Homological characterizations of Minkowski summands

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Received 10 June 1999; received in revised form 20 June 2000

To Benny Rushing

Abstract

The purpose of this paper is to give several topological characterizations of Minkowski summands. These techniques allow us to characterize when two convex sets are homothetic. We use these results to give two characterizations of the solid sphere. © 2002 Elsevier Science B.V. All rights reserved.

AMS classification: 52A20

Keywords: Summands; Homothetic

1. Introduction

The sum or Minkowski Addition of two subsets $A, B \subseteq E^n$ is defined by:

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

For convex bodies $K, L$, we say that $L$ is a summand of $K$ if there exists a convex body $M$ such that $L + M = K$.

Let $L \subseteq E^n$ be a convex body. We are interested in the set of all convex bodies $K \subseteq E^n$ which are unions of translated copies of $L$, or in other words, whose complements are intersection of translated copies of $E^n - L$. That is, we are interested in those convex sets $K$ for which $L$ is a summand of $K$. We shall think of the notion of Minkowski addition as an abstract “convexity”, that we shall call internal $L$-convexity, where the complement of the interior of the sets $K$, for which $L$ is a summand of $K$, play the role of “convex sets”, the boundaries of the translated copies of $L$ play the role of the “hyperplanes” and the complement of the interior of the translated copies of $L$ play the role of closed...
“halfspaces”. There are strong analogies between this internal convexity and the classic theory of convexity. For example, Goodey’s homological characterization of summands [4] is the analogue of the Kosinski–Aumann Theorem [6] for convex sets.

The purpose of this paper is to exploit these analogies, using techniques from Algebraic Topology, as homology theory and the theory of acyclic sections of fiber bundles, to obtain several characterizations of summands. Section 3 is devoted to obtain a characterization of summands in terms of acyclic support sets. In Section 2, we obtain a generalization of Goodey’s Theorem to characterize when two convex sets are homothetic. We use this result to give two characterizations of the solid sphere in the spirit of Fujiwara [2], Bol [1], Yanagihara [13], Goodey [3], Goodey and Woodcock [5] and Mani-Levitska [8].

During this paper, we will use reduced Čech-homology [cohomology] with $\mathbb{Z}_2$-coefficients. A compact space $X$ is acyclic if $H_*(X) = H^*(X) = 0$.

Let us denote by $K^n$ the set of convex bodies contained in Euclidean $n$-space $E^n$.

2. Characterizations of the sphere

The Kosinski–Aumann Theorem [6] states that if every section of a compact set is acyclic, then the set is convex. Analogously, Goodey’s Theorem states that given two convex bodies $K, L \in K^n$ such that $\text{bd} \ K \cap \text{int} \ L'$ is acyclic, for every translate $L'$ of $L$, then $L$ is a summand of $K$. The following characterization of summands which follows from Goodey’s Theorem will be used here. First, we need a definition.

If $K, L \in K^n$, then we say that $L$ is never properly contained in $K$ if given any translated copy $L'$ of $L$ such that $\text{int} \ L' \subset K$, we have that $L' = K$. For example, if $K$ and $L$ have the same $n$-dimensional volume, then one is never properly contained in the other.

**Theorem 2.1.** Let $K, L$ be two convex bodies such that $K$ is never properly contained in $L$ and for every translated copy $L'$ of $L$ such that $\text{int} \ L'$ cuts $\text{bd} \ K$, we have that $\text{bd} \ K \cap \text{bd} \ L'$ is a homological $(n - 2)$-sphere. Then, $L$ is a summand of $K$.

**Proof.** Suppose that $L'$ is a translated copy of $L$ such that $\text{int} \ L'$ intersects $\text{bd} \ K$. Then, since $K$ is not contained in $L'$, we have that $\text{bd} \ K$ is not contained in $L'$. Since $\text{bd} \ K \cap \text{bd} \ L'$ is a homological $(n - 2)$-sphere contained in $\text{bd} \ K$, by Alexander Duality, we have that $\text{bd} \ K - \text{bd} \ L'$ has exactly two nonempty acyclic open components, one of them is $\text{bd} \ K - L'$ and the other $\text{bd} \ K \cap \text{int} \ L'$. Consequently, by Goodey’s Theorem [4], $L$ is a summand of $K$. $\square$

The next characterization of translates follows from the above:

**Theorem 2.2.** Let $K, L$ be two convex bodies such that one is never properly contained in the other. Then, $L$ is a translated copy of $K$ if and only if for every translated copy $L'$ of
$L$, distinct from $K$, that shares a common interior point with $K$, we have that $\text{bd } K \cap \text{bd } L'$ is a homological $(n-2)$-sphere.

**Proof.** Suppose that $L'$ is a translated copy of $L$ such that $\text{int } L'$ intersects $\text{bd } K$. Then $L'$ shares an interior point with $K$ and therefore, by Theorem 2.1, $L$ is a summand of $K$. Analogously, $K$ is a summand of $L$ and therefore, $L$ is a translated copy of $K$. For the converse, note that if $L'$ is a translated copy of $L$, distinct from $L$, by Corollary 1 of [4], we have that $\text{int } L \cap \text{bd } L'$ and $\text{int } L' \cap \text{bd } L$ are both homeomorphic to $E^{n-1}$. Moreover, the $(n-1)$-sphere $\text{bd}(L' \cap L)$ has the property that

$$\text{bd}(L' \cap L) - [(\text{int } L \cap \text{bd } L') \cup (\text{int } L' \cap \text{bd } L)] = \text{bd } L' \cap \text{bd } L.$$

Consequently, $\text{bd } L' \cap \text{bd } L$ is a homology $(n-2)$-sphere. □

The next theorem is an immediate corollary of the above theorem and generalizes results of Fujiwara [2] and Bol [1], Yanagihara [13], Goodey [3,5] and Mani-Levitska [8].

**Theorem 2.3.** Let $K \in \mathbb{K}^n$ be a convex body. Then, $K$ is a Euclidean $n$-ball if and only if for every congruent copy $K'$ of $K$, distinct from $K$, that shares a common interior point with $K$, we have that $\text{bd } K \cap \text{bd } K'$ is a homological $(n-2)$-sphere.

**Proof.** Let $K'$ be a congruent copy of $K$. Since both $K$ and $K'$ have the same $n$-dimensional volume, then one is never properly contained in the other. Suppose that $K''$ is a translated copy of $K'$ that shares a common interior point with $K$. By hypothesis, $\text{bd } K \cap \text{bd } K''$ is a homological $(n-2)$-sphere and therefore, by Theorem 2.2, $K'$ is a translated copy of $K$. Consequently, $K$ is a Euclidean $n$-ball. □

In our next theorem we replace the hypothesis that $\text{bd } K \cap \text{bd } K'$ is a homological $(n-2)$-sphere by the hypothesis that $\text{bd } K - \text{bd } L'$ has exactly two connected components. Although the spirit is the same, the theorem and its proof differs substantially from Goodey’s Theorem.

**Theorem 2.4.** Let $K, L \in \mathbb{K}^n$, such that $L \subset \text{int } K$. Suppose that for every translated copy $L'$ of $L$ such that $\text{bd } L'$ meets $\text{bd } K$ in more than one point, we have that $\text{bd } K - \text{bd } L'$ [or equivalently, $\text{bd } L' - \text{bd } K$] has exactly two connected components. Then $L$ is a summand of $K$.

**Proof.** By Theorem 3.7 of [10], it will be enough to prove that for every translated copy $L'$ of $L$ such that $L'$ meets $\text{bd } K$ and $L' \subset K$, we have that $\text{bd } L' \cap \text{bd } K$ consists of a single point. Let $L'$ be such that $L'$ meets $\text{bd } K$, $L' \subset K$ and $\text{bd } L'$ meets $\text{bd } K$ in more than one point. Then, $\text{bd } K - \text{bd } L'$ has exactly two connected components, and consequently, $H_{n-2}(\text{bd } L' \cap \text{bd } K) = \mathbb{Z}_2$. Since $L \subset \text{int } K$, there is a unit vector $\nu_0$ such that for every $\epsilon > 0$ sufficiently small, $(\text{bd } L' \cap \text{bd } K) + \epsilon \nu_0 \subset \text{int } K$. Let $W$ and $V$ be the connected components of $\text{bd } K - \text{bd } L'$ and suppose that $W$ is such that for $\epsilon > 0$...
sufficiently small, $w + \varepsilon v_0 \subseteq \text{int} K$. Let $X = W \cup (\text{bd} L' \cap \text{bd} K)$. It is not difficult to verify that $H_{n-1}(X) = H_{n-2}(X) = 0$.

Let $Y = X \cup L'$. Note that $X \cap L' = \text{bd} L' \cap \text{bd} K$ and therefore, by Mayer–Vietoris, $H_{n-1}(Y) = Z_2$. This implies that $E^n - Y$ has exactly two components. Let $V_b$ be the bounded component of $E^n - Y$ and $V_u$ the unbounded component.

Let $w_0$ be any point of $W$. Then, for every $t < 0$, $w_0 + tv_0 \in V_u$ and for $\varepsilon > 0$ sufficiently small, $w_0 + \varepsilon v_0 \in V_b$. In fact, let us consider that $\varepsilon > 0$ is so small that $w_0 + \varepsilon v_0 \in \text{int} K - L'$, $\text{bd}(K + \varepsilon v_0) \cap \text{int} L'$ is non empty and $\text{bd}(K + \varepsilon v_0)$ meets $\text{bd} L'$ in more than one point. Therefore, by hypothesis, $\text{bd}(K + \varepsilon v_0) - \text{bd} L'$ has exactly two connected components which are the connected open sets $\text{bd}(K + \varepsilon v_0) \cap \text{int} L'$ and $\text{bd}(K + \varepsilon v_0) - L'$.

Let $w_1$ be a point of $\text{bd} K$ such that for every $t > 0$, $w_1 + tv_0 \notin K$. Then, the points $w_0 + \varepsilon v_0$ and $w_1 + \varepsilon v_0$ belong to $\text{bd}(K + \varepsilon v_0) - L'$ and consequently there is an arc $\gamma \subseteq \text{bd}(K + \varepsilon v_0) - \text{bd} L'$ between the point $w_0 + \varepsilon v_0$ and the point $w_1 + \varepsilon v_0$. It is clear that $\gamma \subseteq E^n - Y$ because $\gamma \cap L' = \emptyset$ and $\gamma \cap X = \emptyset$, otherwise $\text{bd}(K + \varepsilon v_0) \cap X \neq \emptyset$.

The existence of the arc $\gamma$ and the ray $\{w_1 + tv_0 \mid \varepsilon < t\}$ implies that the point $w_0$ is in $V_u$, the unbounded component of $E^n - Y$, which is a contradiction. This completes our proof. \qed

As a corollary of Theorem 2.4, we have a characterization of when two strictly convex bodies are homothetic.

**Theorem 2.5.** Let $K, L \in \mathbb{R}^n$ be two convex bodies. Then $K$ and $L$ are two homothetic strictly convex bodies if and only if for every homothetic copy $L'$ of $L$, distinct from $K$, such that $\text{bd} L'$ meets $\text{bd} K$ in more than one point, we have that $\text{bd} K - \text{bd} L'$ has exactly two connected components.

**Proof.** Assume that $K$ and $L$ are two different homothetic strictly convex bodies, such that $\text{bd} L$ meets $\text{bd} K$ in more than one point. First of all note that one is a summand of the other, let say $L$ is a summand of $K$. By Theorem 3.5 of [10], $\text{int} L \cap \text{bd} K$ is convex in $\text{bd} K$. Therefore, by Lemma 3.3(ii) of [10], $\text{bd} L \cap \text{bd} K$ is a homological $(n - 2)$-sphere and hence $\text{bd} K - \text{bd} L$ has exactly two connected components.

For the converse, first note that by Theorem 2.4, if $L'$ is a homothetic copy of $L$ which is contained in the interior of $K$, then $L'$ is a summand of $K$. Let $L''$ be the homothetic copy of $L$ contained in $K$ with the property that its volume is maximal. Clearly, $L''$ is a summand of $K$. Let us suppose $L'' + I = K$. Then $I$ must be a convex body with empty interior. Note that if we assume that the origin is in the relative interior of $I$, then it is easy to see that $\text{bd}(L'' + I) - \text{bd} I$ has more than one point but only one component, except when $I$ is an interval. By the same argument, there is a homothetic copy $K''$ of $K$ and an interval $I'$ such that $K'' + I' = L$. Consequently, either $K$ and $L$ are homothetic strictly convex bodies or both are parallelepipeds, which is impossible. This concludes the proof of the theorem. \qed
The next theorem characterizes the solid sphere.

**Theorem 2.6.** Let \( K \in \mathbb{K}^n \) be a convex body. Then, \( K \) is a Euclidean \( n \)-ball if and only if for every similar copy \( K' \) of \( K \), distinct from \( K \), such that \( \text{bd} \, K' \) meets \( \text{bd} \, K \) in more than one point, we have that \( \text{bd} \, K - \text{bd} \, K' \) has exactly two connected components.

**Proof.** The proof follows immediately from Theorem 2.5 because every congruent copy \( K' \) of \( K \) is also homothetic to \( K \) and hence a translated copy. \( \square \)

### 3. Internal acyclic support sets

We start this section following the ideas of the characterization of convex sets in terms of acyclic support sets [11], always using the analogy between the notion of convexity and internal \( L \)-convexity.

Let \( L \subset E^n \) be a convex body and let \( K \subset E^n \) be a convex body that contains \( L \). For every unit vector \( v \in S^{n-1} \), let

\[
\gamma(v) = \max \{ h(L + t, v) \mid t \in E^n \text{ and } (L + t) \subset K \},
\]

and

\[
\Gamma(v) = \{ t \in E^n \mid h(L + t, v) = \gamma(v) \text{ and } (L + t) \subset K \},
\]

where \( h(L, \cdot) \) is the support function.

Finally, for every direction \( v \in S^{n-1} \), define

\[ i_L(K, v) := \bigcup_{t \in \Gamma(v)} ((L + t) \cap \text{bd} K). \]

Two points, \( A, B \in \text{bd} \, K \), are called **interior \( L \)-antipodal points** if there is a direction \( v \) such that \( A \in i_L(K, v) \) and \( B \in i_L(K, -v) \). A line \( \Lambda \) is called an **interior \( L \)-diametral line** if \( \Lambda \) passes through two different interior \( L \)-antipodal points.

Let \( (L + t) \) be a translated copy of \( L \) such that \( (L + t) \subset K \) and \( (L + t) \cap \text{bd} \, K \neq \emptyset \). Note that \( t \in \Gamma(v) \), for some \( v \in S^{n-1} \). If this is the case, \( (L + t) \cap \text{bd} \, K \) will be called an **interior \( L \)-support set of \( K \) in the direction \( v \)**.

**Theorem 3.1.** Let \( K, L \in \mathbb{K}^n \), such that \( L \subset K \). Suppose that every interior \( L \)-support set of \( K \) is acyclic and that for every \( v \in S^{n-1} \), \( i_L(K, v) \) and \( i_L(K, -v) \) can be separated by a hyperplane orthogonal to \( v \). Then, \( L \) is a summand of \( K \).

**Proof.** We shall follow the proof of Theorem 4 of [11]. Let us prove first that the set of all interior \( L \)-diametral lines of \( K \) covers \( \text{int} \, K \).

Let us suppose that there is no interior \( L \)-diametral line of \( K \) that passes through the origin, which is an interior point of \( K \). We start the proof with the definition of the following compact subsets of \( E^n \times E^n \). For every \( v \in S^{n-1} \) let

\[ T(v) = \{(t, x) \in E^n \times E^n \mid t \in \Gamma(v) \text{ and } x \in ((L + t) \cap \text{bd} K)\}, \]
which is the disjoint union of all interior $L$-support sets of $K$ in the direction $v$. To see this, define the continuous map $\tau_v : T(v) \to \Gamma(v)$ by $\tau_v(t,x) = t$, for every $(t,x) \in T(v)$. Then, the point inverses of $\tau_v$ are precisely the interior $L$-support sets of $K$ in the direction $v$, because $\tau_v^{-1}(t) = ((L + t) \cap \text{bd} K)$. Consequently, by the Vietoris–Begle Theorem [7], the map $\tau_v : T(v) \to \Gamma(v)$ induces isomorphisms in homology, which implies that $T(v)$ is a compact acyclic set because $\Gamma(v)$ is a compact convex set.

Next, we shall construct a Euclidean bundle appropriated for our purposes. We start with the Euclidean trivial bundle
\[
\pi : S^{n-1} \times E^n \times E^n \times E^n \to S^{n-1},
\]
and then, we construct a Euclidean bundle over $\mathbb{R}P^{n-1}$ identifying, for every $v \in S^{n-1}$, the fibre $\pi^{-1}(v)$ with the fibre $\pi^{-1}(-v)$ through the homeomorphism $h_v : \pi^{-1}(v) \to \pi^{-1}(-v)$ given by $h_v(v,t,x,s,y) = (-v,s,y - w_K(v)v,t,x - w_K(v)v)$. Note that $h_v^{-1} = h_{v}$. Consequently, we obtain a Euclidean bundle over $\mathbb{R}P^{n-1}$.

$\chi : E \to \mathbb{R}P^{n-1}$,

where the total space $E$ is given by
\[
E = \frac{S^{n-1} \times E^n \times E^n \times E^n}{(v,t,x,s,y) \sim (-v,s,y - w_K(v)v,t,x - w_K(v)v)}
\]
and the map $\chi : E \to \mathbb{R}P^{n-1}$ is given by $\chi([v,t,x,s,y]) = [v] \in \mathbb{R}P^{n-1}$, for every $[v,t,x,s,y] \in E$.

Finally, let us consider the Whitney sum
\[
\gamma \oplus \chi : E \oplus E \to \mathbb{R}P^{n-1},
\]
where $\gamma : E \to \mathbb{R}P^{n-1}$ is the standard Euclidean bundle of $(n-1)$-planes over $\mathbb{R}P^{n-1}$.

In order to use the Whitney Sum Acyclic Theorem 2.1, of [11], we must consider the following closed subset of $E \oplus E$
\[
X = \left\{ [x \ast y, v, t, x + w_K(v)v, s, y] \in E \oplus E \mid v \in S^{n-1}, \ t \in \Gamma(v), \ x \in ((L + t) \cap \text{bd} K), \ s \in \Gamma(-v), \ y \in ((L + s) \cap \text{bd} K) \right\}
\]
\[
= \left\{ [x \ast y, -v, s, y - w_K(v)v, t, x] \in E \oplus E \mid -v \in S^{n-1}, \ s \in \Gamma(-v), \ y \in ((L + s) \cap \text{bd} K), \ t \in \Gamma(v), \ x \in ((L + t) \cap \text{bd} K) \right\},
\]
where by $x \ast y$ we denote the point in which the line that passes through $x$ and $y$ cuts the linear subspace of $E^n$ orthogonal to $v$. The existence of the point $x \ast y$ follows from the fact that the sets $i_L(K,v)$ and $i_L(K,-v)$ can be separated by a hyperplane orthogonal to $v$.

Clearly, $X$ is a closed subset of $E \oplus E$ with the property that for every $[v] \in \mathbb{R}P^{n-1}$, $(\gamma \oplus \chi)^{-1}([v]) \cap X$ is homeomorphic to $T(v) \times T(-v)$ which is a compact acyclic set because the product of two compact acyclic sets is a compact acyclic set.

By the Whitney Sum Acyclic Theorem 2.1 of [11], there exists $[v_0] \in \mathbb{R}P^{n-1}$ such that $(\gamma \oplus \chi)^{-1}([v_0]) \cap X \cap E \neq \emptyset$. 

which implies that there are two different interior $L$-antipodal points $x \in i_L(K, v)$ and $y \in i_L(K, -v)$ such that the corresponding interior $L$-diametral line passes through the origin because $x \ast y = 0$, which is a contradiction. This proves that through every point of $\text{int} K$ passes an interior $L$-diametral line.

Let now $K_L$ denote the set of all boundary points of $K$ which lie in a translate $(L + t) \subset K$. Since through every point of $\text{int} K$ passes an interior $L$-diametral line, then the set of all segments determined by points in $K_L$ covers $\text{int} K$. Then, $\text{bd} K = K_L$, because $K_L$ is closed, and hence $L$ slides freely inside $K$, thus proving that $L$ is a summand of $K$. This concludes the proof of Theorem 3.1. $\square$

As a consequence we obtain another proof of Weil’s Theorem [12].

**Theorem 3.2.** Let $L \subset K \subset \mathbb{E}^n$ be two convex bodies. Suppose that for every translated copy $(L + t)$ of $L$ such that $(L + t)$ is not contained in $K$, we have that the set of all points of $(L + t)$ farthest from $K$ is a supporting set of $(L + t)$. Then, $L$ is a summand of $K$.

**Proof.** Let $r > 0$ and let $A$ be an interior $L$-support set of $K_r$ in the direction $v$, where $K_r = K + B_r$ and $B_r$ is the unit ball of radius $r$. Then, $A = (L + t) \cap \text{bd} K_r$, where $(L + t) \subset K_r$, is a translated copy of $L$. By hypothesis, $A$ is a support set of $(L + t)$, which implies that $A = H \cap (L + t)$, where $H$ is a supporting hyperplane of $(L + t)$. Hence, $A$ is convex and therefore acyclic. Let $A \in \text{rel int} \ A \subset \text{bd} K_r$ and let $H(K_r, u)$ be a supporting hyperplane of $K_r$ at $A$ which has unit normal $u$, then $A \subset H(K_r, u) \cap K_r$. This implies that there is $A' \subset \text{bd} K$, such that $A' + u = A$ and therefore, $(A' + B_r) \subset (K + B_r) = K_r$.

Now, it is not difficult to see that if $u = v$, otherwise there would be a translated copy of $L$, $L' \subset K_r$, for which $h((L + t), v) < h(L', v)$, which is a contradiction to the fact that $A$ is an interior $L$-support set of $K_r$ in the direction $v$.

This implies that $i_L(K_r, v) \subset H(K_r, v)$ and similarly that $i_L(K_r, -v) \subset H(K_r, -v)$, but hence $i_L(K_r, v)$ and $i_L(K_r, -v)$ can be separated by a hyperplane orthogonal to $v$. Then, by Theorem 3.1, $L$ is a summand of $(K_r)$, for every $r > 0$, which implies that $L$ is a summand of $K$. $\square$

An important corollary of Theorem 3.1, Theorem 3.7 of [10] and Lemma 3.3 of [10], is the following theorem

**Theorem 3.3.** Let $K, L \in \mathbb{K}^n$ and suppose that $2L \subset \text{int} K$. Then, $L$ is a summand of $K$ if and only if for every translated copy $(L + t)$ of $L$ contained in $K$, we have that $(L + t) \cap \text{bd} K$ is either empty or acyclic.

**Proof.** We just have to show that for every $v \in S^{n-1}$, $i_L(K, v)$ and $i_L(K, -v)$ can be separated by a hyperplane orthogonal to $v$, but this is true, in this case, because $2L \subset \text{int} K$. $\square$
Another corollary is:

**Corollary 3.4.** Let $K, L \in K^n$. Suppose that $2L \subset K$. Then, $L$ is a summand of $K$ if and only if for every translated copy $(L + t)$ of $L$ such that $(L + t)$ is not contained in $K$, we have that the set of all points of $(L + t)$ farthest from $K$ is an acyclic set.

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