# Infinite Dimensional Lie Algebras and Superalgebras.

Nov 19, 2015, Mexico City

## 1. Infinite Dimensional Lie Algebras.

Lie algebras $/\mathbb{C}$

$L = \bigoplus_{i\in\mathbb{Z}} L_i$, $[L_i, L_j] \subseteq L_{i+j}$ graded

<u>Polynomial growth</u>: $\exists\, p(t) \in \mathbb{C}[t]$:

$\dim L_i \leq p(|i|)$

$$\mathrm{GKdim}\, L = \limsup_{n\to\infty} \frac{\ln \dim L_n}{\ln n}$$

Gelfand-Kirillov dimension.

Graded simple : no nontrivial graded ideals.

EX. 1 $g$ finite dimensional simple Lie algebra, $L = g[t^{-1}, t] = \bigoplus_{i\in\mathbb{Z}} g t^i$

$\mathrm{GKdim}\, L = 1$

EX. 1' (twisted version) $g = g_0 + g_1 + \cdots + g_{m-1}$, $\mathbb{Z}/m\mathbb{Z}$-graded, $L = \sum_{i \equiv j \bmod m} g_i t^j$.

EX. 2-3-4-5. $L = W_n = \mathrm{Der}\, F[t_1, \ldots, t_n]$ $= \{\sum f_i \partial/\partial t_i\}$, $\deg t_i = 1$, $\deg \partial/\partial t_i = -1$

$L = L_{-1} + L_0 + L_1 + \cdots$, $\mathrm{GKdim}\, L = n$

$S_n$ special, $H_{2n}$ Hamiltonian,

$K_{2n+1}$ contact

<u>EX. 6.</u> $Vir = \operatorname{Der} F[t^{-1}, t] = \sum_{i \in \mathbb{Z}} F t^{i+1} \frac{d}{dt}$

(centerless) Virasoro algebra.

<u>Theorem</u> (V. Kac, 67; O. Mathieu, 91)

All graded simple Lie algebras of polynomial growth are Ex. 1 - 6.

One can consider $\mathbb{Z}^n = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n}$ - graded algebras. Then we allow $\operatorname{Der} F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, $S$, $H$, $K$.

Recently, Iohara - Mathieu: simple $\mathbb{Z}^n$-graded algebras with all $\dim L$ [illegible] $= 1$

## 2. Superalgebras.

$A = A_{\bar{0}} + A_{\bar{1}}$ $\mathbb{Z}/2\mathbb{Z}$-graded algebras.

Ex. 1. $M_{m+n}(\mathbb{C}) = \begin{array}{c} \left(\begin{array}{c|c} \ast & \\ \hline & \ast \end{array}\right) \end{array}\begin{array}{l} m \\ n \end{array} + \left(\begin{array}{c|c} & \ast \\ \hline \ast & \end{array}\right)$

Ex. 2. $G = \langle 1, \xi_1, \xi_2, \dots \mid \xi_i^2 = 0,$

$\xi_i \xi_j + \xi_j \xi_i = 0 \rangle$

Basis: $1, \xi_{i_1} \cdots \xi_{i_k}, \quad i_1 < i_2 < \dots < i_k.$

$$G = G_{\bar{0}} + G_{\bar{1}}$$

a superalgebra.

Def. $G(A) = A_{\bar{0}} \otimes G_{\bar{0}} + A_{\bar{1}} \otimes G_{\bar{1}} < A \otimes G$

Grassmann envelope (Berezin).

$\mathcal{V}$ = a variety of algebras.

Def. A is a $\mathcal{V}$-superalgebra if $G(A) \in \mathcal{V}$.

$\rightsquigarrow$ Lie Superalgebra.

V. Kac (76) : classification of finite dimensional simple Lie superalgebras.

## 3. Superconformal Algebras.

Neveu, Schwarz, Ramon, Seiberg et al, et al ... :

Superextensions of the Vir

("superconformal algebras")

$L = L_{\bar{0}} + L_{\bar{1}}$ Lie Superalgebra

$Vir \subseteq L_{\bar{0}}$

Kac – Van de Leur (observation):

$L = \sum_{i \in \mathbb{Z}} L_i$ graded simple, $\dim L_i \leq d$

Search for Superconformal Algebras.

- $g[t^{-1}, t] = \sum_i g t^i$ is graded simple and $\dim L_i = \dim g = d$, but $g[t^{-1}, t]$ does not contain $Vir$
- $\Lambda(1:n) = \mathbb{C}[t^{-1}, t, \xi_1, \dots, \xi_n] = \mathbb{C}[t^{-1}, t] \otimes G(n)$

$W(1:n) = \mathrm{Der}\, \Lambda(1:n) = \left\{ \sum_{i=1}^{n} f_i \partial/\partial\xi_i + f \frac{d}{dt} \right\}$

• For $D = f\frac{d}{dt} + \sum_i f_i \partial/\partial\xi_i$ let

$$\operatorname{div}(D) = \frac{\partial f}{\partial t} + \sum_i (-1)^{|f_i|} \frac{\partial f}{\partial \xi_i}$$

$$S(1:n) = \{D \in W(1:n) \mid \operatorname{div} D = 0\}$$

• Poisson brackets $[\ ,\ ]$ on an associative commutative superalgebra $R = R_{\bar 0} + R_{\bar 1}$:

(i) $(R, [\ ,\ ])$ is a Lie superalgebra,

(ii) $[ab, c] = a[b,c] + (-1)^{|b|\cdot|c|}[a,c]b$.

A Poisson bracket on $\Lambda(1:n)$

$\rightsquigarrow$ a ~~superconfor~~ superalgebra of Hamiltonian type.

But (!) we need an even number of $t$'s.

So: no superconformal algebras of Hamiltonian type.

- Contact brackets:

(i) $(R, [\ ,\ ])$ a Lie superalgebra,

(ii) $D(a) = [a, 1]$ is a derivation,

(iii) $[ab, c] = a[b, c] + (-1)^{|b|\cdot|c|}[a, c]b + abD(c)$.

EX. (1) Poisson brackets are contact brackets with $D = 0$;

(2) $R = \mathbb{C}[t]$, $[a, b] = a'(t)b(t) - a(t)b'(t)$;

(3) for $f \in \Lambda(1:n)$ denote $\Delta(f) = 2f - \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial \xi_i}$. Then $[f, g] = \Delta(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\Delta(g) + (-1)^{|f|}\sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_i}$

is a contact bracket.

$\rightsquigarrow$ superconformal algebra $K(1:n)$.

<u>Question:</u> is that all ($W(1:n)$, $S(1:n)$, $K(1:n)$) as in Mathieu's Theorem?

1996 <u>Cheng-Kac</u>, <u>Grozman-Leites-Shchepochkina</u> : a new superconformal algebra $CK(6)$.

<u>C. Martinez - E.Z.</u> (2001): $R = R_{\bar{0}} + R_{\bar{1}}$ associative commutative superalgebra, $d : R \to R$ even derivation $\rightsquigarrow$

$CK(R, d)$ , $CK(6) = CK(F[t^{-1}, t], \frac{d}{dt})$.

$$W = W(R, d) = \sum_{i=0}^{\infty} R d^i, \quad da - ad = d(a)$$

$$CK(R, d) \subset M_8(W).$$

Current Conjecture: $W(1:n)$, $S(1:n)$, $K(1:n)$, $CK(6)$.

Kac-Martinez-Z. (2001): "Jordan" superconformal algebras: $K(1:2n+1)$, $CK(6)$.

Kac-Fattori (2002): conformal case.

In all superalgebras above $\exists$ bases $e_{i1}, \dots, e_{id} \in L_i$, $i \in \mathbb{Z}$ such that

$$[e_{ip}, e_{jq}] = \sum_{r=1}^{d} \gamma_{pqr}(i,j)\, e_{i+j,r},$$

$\gamma_{pqr}(i,j)$ are polynomials in $i,j$; $1 \le p,q,r \le d$.

## Different language:

$$e_p(z) = \sum_{i \in \mathbb{Z}} e_{ip} z^{-i-1}$$

a formal distribution.

S. Sobolev, L. Schwartz

Locality: formal distributions $a(z), b(z) \in L[[z^{-1}, z]]$ are mutually local if $\exists\ N \geq 1$:

$$[a(z), b(w)](z-w)^N = 0.$$

Partial product:

$$a(z) \circ_k b(z) = \mathrm{Res}_w [a(z), b(w)](z-w)^k$$

$C \subseteq L[[z^{-1}, z]]$ is a conformal algebra of formal distributions if: (i) $\forall\, a(z), b(z) \in C$ are mutually local; (ii) $C$ is closed under all operations $o_k$, $k \in \mathbb{Z}_{\geq 0}$; (iii) $C$ is closed under $\frac{d}{dz}$.

Def. $C$ is of <u>finite type</u> if it is a finitely generated $\mathbb{C}[\frac{d}{dz}]$-module.

<u>Kac-Fattori</u>: classification of conformal Lie superalgebras of finite type $\Rightarrow$ classification of superconformal algebras, that lead to conformal ... algebras of finite type.

**Theorem** (Martinez - Z.)

Let $L$ be a superconformal algebra that contains $S(1:2)$. Then $L = W(1:n)$, $S(1:n)$, $K(1:n)$ or $CK(6)$.

## Representations.

What kinds of representations?

Harish-Chandra     Conformal

### 1) Harish-Chandra

Superconformal algebras are $\mathbb{Z}$-graded.

**Def.** A module $V$ is H-C if

$$V = \sum_{i \in \mathbb{Z}} V_i \quad , \quad \dim V_i < \infty \quad \forall i$$

O. Mathieu (early success, 91):

$$Vir = Der\, F[t^{-1}, t] = \sum_{i \in \mathbb{Z}} F\, t^{i+1} \frac{d}{dt}$$

All Harish-Chandra irreducible modules are either highest weight

$V = \sum_{i=-\infty}^{N} V_i$ or lowest weight

$V = \sum_{i=-N}^{\infty} V_i$ or the so called intermediate

modules (Kaplansky-Santharoubane, Feigin-Fuchs):

$$V = \overline{F[t^{-1}, t]} \; ; \; \alpha, \beta \in F$$

$$\left( f(t) \frac{d}{dt} \right) \overline{g(t)} = \overline{f g' + \alpha f' g + \beta \frac{1}{t} f g}$$

Irreducible unless $\alpha = 0$ or $1$, $\beta \in \mathbb{Z}$.

Recently Y. Billig - V. Futorny :
irreducible Harish-Chandra modules
over $\mathbb{Z}^n$-graded $W(n) = \mathrm{Der}\, \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

2) Conformal modules = polynomial structural constants = formal distributions, locality.

In the last 25 years V. Kac + his students (D'Andrea, Bakalov, Cheng, Fattori et el): classification of irreducible conformal modules of finite type.

C. Martinez - Z. : another approach for CK(6).

C. Martinez - Z. : classification of Harish-Chandra modules over all superconformal algebras $W(1:n)$, $n \geq 2$; $S(1:n)$; $K(1:n)$, $n \geq 3$; except for small cases.

Idea: Harish-Chandra = Conformal.

We construct formal distributions over Harish-Chandra modules and prove locality.