x) + x^2 \psi_n(x) = (H_- H_+ + 1) \psi_n(x) = \lambda_n \psi_n(x). \]

\[ a := H_+ = x + \frac{d}{dx}, \quad a^+ := H_- = x - \frac{d}{dx} m \]
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\[ H_+\psi_0 = 0, \quad H_+\psi_n = \sqrt{2n} \psi_{n-1}, \quad H_-\psi_n = \sqrt{2n + 2} \psi_{n+1}. \]

How it works?
The typical example is the quantum harmonic oscillator

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$$H_+ \psi_0 = 0, \quad H_+ \psi_n = \sqrt{2n} \psi_{n-1}, \quad H_- \psi_n = \sqrt{2n + 2} \psi_{n+1}.$$  

How it works?

$$x \psi_0(x) + \psi'_0(x) = 0 \quad \Rightarrow \quad \psi_0(x) = \frac{1}{\sqrt[4]{\pi}} e^{-x^2/2},$$

and $$[H_-]^n \psi_0(x) = \sqrt{(2n)!!} \psi_n(x),$$  

thus

$$\psi_n(x) = \frac{1}{\pi^{1/4} \sqrt{(2n)!!}} [H_-]^n e^{-x^2/2} = \frac{1}{\pi^{1/4} \sqrt{(2n)!!}} \left[ x I - \frac{d}{dx} \right]^n e^{-x^2/2}. $$

This the classical algebraic realization of the quantum oscillator.
Factorization of the NU equation

- Bankerezako (Askey-Wilson Case, JCAM 1999)
- Lorente (Continuous and discrete classical pols., JPA 2001)
- RAN, Costas Santos (NU Eq. JPA 2001):

\[ \phi_n(s) = \sqrt{\rho(s)} d_n P_n(x(s)) q, \quad H(s,n) \phi_n(s) = 0 \]

\[ H(s,n) \equiv \sqrt{\sigma(-s-\mu+1)} \sigma(s) \nabla x(s) e^{-\partial s} + \sqrt{\sigma(-s-\mu)} \sigma(s+1) \Delta x(s) e^{\partial s} - \lambda n \Delta x(s-1/2) \]

Main properties:
1. The orthonormal functions \( \phi_n \) satisfy a 2o diff Eq. & TTRR
2. There exist two ladder operators:
   \[ L^+(s,n) \phi_n(s) = A_n \phi_{n+1}(s) \] and
   \[ L^-(s,n) \phi_n(s) = B_n \phi_{n-1}(s) \]

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\[ \varphi_n(s) = \sqrt{\frac{\rho(s)}{d_n^2}} P_n(x(s))_q, \quad \tilde{\mathcal{H}}(s, n)\varphi_n(s) = 0 \]

\[
\tilde{\mathcal{H}}(s, n) \equiv \sqrt{\frac{\sigma(-s-\mu+1)\sigma(s)}{\nabla x(s)}} e^{-\partial_s} + \sqrt{\frac{\sigma(-s-\mu)\sigma(s+1)}{\Delta x(s)}} e^{\partial_s} - \\
\left( \frac{\sigma(-s-\mu)}{\Delta x(s)} + \frac{\sigma(s)}{\nabla x(s)} - \lambda_n\Delta x(s - 1/2) \right) I.
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Factorization of the NU equation

\[ H(s, n) \equiv \sqrt{\sigma(-s-\mu+1)\sigma(s)} \frac{1}{\nabla x(s)} e^{-\partial_s} + \sqrt{\Theta(s)\sigma(s+1)} \frac{1}{\Delta x(s)} e^{\partial_s} - \left( \frac{\sigma(-s-\mu)}{\Delta x(s)} + \frac{\sigma(s)}{\nabla x(s)} - \lambda_n \Delta x(s - 1/2) \right) I. \]

Theorem: The operator \( H(s, n) \) admits the following factorization

\[ u(s + 1, n)H(s, n) = L^{-}(s, n + 1)L^{+}(s, n) - h^{\mp}(n)I, \]
\[ u(s, n)H(s, n + 1) = L^{+}(s, n)L^{-}(s, n + 1) - h^{\mp}(n)I, \]

respectively, where

\[ h^{\pm}(n) = \frac{\lambda_{2n-2}}{[2n-2]_q} \frac{\lambda_{2n}}{[2n]_q} \alpha_{n-1} \gamma_n, \quad u(s, n) = \frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau'_n} - \frac{\sigma(s)}{\nabla x(s)} \]

where \( \alpha \) and \( \gamma \) are the coeff. of the TTRR.
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This is not a good solution to the problem. Why?
Motivation

Going further \([a, a^+] := aa^+ - a^+a = I\)
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Thus \(a, a^+, H\) and \(l\) form a closed Lie algebra:
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**THE DYNAMICAL ALGEBRA** of the HO and the SODE

The problem of build such dynamical algebra from the NU Eq. was proposed by Atakishiyev in 2002 motivated by the previous works of MacFarlane 1989, Biedenharn 1989, Atakishiyev et at 1991, 1994, 1996, ...
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\]

\[
\alpha = 1 \quad \beta = -1 \quad \gamma = 1
\]

\[ [J_0, J_\pm] = \pm J_\pm \quad [J_+, J_-] = 2J_0 \quad \text{SU}(2) \]
Motivation

Going further \( [a, a^+] := aa^+ - a^+ a = l \)

In fact we have: \( [a, \mathcal{H}] = a, \quad [a^+, \mathcal{H}] = -a^+ \)

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\[
[J_0, J_\pm] = \pm J_\pm \quad [J_+, J_-] = -2J_0 \quad \text{SU}(1,1)
\]
The $q$-wave functions $\varphi_n$ and the $q$-Hamiltonian $\mathcal{H}_q$

$$\mathcal{H}_q(s)\varphi_n(s) = \lambda_n \varphi_n(s),$$

$$\mathcal{H}_q(s) := \frac{1}{\nabla x_1(s)} A(s) H_q(s) \frac{1}{A(s)}, \quad \varphi_n(s) = \frac{A(s) \sqrt{\rho(s)}}{d_n} P_n(s; q),$$

where

$$H_q(s) := -\frac{\sqrt{\sigma(-s-\mu+1)\sigma(s)}}{\nabla x(s)} e^{-\partial_s} - \frac{\sqrt{\sigma(-s-\mu)\sigma(s+1)}}{\Delta x(s)} e^{\partial_s}$$

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\[ + \left( \frac{\sigma(-s-\mu)}{\Delta x(s)} + \frac{\sigma(s)}{\nabla x(s)} \right) I, \]

The next step is to find two operators $a(s)$ and $b(s)$ such that

\[ \mathcal{H}_q(s) = b(s) a(s) \]

We will follow an original idea by Atakishiyev:
The $\alpha$-operators

In order to factorize an arbitrary difference equation, one should express it explicitly in terms of the shift operators $\exp(a \frac{d}{ds})$, defined as $\exp(a \frac{d}{ds}) f(s) = f(s + a)$, $a \in \mathbb{C}$. 
The $\alpha$-operators

In order to factorize an arbitrary difference equation, one should express it explicitly in terms of the shift operators $\exp(a \frac{d}{ds})$, defined as $\exp(a \frac{d}{ds}) f(s) = f(s + a)$, $a \in \mathbb{C}$.

Let $\alpha \in \mathbb{R}$ and $A(s)$ and $B(s)$ are continuous functions. We define a family of $\alpha$-down and $\alpha$-up operators by

$$
a_{\alpha}^\downarrow(s) := \frac{B(s)}{\sqrt{\nabla x_1(s)}} e^{-\alpha \partial_s} \left( e^{\partial_s} \sqrt{\frac{\sigma(s)}{\nabla x(s)}} - \sqrt{\frac{\sigma(-s - \mu)}{\Delta x(s)}} \right) \frac{1}{A(s)},$$

$$
a_{\alpha}^\uparrow(s) := \frac{A(s)}{\nabla x_1(s)} \left( \sqrt{\frac{\sigma(s)}{\nabla x(s)}} e^{-\partial_s} - \sqrt{\frac{\sigma(-s - \mu)}{\Delta x(s)}} \right) e^{\alpha \partial_s} \frac{\sqrt{\nabla x_1(s)}}{B(s)}.
$$

$$
\mathcal{H}_q(s) = a_{\alpha}^\uparrow(s) a_{\alpha}^\downarrow(s), \quad \forall \alpha \in \mathbb{R}, \text{ and } B(s).
$$
**Definition:** Let $\varsigma$ be a complex number, and let $a(s)$ and $b(s)$ be two operators. We define the $\varsigma$-commutator of $a$ and $b$ as

$$[a(s), b(s)]_{\varsigma} = a(s)b(s) - \varsigma b(s)a(s), \quad \varsigma = q^{\gamma} \neq 0.$$
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Proposition: Let $\mathcal{H}_q(s)$ be an operator, such that $\exists a(s), b(s)$ and $\varsigma, \Lambda \in \mathbb{C}$, that $\mathcal{H}_q(s) = b(s)a(s)$, and $[a(s), b(s)]_{\varsigma} = \Lambda$. 

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**Definition:** Let $\varsigma$ be a complex number, and let $a(s)$ and $b(s)$ be two operators. We define the $\varsigma$-commutator of $a$ and $b$ as

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The Dynamical Algebra

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- $\mathcal{H}_q(s)\{a(s)\Phi(s)\} = \varsigma^{-1}(\lambda - \Lambda)\{a(s)\Phi(s)\}$, $a(s)$ lowering op.
- $\mathcal{H}_q(s)\{b(s)\Phi(s)\} = (\Lambda + \varsigma \lambda)\{b(s)\Phi(s)\}$, $b(s)$ raising op.
Let us assume that \([a(s), b(s)]_\varsigma = I\), \(\varsigma = q^2\), and \(b(s) = a^+(s)\).
Let us assume that 
\[ [a(s), b(s)] = l, \varsigma = q^2, \text{ and } b(s) = a^+(s). \]
Then one can rewrite the \( q^2 \)-commutator as follows
\[
[a(s), a^+(s)] = l - (1 - q^2) a^+(s) a(s) \equiv q^2 N(s),
\]
where, 
\[
N(s) = \ln[l - (1 - q^2) a^+(s) a(s)] / \ln q^2.
\]
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where, \(N(s) = \ln[I - (1 - q^2) a^+(s) a(s)]/\ln q^2\). \(\Rightarrow \)

\[
[N(s), a(s)] = -a(s), \quad [N(s), a^+(s)] = a^+(s),
\]

i.e., \(N(s)\) is the “number” operator.
The Dynamical Algebra

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\[
[N(s), a(s)] = -a(s), \quad [N(s), a^+(s)] = a^+(s),
\]

i.e., \(N(s)\) is the “number” operator. Next we introduce

\[
\tilde{b}(s) := q^{-N(s)/2} a(s), \quad \tilde{b}^+(s) := a^+(s) q^{-N(s)/2},
\]

which satisfy \(\tilde{b}(s) \tilde{b}^+(s) - q \tilde{b}^+(s) \tilde{b}(s) = q^{-N(s)}\).
The Dynamical Algebra

The operators $\tilde{b}(s)$, $\tilde{b}^+(s)$, and $N(s)$ lead to the dynamical algebra $su_q(1, 1)$ with the generators ($\beta^{-1} = q + q^{-1}$.)

\[
K_0(s) = \frac{1}{2} \left( N(s) + 1/2 \right), \quad K_+(s) = \beta (\tilde{b}^+(s))^2, \quad K_-(s) = \beta \tilde{b}^2(s),
\]

\[
[K_0(s), K_\pm(s)] = \pm K_\pm(s), \quad [K_-(s), K_+(s)] = [2K_0(s)]q^2,
\]

of the algebra $su_q(1, 1)$. 

On q-polynomials and some of their applications Renato Álvarez-Nodarse 28/37
The operators \( \tilde{b}(s) \), \( \tilde{b}^+(s) \), and \( N(s) \) lead to the dynamical algebra \( su_q(1, 1) \) with the generators (\( \beta^{-1} = q + q^{-1} \)).

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of the algebra \( su_q(1, 1) \).

Similarly we can derive the dynamical algebra \( su_q(2) \).
Let us now pose the **problem 1**:

Given the operator $\mathcal{H}_q(s)$

To find two operators $a(s)$ and $b(s)$ and $\varsigma \in \mathbb{C}$ such that:

1) the Hamiltonian $\mathcal{H}_q(s) = b(s)a(s)$
2) $[a(s), b(s)]\varsigma = I$. 

For the first question we already have the answer: the operators $b(s) = a^{\uparrow \alpha}(s)$ and $a(s) = a^{\downarrow \alpha}(s)$ factorize the Hamiltonian $\mathcal{H}_q(s) = a^{\uparrow \alpha}(s)a^{\downarrow \alpha}(s)$. 

The question is under which conditions they also satisfy the commutation relation.

In the following we assume that $A(s) = B(s)$. 

⇒ On q-polynomials and some of their applications Renato Álvarez-Nodarse 29/37
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**Theorem 1: NECESSARY CONDITION**

Let \((\varphi_n)_n\) the eigenfunctions of \(\mathcal{H}_q(s)\) corresponding to the eigenvalues \((\lambda_n)_n\) and suppose that the problem 1 has a solution for \(\Lambda \neq 0\). Then, the eigenvalues \(\lambda_n\) of the NU \(q\)-equation are \(q\)-linear or \(q^{-1}\)-linear functions of \(n\), i.e.,

\[
\lambda_n = C_1 q^n + C_3 \quad \text{or} \quad \lambda_n = C_2 q^{-n} + C_3,
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The Dynamical Algebra: Problem 1

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\]

Impossible to solve: general Askey-Wilson, \(q\)-Racah, big and little \(q\)-Jacobi polynomials.
**Theorem 2:** Necessary and sufficient condition

Let be $\mathcal{H}_q(s)$ the $q$-Hamiltonian defined from the NU Eq. The operators $a_{\alpha}^\uparrow(s)$ and $a_{\alpha}^\downarrow(s)$ factorize the Hamiltonian $\mathcal{H}_q(s)$ and satisfy the relation $[a_{\alpha}^\downarrow(s), a_{\alpha}^\uparrow(s)]_\varsigma = \Lambda$ for $\varsigma \in \mathbb{C}$ iff the following two conditions hold:

$$\frac{\nabla x(s)}{\nabla x_1(s-\alpha)} \sqrt{\frac{\nabla x_1(s-1)\nabla x_1(s)}{\nabla x(s-\alpha)\Delta x(s-\alpha)}} \sqrt{\frac{\sigma(s-\alpha)\sigma(-s-\mu+\alpha)}{\sigma(s)\sigma(-s-\mu+1)}} = \varsigma, \quad \text{and}$$

$$\frac{1}{\Delta x(s-\alpha)} \left(\frac{\sigma(s-\alpha+1)}{\nabla x_1(s-\alpha+1)} + \frac{\sigma(-s-\mu+\alpha)}{\nabla x_1(s-\alpha)}\right) - \varsigma \frac{1}{\nabla x_1(s)} \left(\frac{\sigma(s)}{\nabla x(s)} + \frac{\sigma(-s-\mu)}{\Delta x(s)}\right) = \Lambda.$$

The values $\varsigma$ and $\Lambda$ are uniquely determined!
To find two operators \( a(s) \) and \( b(s) \) and a constant \( \varsigma \) such that the Hamiltonian \( \mathcal{H}_q(s) = b(s)a(s) \) and \( [a(s), b(s)]_\varsigma = I \) and such that \( a(s) \) and \( b(s) \) are the lowering and raising operators, i.e.,

\[
a(s)\varphi_n(s) = D_n \Phi_{n-1}(s) \quad \text{and} \quad b(s)\varphi_n(s) = U_n \Phi_{n+1}(s).
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**Answer:** $\lambda_n$ should be a $\varsigma$-linear function, i.e., $\lambda_n$ has the form $\lambda_n = A\varsigma^n + D$ and this is again a necessary condition.
The Dynamical Algebra: Problem 2

To find two operators $a(s)$ and $b(s)$ and a constant $\varsigma$ such that the Hamiltonian $\tilde{H}_q(s) = b(s)a(s)$ and $[a(s), b(s)]_\varsigma = I$ and such that $a(s)$ and $b(s)$ are the lowering and raising operators, i.e.,

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To find two operators \( a(s) \) and \( b(s) \) and a constant \( \varsigma \) such that the Hamiltonian \( \mathcal{H}_q(s) = b(s)a(s) \) and \([a(s), b(s)]_\varsigma = I\) and such that \( a(s) \) and \( b(s) \) are the lowering and raising operators, i.e.,

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**Answer:** \( \lambda_n \) should be a \( \varsigma \)-linear function, i.e., \( \lambda_n \) has the form \( \lambda_n = A\varsigma^n + D \) and this is again a necessary condition

When the \( \alpha \)-operators are mutually adjoint?

**Answer:** It depends of the “scalar product”. E.g. Discrete case: \( \alpha = 0 \) is a sufficient condition
Example 1: The Al-Salam & Carlitz I

The \( q \)-Hamiltonian is, in this case, \( (x = q^s) \).

\[
\mathcal{H}_q(s) = -\frac{q^2 \sqrt{a(x-1)(x-a)}}{(q-1)^2 x^2} e^{-\partial_s} - \frac{\sqrt{a(1-qx)(a-qx)}}{x^2} e^{\partial_s} + \left( \frac{\sqrt{q(x-1)x+a(1+q-qx))}}{(q-1)^2 x^2} \right) l
\]

Then, \( \mathcal{H}_q(s) \varphi_n(s) = q^{\frac{3}{2}} \frac{1-q^{-n}}{(1-q)^2} \varphi_n(s) \) and the operators

\[
a^\dagger(s) \equiv a^\dagger_0(s) = \frac{1}{q^{-\frac{1}{2}}} \left( \frac{1}{q^4 x^{-1}} \left( \sqrt{(x-1/q)(x-a/q)} e^{\partial_s} - \sqrt{a} l \right) ,
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\]

are such that
Example 1: The Al-Salam & Carlitz I

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Notice that since this is a discrete case when \( \alpha = 0 \) the operators \( a^\uparrow(s) \) and \( a^\downarrow(s) \) are mutually adjoint.

Other related cases:
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- from where putting \( a = -1 \) and \( x \rightarrow ix \) follows the solution for the discrete Hermite \( q \)-polynomials \( \tilde{h}_n(x; q) \)
### Other examples in the \( q \)-Askey Tableau

<table>
<thead>
<tr>
<th>( x(s) )</th>
<th>( P_n(s)_q )</th>
<th>( \sigma(s) + \tau(s)\nabla x_1(s) )</th>
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<td>( x^2 - 1 )</td>
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<tr>
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<td>( 1 + x^2 )</td>
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<tr>
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<td>( q^{-1}x )</td>
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<tr>
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<tr>
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Al-Salam & Carlitz I, II, discrete \( q \)-Hermite I, II, \( q \)-Charlier-type, Stieltjes-Wigert, Wall polynomials, discrete \( q \)-Laguerre, \( q \)-Charlier.
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The Askey-Wilson case: Only for some special cases. continuous $q$-Laguerre and continuous $q$-Hermite polynomials.
Open problems

- What about the lowering and raising properties of the $\alpha$-down and $\alpha$-up operators?
- What happen for the general Askey-Wilson case and for the general big $q$-Jacobi?
- There exists a more general dynamical algebra? Of which kind?
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Last but not least ...

Some other relevant results related with the $q$-polynomials by Natig: Classical-type integral transform formulas: Mellin transforms, Fourier-Gauss transforms, etc.
Natig Atakishiyev’s secret: “Los problemas matemáticos hacen que canalicemos nuestros esfuerzos y nos ayudan a sobrevivir. Solucionar un enigma es lo más gratificante que hay. Si no tuviéramos este tipo de incógnitas esperándonos al día siguiente, los matemáticos no viviríamos tanto”