Geometric variants of Hall’s Theorem through Sperner’s Lemma

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Abstract

Hall’s marriage theorem can be regarded as a condition for the existence of a rainbow set of distinct points. Motivated by this interpretation, we introduce the notion of geometric Hall-type problems. We prove that a linear Hall-type condition guarantees the existence of a rainbow set of pairwise disjoint unit balls in $\mathbb{R}^n$. We also prove that a quadratic Hall-type condition guarantees the existence of a rainbow set of points in general position on the plane. To prove these results, we present and extend a topological technique by Aharoni and Haxell that uses Sperner’s lemma in tight triangulations of the $k$-dimensional simplex.

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1 Geometric Hall-type Problems

Let $G$ be a finite bipartite graph with vertex classes $X$ and $Y$. Hall’s marriage theorem states that if for every $S \subseteq X$ the neighbourhood of $S$ has at least $|S|$ vertices, then there exists a matching of $G$ that saturates the vertices of $X$. We are interested in finding Hall-type theorems within a geometric context.

We can restate Hall’s theorem in terms of colors and points of $\mathbb{R}$ as follows. Let $k$ be a positive integer and $C_1, C_2, \ldots, C_k$ be some finite subsets of $\mathbb{R}$ (we think of $C_j$ as “the points of color $j$”). Suppose that for every $\beta \subseteq [k] := \{1, 2, \ldots, k\}$ we have at least $|\beta|$ points in $M_\beta := \bigcup_{r \in \beta} C_r$. Then, there exist distinct points $x_1, x_2, \ldots, x_k$ such that $x_j \in C_j$. In other words, we have found a rainbow set of points in general position in $M[k]$. This motivates the following problem in higher dimensions:

**Problem 1.1** Let $C_1, C_2, \ldots, C_k$ be some finite subsets of $\mathbb{R}^n$. Suppose that for every $\beta \subseteq [k]$ we have at least $|\beta|$ points in general position in $M_\beta$.

Does it follow that there exists a rainbow subset of $M[k]$ in general position?

If this is not the case, can we find a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ that does not depend on $k$, or the $C_j$, such that if there are at least $f(|\beta|)$ points in general position in $M_\beta$ for every $\beta \subseteq [k]$, then there exists a rainbow subset in general position?

In the cases for which such a function $f$ exists, we will call it a Hall function for the general position relation in $\mathbb{R}^n$. Clearly, if such an $f$ exists, then any higher valued function also satisfies the conditions.

**Problem 1.2** What can we say about the order of the smallest Hall function $f$?

We can further explore geometric Hall-type problems by considering other geometric relations, for example, pairwise disjointness. Let $\Omega_n$ be the family of unit balls in $\mathbb{R}^n$. We say that a function $f_{\Omega_n} : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a Hall function for the pairwise disjointness relation of the sets in the family $\Omega_n$ if the following happens for every positive integer $k$ and every family of sets of unit balls $\{C_j\}_{j \in [k]}$:

If for every $\beta \subseteq [k]$ we have at least $f_{\Omega_n}(|\beta|)$ pairwise disjoint balls in $M_\beta$, then there exists a rainbow subset $H$ of $M[k]$ such that the balls in $H$ are pairwise-disjoint.

**Problem 1.3** Does there exist a Hall function for the pairwise disjointness relation of the unit balls of $\mathbb{R}^n$? If so, which is the smallest such $f$?
Finally, we state a more general version of the previous problem. Let $\mathcal{F}$ be a family of convex subsets of $\mathbb{R}^n$. We similarly define a Hall function for the pairwise disjointness relation of the sets in the family $\mathcal{F}$.

**Problem 1.4** Does there exist a Hall function for the pairwise disjointness relation of the sets of a given family $\mathcal{F}$?

### 2 Hall’s Theorem and Sperner’s Lemma

There are many proofs of Hall’s theorem in the literature, for example in [2,3,4,5]. When we try to prove results in a geometric context, most of the ideas in these proofs present some kind of difficulty.

For this reason, we are particularly interested in the technique used in [1]. In that paper, Aharoni and Haxell prove a Hall’s theorem for hypergraphs using topological methods. More explicitly, they define the notion of an *economically hierarchic triangulation* of $\Delta_k$, the $k$–dimensional simplex. The intuition behind these kind of triangulations is that every point is connected with “few points” that have a support of lower dimension in the triangulation.

By using Hall’s condition, Aharoni and Haxell find a Sperner coloring of such a triangulation of $\Delta_k$. Finally, by using Sperner’s lemma, they find the desired system of disjoint representatives.

The key idea here is that we can prove Hall-type results by using topological methods. To illustrate how we can extend these techniques to our geometric context, we present the following result which solves Problem 1.3 for the case $n = 2$. We give a sketch of the proof.

**Theorem 2.1** The function $f_{\Omega_2} : \mathbb{Z}^+ \to \mathbb{Z}^+$ given by

$$f_{\Omega_2}(j) = 5j - 4$$

is a Hall function for the pairwise disjointness relation of the unit disks of $\mathbb{R}^2$.

**Proof.** Let $C_1, C_2, \ldots, C_k$ be some sets of unit disks. Suppose that for every $\beta \subseteq [k]$ we have at least $5|\beta| - 4$ pairwise disjoint disks in $M_{\beta} := \bigcup_{r \in \beta} C_r$. Let $\Delta_k$ be the $k$–dimensional simplex and $T$ an economically hierarchic triangulation (which exists by [1]).

We would like to label the vertices of $T$ with pairs $(D, r) \in M_{[k]} \times [k]$ so that

a) $D \in C_r$. 
b) The second coordinate of these labels form a Sperner coloring of $T$.

c) If $r_1$ and $r_2$ are different colors and $v_1$ and $v_2$ are adjacent vertices of $T$ with labels $(D, r_1)$ and $(E, r_2)$, then $D$ and $E$ are disjoint.

Let dim$(supp(v))$ be the dimension of the support of the vertex $v$ in $\Delta_k$. We enlist the key ideas to prove inductively on dim$(supp(v))$ that such a labeling exists:

- We arbitrarily label the vertices of $\Delta_k$ so that each one gets a different color.
- We suppose that we have labeled the vertices $v$ with dim$(supp(v)) = m < k$.
- Take a vertex $v$ with dim$(supp(v)) = m + 1$. Since $T$ is economically hierarchic, $v$ has at most $m$ previously labeled neighbours in $T$. On the other hand, dim$(supp(v)) = m + 1$ implies that we have $m + 1$ available colors to continue the Sperner’s labeling, and thus at least $f(m+1) = 5m+1$ disjoint disks that use those colors.
- Given a unit disk $D$, there are at most 5 pairwise disjoint disks that can intersect $D$. Therefore, each previously labeled neighbour of $v$ in $T$ discards at most 5 disks. Since this implies that at most $5m$ disks have been discarded, then we still have an available label for $v$.

After labeling the vertices of $T$ in such a way, we get a Sperner’s coloring of $T$. Then, by Sperner’s lemma we can find a $k$–simplex with vertices of different colors. By c) above, the corresponding disks are pairwise disjoint. By a), we have found a set of $k$ pairwise disjoint disks, one for each color.

This proof scheme works very neatly with geometric relations that only have to be verified for each pair of elements of the set. This is not always the case. For example, to check that a given set of points in the plane is in general position we have to verify that every three of them are not collinear. Then, the proof scheme has to be adapted to consider the 2–skeleton of $T$.

In the following section we present some of the work we have done in this direction.

### 3 A Couple of Geometric Hall-type Theorems

We start by extending our result of unit disks to higher dimensions. Let $\lambda(n)$ be the greatest number of pairwise disjoint unit balls in $\mathbb{R}^n$ that can intersect a given unit ball.

Theorem 2.1 can be generalized to any dimension as follows:
Theorem 3.1 The function $f_{\Omega_n} : \mathbb{Z}^+ \to \mathbb{Z}^+$ given by

$$f_{\Omega_n}(j) = \lambda(n)(j - 1) + 1$$

is a Hall function for the pairwise disjointness relation in $\Omega_n$.

Let $\kappa(n)$ be the greatest number of unit balls in $\mathbb{R}^n$ with pairwise disjoint interiors that can touch another given unit ball. It can be proved that $\lambda(n)$ is either $\kappa(n)$ or $\kappa(n) + 1$. In the literature, $\kappa(n)$ is known as the kissing number and has been studied extensively (for example, in [6,7,8])

Notice that for each function in Theorem 3.1, we have that $f_{\Omega_n}(j) = O(j)$. We now present a lower bound, which provides an answer for Problem 1.3.

Theorem 3.2 A constant function cannot be a Hall function for the pairwise disjointness relation in $\Omega_n$.

Let us now state some results related to Problems 1.1 and 1.2. We may assume $n \geq 2$. A simple example shows that $f(j) = j$ is not a Hall function for the general position relation in $\mathbb{R}^n$. Do there exist such Hall functions? The next result answers this question affirmatively.

Theorem 3.3 The function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ given by

$$f(j) = \max \left\{ jm \binom{j - 1}{m} \right\}_{m \in [n]} + 1$$

is a Hall function for the general position relation in $\mathbb{R}^n$.

Notice that for a given $n$, we have $f(j) = O(j^{n+1})$. We know that Imre Barany has also proven a $O(j^{n+1})$ bound\(^4\). Our proof uses the pidgeon hole principle based on a very rough estimate, so we are inclined to believe that this is not the best order possible. We conjecture that $f(j) = O(j^n)$.

We now present the main theorem of this paper, which provides a quadratic variant for the plane.

Theorem 3.4 Consider the function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ given by

$$f(j) = 2 \binom{j}{2} + 1.$$

Let $k$ be a positive integer and $C_1, C_2, \ldots C_k$ some sets of points in the plane. If for every $\beta \subseteq [k]$ we have at least $f(|\beta|)$ points in generic position in $M_\beta := \bigcup_{r \in \beta} C_r$, then $M_{[k]}$ has a rainbow set of points in general position.

\(^4\) Personal communication
The proof is based on a refinement of Aharoni and Haxell’s technique which is more sensitive to the 2−skeleton of the triangulation.

4 Current work and conclusions

The theorems we have presented are the result of an ongoing doctoral research. There is still a lot of work to do. We are currently working on a version of Theorem 3.4 for higher dimensions. We would also like to find other ways to improve the bound given by Theorem 3.3.

The ideas behind Theorem 3.1 extend to other relations that have a notion of kissing number. This suggests the study of a family of related theorems and this also provides a tool for tackling Problem 1.4.

Finally, it would be fantastic to replace the word “generic” by “general” in Theorem 3.4. This seems to be quite challenging with our current tools. We expect to get a better understanding of why certain triangulations of $\Delta_k$ help to get Hall-type results in order to use this tool more flexibly.

References


