PROBLEMS FOR THE 11th RESEARCH WORKSHOP ON
CONTINUUM THEORY AND HYPERSPACES

Problem by Alejandro Illanes and Norberto Ordoñez

**Problem 1.** Let $X$ be a hereditarily indecomposable continuum and $p \in X$. Does there exist a proper subcontinuum $X$ containing $p$ in its interior?

Problems by Verónica Martínez de la Vega and Edder Y. Valeriano

A continuum $X$ is selectable provided that there exists a mapping $s : C(X) \to X$ such that $s(A) \in A$ for each $A \in C(X)$, where $C(X)$ is the hyperspace of subcontinua of $X$.

**Problem 1.** Suppose that $X$ and $Y$ are selectable subcontinua of a continuum $Z$ such that $X \setminus Y$ is a one-point set. Is $X \cup Y$ selectable?

Problems by Eiichi Matsumahshi

**Arwise increasing maps**

A surjective continuous map $f : [0,1] \to X$ is called an *arcwise increasing map* if for any two closed subintervals $A$ and $B$ of $[0,1]$ such that $A \subseteq B$, $f(A) \subset f(B)$. If $X$ is a finite graph, then the notions of arcwise increasing map and Eulerian path coincide ([3]). The notion of arcwise increasing map was introduced in [5] as a generalization of Eulerian path for Peano continua.

I think it is natural to ask the following problem because as is well known, a characterization of the graphs which admit Eulerian paths is already obtained.

**Problem 1.** ([3, Problem 6.6]) Characterize the Peano continua that admit arcwise increasing maps.

Espinoza and I characterized the dendrites which admit arcwise increasing maps.

**Problem 2.** ([3, Problem 6.6]) Is there a characterization of local dendrites that admit arcwise increasing maps?

**Remark.** A map $g : X \to Y$ between compacta is called a *hereditarily irreducible map* (see [7, p. 204]) if for each $A, B \in C(X)$ with $A \subseteq B$, $f(A) \subset f(B)$. Note that every arcwise increasing map is a hereditarily irreducible map.

**Indecomposable continua**
In [2, Corollary 4] Bellamy proved that if \( X \) is a continuum, then there exists an indecomposable continuum \( Y \) such that \( \text{dim}X = \text{dim}Y \) and \( Y \) contains \( X \) as a retract. In [6], van Mill proved that for each homogeneous continuum \( X \), there exists a non-metrizable indecomposable homogeneous continuum \( Y \) such that \( \text{dim}X = \text{dim}Y \) and \( X \) is an open retract of \( Y \). Recently, in [4] Fukaishi and I proved that for each continuum \( X \) there exist an indecomposable continuum \( Y \) which contains \( X \) and an open retraction \( r : Y \to X \) such that each fiber of \( r \) is homeomorphic to the Cantor set. Furthermore, \( Y \) is homeomorphic to the closure of the countable union of topological copies of \( X \) in some continuum.

In particular, by the results above, we see that for each continuum \( X \) there exists an indecomposable continuum \( Y \) such that \( \text{dim}X = \text{dim}Y \).

**Problem 2.1.** For each continuum \( X \), is there an uncountable family \( \{Y_\lambda\}_{\lambda\in\Lambda} \) of indecomposable continua such that \( \text{dim}X = \text{dim}Y_\lambda \) and \( Y_\lambda \) contains \( X \) (as a retract) for each \( \lambda \in \Lambda \)?

**References**


**Problems by David Herrera and Fernando Macías**

Let \( \mathcal{C} \) be a class of continua, let \( X \in \mathcal{C} \) and \( \mathcal{H}(X) \in \{2^X, C_n(X), F_n(X), HS_n(X)\} \). The class \( \mathcal{C} \) is \( \mathcal{H} \)-closed provided that given continua \( X \) and \( Y \), with \( X \in \mathcal{C} \) the following implications holds: if \( Y \) is a continuum such that \( \mathcal{H}(X) \) is homeomorphic to \( \mathcal{H}(Y) \), then \( Y \in \mathcal{C} \) [4, p. 4].

(a) It is known that for each \( n \in \{2, 3\} \), the class of wired continua is \( F_n \)-closed [2, Theorem 3.10].

(b) The class of wired continua is \( HS_n \)-closed \( n \in \mathbb{N} \), [3, Theorem 3.2],

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(c) The class of almost meshed locally connected continua is $F_n$-closed for each $n \in \mathbb{N}$, [4, Theorem 3.1].

**Problem 1** [6, Question 15]. Have the compactifications of the ray $[0,1)$ unique hyperspace $C_n(X)$, for each $n > 1$?

**Problem 2** [6, Question 17]. Have the compactifications of the real line with connected remainder unique hyperspace $C_2(X)$?

**Problem 3** [6, Question 18]. Have the compactifications of the real line unique hyperspace $C_2(X)$?

**Problem 4.** [5, Question 3.7]. If $n \in \{1, 2\}$, is the class of wired locally connected continua $HS_n$-determined?

**Problem 5.** Given $n \in \mathbb{N}$, if $X$ does not have unique hyperspace $C_n(X)$, is it true that $X$ has unique hyperspace $HS_n(X)$?, does the other implication hold?

**Problem 6.** Does there exist a finite-dimensional continuum $X$ without unique hyperspace $F_n(X)$?

**Problem 7.** [1, Question 1.1]. Let $X$ be a dendrite, does $X$ have unique hyperspace $F_n(X)$?

**Problem 8.** Have the local dendrites unique hyperspace $F_n(X)$?

**Problem 9.** Have the locally connected finite-dimensional continua unique hyperspace $F_n(X)$?

**Problem 10.** Have the finite graphs (meshed, wired continua) unique hyperspace $F_n(X)/F_1(X)$?

**Problem 11.** Have the finite graphs (meshed, wired continua) unique hyperspace $F_n(X)/F_m(X)$, where $m < n$?

**Problem 12.** Have the wired continua unique hyperspace $F_2(X)$?

**Problem 13.** Have the wired continua unique hyperspace $F_3(X)$?

**References**


**Problems by Mauricio Esteban Chacón**

Given $M \subset [0,1]^2$, the generalized inverse limit of $M$ is defined as:

$$\lim_{\rightarrow} M = \{(x_1, x_2, \ldots) \in [0,1]^\infty : (x_{n+1}, x_n) \in M \text{ for each } n \in \mathbb{N}\}$$

This set is considered with the topology as subspace of the Hilbert cube $Q = [0,1]^\infty$.

The problems here are: find models for $\lim_{\rightarrow} M$ when $M$ is one of the following sets.

(a) $M_1 = ([0,1) \times [0,1]) \cup ([0,1] \times \{0\})$ (the boundary of the square),

(b) $M_2 = ([0,1] \times \{0\}) \cup ([0,1] \times [0,1])$ ($M_2$ has the shape of the letter "U").

Verónica and Mauricio have shown that the inverse limit of $M_1$ is not the Hilbert cube (it has many holes), but they were unable to find an exact model. On the other hand, they do believe that the inverse limit of $M_2$ is a Hilbert cube.

**Problema propuesto por Ivon Escobar**

Given a sequence $\{M_n\}_{n=1}^{\infty}$ of closed subsets of $[0,1]^2$, its generalized inverse limit is defined as:

$$\lim_{\rightarrow} M_n = \{(x_1, x_2, \ldots) \in [0,1]^\infty : \text{for each } n \in \mathbb{N}, (x_{n+1}, x_n) \in M_n\}.$$ 

Given $n \in \mathbb{N}$, define $\pi_{n,n+1} : \lim_{\rightarrow} M_n \rightarrow [0,1]^2$ as

$$\pi_{n,n+1}(x_1, x_2, \ldots) = (x_{n+1}, x_n).$$
Suppose that $X = \lim_{n=1} M_n$ is a continuum and $Y$ is a closed subset of $X$.

**Problem** (Problem 6.17 of the book by W. T. Ingram, *An Introduction to Inverse Limits with Set-valued Functions*). Find conditions on:

1. the closed sets $M_n$,
2. the continuum $X$,
3. the closed set $Y$.

In order that the equality $Y = \lim_{n=1} K_n$ holds, where $K_n = \pi_{n,n+1}(Y)$ for each $n \in \mathbb{N}$.

**Problemas by Florencio Corona, Rusell Aarón Quiñones, Javier Sánchez and Hugo Villanueva**

Let $X$ be a continuum and $C(X)$ the hyperspace of subcontinua of $X$. Define $M(X) \subset C(X)$ as the subspace of meager subcontinua (those with empty interior).

Define

$$FC(X) = \{A \in C(X) - \{X\} : Fr_X(A) \text{ is connected}\}.$$ 

This hyperspace is called the **hyperspace of the proper subcontinua with connected boundary**.

1. **Basic observations.**

   Note that $A \in M(X)$ if and only if $Fr(A) = A$. Then $M(X) \subset FC(X)$. It is easy to find examples for which $FC(X)$ is not connected, it is known that $M(X)$ always is connected. We can consider the component $FC(X)_M$ of $FC(X)$ containing $M(X)$.

   The operator taking boundary defined as

   $$Fr : FC(X)_M \rightarrow C(X),$$

   that sends $A$ in $Fr(A)$,

   the image of this map is contained in the see $M(X)$ and, in fact, its restriction to $M(X)$ is the identity.

   So the map $Fr$ is an extension of the identity map from $M(X)$ to $M(X)$.

   Since $Fr$ extends a continuous function, we can ask:

   What conditions must $X$ have to assure that $Fr$ is continuous?

   We have the following necessary condition.

   **Proposition.** If $Fr$ is continuous, then $M(X)$ is closed in $FC(X)$. 


Proof. Let \( \{A_n\}_{n=1}^{\infty} \) be a sequence in \( M(X) \), converging to an element \( A \in FC(X) \). Then \( A_n = Fr(A_n) \) converges to both sets \( A \) and \( Fr(A) \). Thus, \( A = Fr(A) \). This implies that \( A \in M(X) \). ■

2. Examples

(a) The proposed problem is only interesting for decomposable continua since:
   If \( S \) is indecomposable, then \( M(X) = C(X) - \{X\} \), and hence \( FC(X)_M = M(X) \), so \( Fr \) is the identity (so we do not have an extension).
   (b) An example of a decomposable continuum for which \( FC(X) = M(X) \) (and then \( Fr \) is continuous) is the continuum constructed by joining two harmonic fans by their end points, one by one, in the most natural possible way.
   (c) The harmonic fan shows that the necessary condition given in the proposition is not a sufficient condition.
   (d) For \( X = [0, 1] \), \( M(X) \subsetneq FC(X)_M \), so \( Fr \) is a proper extension of the identity, in this case, such a extension is indeed continuous.
   (e) It seems to be that for the finite graphs, \( Fr \) is an extension (proper, if the continuum is not \( S^1 \)) of the identity, which is continuous.

Problems by María Elena Aguilera and Norberto Ordoñez

Given a continuum \( X \) and a point \( p \in X \), we define the meager composant of \( p \) in \( X \) as:

\[
M_p = \{x \in X : \text{there exists a subcontinuum with nonempty interior } Y \text{ of } X \text{ such that } p, x \in Y \}.
\]

The meager composants were introduced by D. P. Bellamy in [1]. Later, C. Mouron y N. Ordoñez made a detailed study of them in [2].

Clearly, the meager composants are connected sets.

Problem 1. [1, problem 24]. Does there exist a continuum \( X \) containing a dense and open meager composant?

Problem 2 [2, Conjecture 8.4]. Are the meager composants \( F_\sigma \)-sets?

Problem 3 [2, Problem 8.5]. For a continuum \( X \), define

\[
\mathcal{M}_X = \{M_p : p \in X\}.
\]

Is it true that the cardinality of \( \mathcal{M}_X \) is always one or infinity?

Comments to Problem 1
In [2, Theorem 8.7] it is shown that if there exists a continuum with an open and dense meager composant, then it is possible to construct a continuum $X$ for which there exist two distinct points $p, q \in X$ such that $M_p = \{p\}$ and $M_q = X - \{p\}$.

For this reason, Problem 2 can be formulated as follows.

**Problem.** Does there exist a continuum $X$ and distinct points $p, q \in X$ such that $M_p = \{p\}$ and $M_q = X - \{p\}$?

**Comments to Problem 2**

In [2] it is shown that if $X$ is a continuum belonging to one of the following classes:
(a) locally connected continua,
(b) hereditarily arcwise connected continua,
(c) irreducible type $\lambda$ continua, and
(d) irreducible hereditarily decomposable continua.

Then the meager composants are closed subsets of $X$.

Clearly, in indecomposable continua, the meager composants coincide with the composants. So, in the class of indecomposable continua, the meager composants are $F_\sigma$-sets.

**Comments to Problem 3**

We think that the cardinality of $\mathcal{M}_X$ is always one or infinity. If this is true, then we will have a positive answer to Problem 1.

**References**


**Problems by Alejandro Illanes**

For a continuum $X$, we denote by $2^X$ the hyperspace of nonempty closed subsets of $X$, with the Hausdorff metric. We also denote by $F(X)$ the hyperspace of finite subsets of $X$.

Given $A, B \in 2^X$, we say that $A$ *does not block* $B$ if $A \cap B = \emptyset$ and the union of all subcontinua of $X$ intersecting $A$ and contained in $X - B$ is dense in $X$. Given a subset $\mathcal{H}$ of $2^X$, we define:
\[ \mathcal{B}(\mathcal{H}) = \{ B \in 2^X : B \text{ blocks each element of } \mathcal{H} \}. \]

In [2] it was shown that if \( X \) is locally connected, then \( \mathcal{B}(\mathcal{F}(X)) = \mathcal{B}(2^X) \).

In [1], the authors studied conditions under which the this equality holds. The following questions appeared in [1].

**Problem 1.** Does the equality \( \mathcal{B}(\mathcal{F}(X)) = \mathcal{B}(2^X) \) hold for the pseudo-arc?

**Problem 2.** If \( X \) is a nonlocally connected dendroid, does the equality \( \mathcal{B}(\mathcal{F}(X)) = \mathcal{B}(2^X) \) fail?

**Bibliografía**


**Problem by Jorge M. Martínez**

Given a continuum \( X \), a continuous function \( f : X \rightarrow C(X) - (F_1(X) \cup \{X\}) \) is a *coselection for \( X \)* if for each \( x \in X \), \( x \in f(x) \).

Given a coselection \( f \), we define

\[ \delta(f) = \sup\{ \text{diameter}(f(x)) : x \in X \} \].

If for each \( \varepsilon > 0 \) there exists a coselection \( f \) for \( X \) such that \( \delta(f) < \varepsilon \), we say that \( X \) is coselectible.

In [2, p.266], Sam B. Nadler, Jr. asked:

**Question.** When a continuum \( X \) is coselectible?

During the preparation of my dissertation, when I was working with \( Z \)-sets and drinking a beer, I proved the following result [1, Example 4].

**Theorem.** If \( C(X) \) is contractible, then \( X \) is coselectible.

I also used the Exxample 4 of [1] to show that the converse in Theorem 3 does not hold.

**Example.** There exists a coselectible non-contractible continuum \( X \).

I propose to continue considering Nadler’s question and to give families of coselectible continua, or to try to characterize coselectible continua.
References
