A QUICK PROOF OF SINGHOF'S THEOREM \( \text{CAT}(M \times S^1) = \text{CAT}(M) + 1 \)

Luis Montejano

For a topological space \( X \), the Lusternik-Schnirelmann category, \( \text{cat}(X) \), is the smallest number \( N \) such that \( X \) can be covered by \( N \) open subsets each of which is contractible in \( X \). W. Singhof [6] proved that the minimal number of \( n \)-balls which suffice to cover a closed PL \( n \)-manifold \( M \) coincides with the Lusternik-Schnirelmann category if the latter is not too small compared with the dimension of \( M \). As a consequence, one obtains in this case that \( \text{cat}(M \times S^1) = \text{cat}(M) + 1 \), thus establishing a special case of a long-standing conjecture. The purpose of this paper is to give a quick proof of Singhof's results by exploiting the linear structure between the \( k \)-skeleton of a polyhedron and its dual skeleton.

**MAIN LEMMA.** Let \( M \) be a closed \( p \)-connected PL \( n \)-manifold. Let \( \{P_1, \ldots, P_N\} \) be a polyhedral cover of \( M \). Then for every \( 0 \leq q < p \) there exists a polyhedral cover \( \{R_1, \ldots, R_N\} \) of \( M \) such that:

- a) \( R_i \) can be deformed in \( M \) into \( P_i \), \( 1 \leq i \leq N \), and
- b) \( R_i \) is a regular neighborhood of \( N_i \), where \( \dim N_i \leq \text{Max} \{n-(N-1)(q+1), q\} \), \( 1 \leq i \leq N \).

**PROOF.** We start the proof by proving the following fact: Let \( X, Y \) be subpolyhedra of \( M \), \( \{P_1, \ldots, P_N\} \) be a polyhedral cover of \( X \) in \( M \), and \( \dim Y < N(q+1) \). Then there exists a polyhedral cover \( \{R_1, \ldots, R_N\} \) of \( X \cup Y \) in \( M \) such that \( R_i \) is a regular neighborhood of \( P_i \cup N_i \) where \( \dim N_i \leq q \), \( 1 \leq i \leq N \). The proof is by induction on \( N \). If \( N=1 \) then there is nothing to prove. We will suppose it is true for \( N-1 \), and prove it for \( N \). Let \( T \) be a triangulation of \( M \) such that \( K, L, T_1, \ldots, T_N \) are subcomplexes of \( T \) which triangulate \( X, Y, P_1, \ldots, P_N \) respectively.
Without loss of generality, we may assume that \( L \cap K \) is a subcomplex of dimension at most \( N(q+1)-2 \). Let \( L' \) be the \((N-1)(q+1)-1\)-skeleton of \( L \) and let \( L'' \) be its dual skeleton. Note that \( \dim L'' \leq q \). Let \( Q = X \cap(N-1)(\cup_i P_i) \). By induction, since \( \dim L' < (N-1)(q+1) \), there is a polyhedral cover \( \{J_1', \ldots, J_{N-1}'\} \) of \( Q \cup \mid L' \mid \) in \( M \) such that \( J_i' \) is a regular neighborhood of \( P_i \cup N_i \), where \( \dim N_i \leq q \), \( 1 \leq i \leq N-1 \). Let \( J_i'' \) be a regular neighborhood of \( J_i' \), \( 1 \leq i \leq N-1 \), and let \( H \) be a subpolyhedron of \( T_N \cup L \) such that \( H \) collapses to \( T_N \cup L'' \) and \( X \cup Y \subset H \cup (\cup_i J_i') \). Let \( J_i'' \) be a regular neighborhood of \( H \). Hence \( \{J_1', \ldots, J_{N-1}'\} \) is our desired polyhedral cover. This completes the inductive step.

We now return to the proof of the Main Lemma. Let \( T \) be a triangulation of \( M \) such that there are subcomplexes \( T_1', \ldots, T_N' \) which triangulate \( P_1', \ldots, P_N \) respectively. Let \( T_N'' \) be the \((n-(N-1)(q+1))\)-skeleton of \( T_N \) and let \( T_N'' \) be its dual skeleton. Note that \( \dim T_N'' < (N-1)(q+1) \). By the first part of the proof, there is a polyhedral cover \( \{R_1'', \ldots, R_{N-1}''\} \) of \( (\cup_i P_i) \cup T_N'' \) in \( M \) such that \( R_i'' \) is a regular neighborhood of \( P_i \cup N_i \), where \( \dim N_i \leq q \), \( 1 \leq i \leq N-1 \). Let \( R_1'', \ldots, R_{N-1}'' \) be regular neighborhoods of \( R_1'', \ldots, R_{N-1}'' \) respectively and let \( R_N'' \) be a regular neighborhood of \( T_N'' \) such that \( \cup_i R_i'' = M \). Since \( M \) is \( p \)-connected, \( R_i'' \) can be deformed in \( M \) into \( P_i' \), \( 1 \leq i \leq N \). Furthermore, \( R_N'' \) is a regular neighborhood of \( T_N'' \) where \( \dim T_N'' < n-(N-1)(q+1) \). By repeating this process, and using Lemma 1.63 of [5] in order to preserve property b) in the process, we obtain our desired polyhedral cover.

**ZEEMAN'S ENGULFING THEOREM.** Let \( M \) be a closed \( p \)-connected PL \( n \)-manifold and let \( X \) be a compact subpolyhedron of dimension \( q \), \( q \leq n-3 \) and \( 2q \leq n+p-2 \). Then \( X \) is contractible in \( M \) if and only if there exists an \( n \)-ball \( B \) with \( X \subset B \subset M \).
THEOREM 1. (Singhof). Let M be a closed p-connected PL n-manifold. Either $N := \text{cat}(M) \geq 2$ and $n \geq 5$, or $N \geq 3$ and $n \geq 4$. If $N \geq \frac{(n+p+4)\sqrt{p+1}}{2}$, then there are $N$ $n$-balls which cover $M$. (see Theorem 6.1 of [6]).

PROOF. Let $\{P_1, \ldots, P_N\}$ be a polyhedral cover of $M$ such that each $P_i$ is contractible in $M$. By the Main Lemma, we may assume without loss of generality that $P_i$ is a regular neighborhood of $N_i$, where $\dim N_i \leq \max\{n-(N-1)(q+1), q\}$ with $q = \min\{p, n-3\}$, $1 \leq i \leq N$. Since $\dim N_i \leq n-3$ and $2 \dim N_i \leq n+p-2$, by the Zeeman Engulfing Theorem there are $n$-balls $B_1, \ldots, B_N$ such that $N_i \subset B_i$. Since $P_i$ collapses to $N_i$, $1 \leq i \leq N$, we may assume without loss of generality that $P_i \subset B_i$, $1 \leq i \leq N$. This concludes the proof of the theorem.

REMARK. Since the category of any homotopy sphere is two, Theorem 1 for $N=2$, $n \geq 5$, implies the Generalized Poincaré Theorem. In fact, it is possible to work in Theorem 1 with $\text{cat}_M(X)$ (see [1]) instead of $\text{cat}(M)$, thus obtaining a theorem which includes Zeeman's Engulfing Theorem for $N=1$, the Generalized Poincaré Theorem for $N=2$, and Singhof's Theorem for $N \geq 3$.

THEOREM 2. (Singhof [6]). Let $M$ be a closed PL n-manifold. If $\text{cat}(M) \geq \frac{(n+5)}{2}$, then $\text{cat}(M \times S^1) = \text{cat}(M) + 1$.

PROOF. It is a classical result (see [4] for a survey) that $\text{cat}(M) + 1 \geq \text{cat}(M \times S^1)$. Suppose $\text{cat}(M) = \text{cat}(M \times S^1) = N \geq \frac{(n+5)}{2}$. By Theorem 1, there exists a cover $\{B_1, \ldots, B_N\}$ of $M \times S^1$ where each $B_i$ is an $(n+1)$-ball. By means of a homeomorphism, we can assume $B_1$ is so small that $B_1 \cap (M \times \{a\}) = \emptyset$ for some $a \in S^1$. Then $\{B_2 \cap (M \times \{a\}), \ldots, B_N \cap (M \times \{a\})\}$ is a categorical covering of $M \times \{a\}$, which is impossible.

REMARK. Since the proof of the Main Lemma can be easily adapted for manifolds with boundary, we may use Stalling's Embedding Theorem (Theorem 1.2 of [3]) to obtain a sufficient condition for the equality
of \text{cat}(X)$ and the strong category of a polyhedron $X$. This condition, compared with the one given by Ganea in Theorem 3.1 of [2], is slightly stronger.

REFERENCES


Luis Montejano
Instituto de Matemáticas
Universidad Nacional A. de México
04510 México D. F.
México

(Received November 11, 1982)