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Perturbations of the Landau Hamiltonian: Asymptotics of Eigenvalue Clusters

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Abstract. We consider the asymptotic behavior of the spectrum of the Landau Hamiltonian plus a short-range continuous potential. The spectrum of the operator forms eigenvalue clusters. We obtain a Szegő limit theorem for the eigenvalues in the clusters as the cluster index and the field strength B tend to infinity with a fixed ratio $\mathcal{E}$. The answer involves the averages of the potential over circles of radius $\sqrt{\mathcal{E}/2}$ (classical orbits). After rescaling, this becomes a semiclassical problem where the role of Planck's constant is played by $2/B$. We also discuss a related inverse spectral result.

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1. Introduction

The Landau Hamiltonian, in the symmetric gauge, is the operator on $L^2(\mathbb{R}^2)$

$$\widetilde{\mathcal{H}}_0(B) = \frac{1}{2}\left(\frac{1}{i}\frac{\partial}{\partial x_1} + \frac{B}{2}\widehat{Q}_2\right)^2 + \frac{1}{2}\left(\frac{1}{i}\frac{\partial}{\partial x_2} - \frac{B}{2}\widehat{Q}_1\right)^2. \tag{1.1}$$

It is the quantum Hamiltonian of a particle on the plane subject to a constant magnetic field perpendicular to the plane and of intensity B. Here, $\widehat{Q}_j =$ multiplication by x_j and we are taking the Planck's parameter $\hbar = 1$ at this point. It is well known that the spectrum of the operator $\widetilde{\mathcal{H}}_0(B)$ is given by the set of Landau levels

$$\lambda_q(B) = \frac{B}{2}\,(2q+1)\,, \quad q = 0, 1, \ldots, \tag{1.2}$$

where each Landau level has infinite multiplicity.

In [17], A. Pushnitski, G. Raikov and C. Villegas-Blas obtained a limiting eigenvalue distribution theorem for perturbations of the Landau Hamiltonian $\widetilde{\mathcal{H}}_0(B)$ by a potential $V : \mathbb{R}^2 \to \mathbb{R}$. More precisely, they studied perturbations of $\widetilde{\mathcal{H}}_0$ of the form

$$\widetilde{\mathcal{H}}(B) = \widetilde{\mathcal{H}}_0(B) + V, \tag{1.3}$$

where $V \in C(\mathbb{R}^2)$ and V is short-range, that is, it satisfies

$$\exists\, C > 0, \sigma > 1 \quad \text{such that} \quad \forall x \in \mathbb{R}^2 \quad |V(x)| \leq C\langle x\rangle^{-\sigma}, \tag{1.4}$$

where $\langle x\rangle = \sqrt{1+|x|^2}$. The authors show that, outside of a finite interval, the spectrum of the operator $\widetilde{\mathcal{H}}(B)$ consists of clusters of eigenvalues around the Landau levels. More precisely, the eigenvalues of $\widetilde{\mathcal{H}}(B)$ can be written in the form

$$\lambda_q(B) + \tau_{q,j}(B), \qquad q, j \in \mathbb{N}, \tag{1.5}$$

where, as it turns out, $|\tau_{q,j}(B)| = O(q^{-1/2})$ (see Proposition 1.1 in [17]). In the limit as $q \to \infty$ with B fixed, the scaled eigenvalue shifts $\tau_{q,j}$ distribute according to a measure $d\mu$ which we now describe. Consider the function $\breve{V} : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, where $\mathbb{T}$ is the unit circle, given by

$$\begin{aligned}&\breve{V}(\omega, b) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(b\omega + t\omega^{\perp})\,\mathrm{d}t, \quad \omega = (\omega_1, \omega_2) \in \mathbb{T},\\ &\omega^{\perp} = (-\omega_2, \omega_1), \quad b \in \mathbb{R},\end{aligned} \tag{1.6}$$

($\mathbb{T} \times \mathbb{R}$ parametrizes the manifold of straight lines on $\mathbb{R}^2$, and $\breve{V}(\omega, b)$ is the integral of V along the corresponding straight line.) Their main result is:

Theorem 1.1 [Pushnitski, Raikov, Villegas-Blas]. *Let $d\mu$ be the push-forward measure*

$$d\mu = \breve{V}_* \left(\frac{1}{2\pi} dm \right),$$

where dm is the Lebesgue measure on $\mathbb{T} \times \mathbb{R}$. Then, for $B > 0$, $\rho \in C_0^\infty(\mathbb{R}\backslash\{0\})$ and V as above, one has

$$\lim_{q \to \infty} \lambda_q(B)^{-1/2} \sum_j \rho \left(B^{-1} \sqrt{\lambda_q(B)}\, \tau_{q,j}(B) \right) = \int_{\mathbb{R}} \rho(\lambda) \mathrm{d}\mu(\lambda). \tag{1.7}$$

In this paper, we establish a different limiting eigenvalue distribution theorem for the same class of perturbations of the Landau Hamiltonian, taking a limit as *both* q and B tend to infinity along certain values. More precisely, we will fix the ratio

$$\mathcal{E} := \frac{4q+2}{B}, \tag{1.8}$$

which we will refer to as the "classical energy," for reasons that will be clarified below, and will compute the asymptotics of $\sum_j \rho(\tau_{q,j}(B))$ as $q, B \to \infty$ with $\mathcal{E}$ fixed, for suitable test functions ρ.

To state our main theorem, consider the classical Hamiltonian $H_0 : T^*\mathbb{R}^2 \to \mathbb{R}$ of a charged particle moving on the plane $\{(x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R})\}$ under the influence of the constant magnetic field $(0, 0, 2)$ corresponding to the quantum Hamiltonian $\mathcal{H}_0(\hbar)$:

$$H_0(\mathbf{x}, \mathbf{p}) = \frac{1}{2}(p_1 + x_2)^2 + \frac{1}{2}(p_2 - x_1)^2, \quad \mathbf{x} = (x_1, x_2), \quad \mathbf{p} = (p_1, p_2). \tag{1.9}$$

It can be shown that, for a fixed value $\mathcal{E}$ of the energy H_0, the classical orbits of H_0 in configuration space are circles with radius $\sqrt{\frac{\mathcal{E}}{2}}$ and period π. Any given point in $\mathbb{R}^2$ can be the center of one of those circles. More explicitly, if we denote by t the time evolution parameter, we have

$$\begin{aligned} x_1(t) &= \frac{P_2}{\sqrt{2}} + \sqrt{\frac{\mathcal{E}}{2}} \sin(2(t+\phi)) \\ x_2(t) &= \frac{X_2}{\sqrt{2}} + \sqrt{\frac{\mathcal{E}}{2}} \cos(2(t+\phi)) \end{aligned} \tag{1.10}$$

where $P_2 := (x_1 + p_2)/\sqrt{2}$ and $X_2 := (x_2 - p_1)/\sqrt{2}$ are integrals of motion whose particular values are determined by the initial conditions $\mathbf{x}(0)$, $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(0)$, and the equations $(p_1(0), p_2(0)) = \left(\frac{\mathrm{d}x_1}{\mathrm{d}t}(0) - x_2(0), \frac{\mathrm{d}x_2}{\mathrm{d}t}(0) + x_1(0)\right)$. The angle ϕ is a solution of the equation $\exp(2\imath\phi) = \frac{1}{\sqrt{2\mathcal{E}}}\,(\,p_1(0) + x_2(0) - \imath(p_2(0) - x_1(0))\,)$. We denote by $\widetilde{V}(X_2, P_2; \mathcal{E})$ the average of V along the circle with center $(\frac{P_2}{\sqrt{2}}, \frac{X_2}{\sqrt{2}})$ and radius $\sqrt{\frac{\mathcal{E}}{2}}$, that is

$$\widetilde{V}(X_2, P_2; \mathcal{E}) = \frac{1}{\pi} \int_0^\pi V \left(\frac{P_2}{\sqrt{2}} + \sqrt{\frac{\mathcal{E}}{2}} \sin(2t)\, , \, \frac{X_2}{\sqrt{2}} + \sqrt{\frac{\mathcal{E}}{2}} \cos(2t) \right) \mathrm{d}t. \tag{1.11}$$

Our main result is then the following:

Theorem 1.2. *Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a continuous short-range potential, that is, satisfying (1.4). Consider a test function $\rho : \mathbb{R} \to \mathbb{R}$ of the form $\rho(t) = t^\beta g(t)$, where β is the smallest even integer greater than $1/(\sigma - 1)$, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function of compact support. Fix a positive number $\mathcal{E}$.*

Then, $\rho\left(\widetilde{V}(X_2, P_2; \mathcal{E})\right) \in L^1(\mathbb{R}^2)$ and

$$\lim_{q,B\to\infty\, \frac{4q+2}{B}=\mathcal{E}} B^{-1} \sum_j \rho(\tau_{q,j}(B)) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \rho\left(\widetilde{V}(X_2, P_2; \mathcal{E})\right) dX_2\, \mathrm{d}P_2. \tag{1.12}$$

In Theorem 1.1, the shifts $\tau_{q,j}$ need to be rescaled; not so in our regime. Nonetheless, Theorem 1.1 corresponds to the limit as $\mathcal{E} \to \infty$ and, interestingly, the right-hand side of (1.7) is the $\mathcal{E} \to \infty$ limit of normalized integrals of V along circles with energy $\mathcal{E}$, see Eq. (1.16) in [17].

As we now explain, the regime considered in the previous Theorem is really the semi-classical limit for a suitable $\hbar$-differential operator. To see this, we introduce the small parameter

$$\hbar = \frac{2}{B}, \tag{1.13}$$

and define the operator

$$\mathcal{H}(\hbar) := \hbar^2 \widetilde{\mathcal{H}}(B = 2/\hbar) = \mathcal{H}_0(\hbar) + \hbar^2 V, \tag{1.14}$$

where

$$\mathcal{H}_0(\hbar) := \hbar^2 \widetilde{\mathcal{H}}_0(B = 2/\hbar) = \frac{1}{2}\left(\widehat{P}_1 + \widehat{Q}_2\right)^2 + \frac{1}{2}\left(\widehat{P}_2 - \widehat{Q}_1\right)^2, \quad \widehat{P}_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j}. \tag{1.15}$$

$\mathcal{H}(\hbar)$ is a semi-classical differential operator with principal symbol H_0 and, up to an overall factor of $\hbar^2$, the large B asymptotics of the operator $\widetilde{\mathcal{H}}(B)$ is equivalent to the semi-classical asymptotics of the operator $\mathcal{H}(\hbar)$, where B and $\hbar$ are related as above. The eigenvalues of $\mathcal{H}(\hbar)$ are

$$\hbar(2q+1) + \hbar^2\, \tau_{q,j} \qquad q, j = 0, 1, \ldots. \tag{1.16}$$

We will focus on the study of the distribution of the eigenvalues inside clusters of $\mathcal{H}(\hbar = \frac{\mathcal{E}}{2n+1})$ around a fixed classical energy $\mathcal{E}$, in the semi-classical limit $\hbar \to 0$. More precisely, let us take $\mathcal{E}$ fixed and consider $\hbar$ taking discrete values along the sequence

$$\hbar = \frac{\mathcal{E}}{2n+1}, \qquad n = 0, 1, \ldots. \tag{1.17}$$

Then, $\mathcal{E}$ is an eigenvalue of each member of the family of operators $\mathcal{H}_0(\hbar = \frac{\mathcal{E}}{2n+1})$, $n = 0, 1, \ldots$, corresponding to the quantum number $q = n$ in (1.16), and we will study the distribution of eigenvalues that cluster around $\mathcal{E}$ when $n \to \infty$ (or, equivalently, $\hbar \to 0$). We will actually prove the following, which is equivalent to Theorem 1.2:

Theorem 1.3. *With V and ρ as in Theorem 1.2,*

$$\lim \hbar \,\mathrm{Tr}\left(\rho\left(\frac{\mathcal{H}(\hbar)-\mathcal{E}}{\hbar^2}\right)\right) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \rho\left(\widetilde{V}(X_2,P_2;\mathcal{E})\right) \mathrm{d}X_2\, \mathrm{d}P_2, \tag{1.18}$$

the limit as $\hbar \to 0$ along the values (1.17).

To check that (1.18) is equivalent to (1.12) note that, for each n, the eigenvalues of $\frac{\mathcal{H}(\hbar)-\mathcal{E}}{\hbar^2}$ are

$$\hbar^{-2}\left(\hbar(2q+1) + \hbar^2\,\tau_{q,j} - \hbar(2n+1)\right) = \tau_{q,j} + 2\hbar^{-1}(q-n), \quad q,j = 1,2,\ldots.$$

Therefore,

$$\begin{aligned}\mathrm{Tr}\left(\rho\left(\frac{\mathcal{H}(\hbar)-\mathcal{E}}{\hbar^2}\right)\right) &= \textstyle\sum_{q,j} \rho\left(\tau_{q,j} + 2\hbar^{-1}(q-n)\right) &(1.19)\\ &= \textstyle\sum_j \rho\left(\tau_{n,j}\right) \quad \text{if} \quad \hbar \ll 1, &(1.20)\end{aligned}$$

where the final equality holds because ρ has compact support.

We now place this result in the context of previous works. There are many results in the literature of the following type: small perturbations of quantum Hamiltonians with degenerate spectrum (and periodic classical flow) yield eigenvalue clusters, and, in appropriate asymptotic regimes, the distribution of eigenvalues in the clusters is described by a Szegő-type theorem involving the average of the symbol of the perturbation over the classical trajectories of the unperturbed problem. References include: asymptotics of eigenvalue clusters for the Laplacian plus a potential on spheres and other Zoll manifolds (see [20] for the seminal work on this type of theorems), bounded perturbations of the n-dimensional isotropic harmonic oscillator with $n \geq 2$, [9] and [15], and both bounded and unbounded perturbations of the quantum hydrogen atom Hamiltonian, [2,11,19]. Previous results specific to perturbations of the Landau Hamiltonian can be divided into two classes: (a) Those where B is held constant and $q \to \infty$, and (b) the opposite scenario where the quantum number q remains fixed and $B \to \infty$. In the notation of (1.12), these correspond to $\mathcal{E} \to \infty$ and $\mathcal{E} \to 0$, respectively (see Fig. 1).

Results in the regime $q \to \infty$ with B constant include:

- [17] (Theorem 1.1 above). As we noted above, $\breve{V}$ (appearing on the right-hand side of (1.7)) should be thought of the average of the potential V over circles of infinite radius, that is, straight lines.
- [16] in the case of long-range potentials that can be approximated in a neighborhood of infinity by homogeneous functions $V \in C^\infty(\mathbb{R}^2\backslash\{0\})$. The authors obtain the asymptotics of the left-hand side of (1.7) with a different rescaling. Interestingly, the limit involves the circular Radon transform of V on circles of radius one.

Results in the regime $B \to \infty$ with q constant include:

- [18], where it is shown that

$$\sum_j \rho(\tau_{q,j}) \sim \frac{B}{2\pi}\int_{\mathbb{R}^2} \rho(V(x))\,\mathrm{d}x, \tag{1.21}$$

and

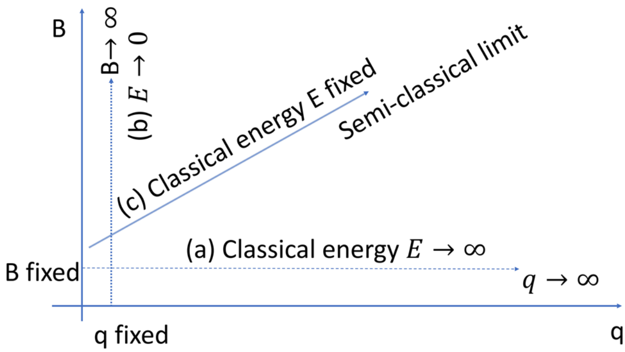

FIGURE 1. Diagram showing three different regimes. Case (a): limit as in [16,17], corresponding to $\mathcal{E} \to \infty$. Case (b): The limit as in [6,18] corresponds to $\mathcal{E} \to 0$. Case (c) is the regime considered in this work, where $\mathcal{E}$ is held constant and both q, B tend to infinity

- [6], where a complete asymptotic expansion of the left-hand side of (1.21) is obtained, provided V belongs to a certain class.

The regime studied in this work interpolates continuously between (1.7) ($\mathcal{E} = \infty$) and (1.21) ($\mathcal{E} = 0$), for short-range continuous potentials. Intuitively, Theorem 1.2 can be thought of as describing the q-th cluster of the spectrum of $\widetilde{\mathcal{H}}$ when the magnetic field is intense B and q is also large, with $\mathcal{E} = (4q+2)/B$.

We should also mention the treatises [12,13] that include many other results on Schrödinger operators with strong magnetic fields.

We now describe the organization of the paper. We begin in the next section by showing that one can replace the perturbation by an "averaged" version of it. For this, we re-examine estimates derived in section 4 of [17], in order to keep track of the dependence on B. Using this result, in Sect. 3 we reduce the problem to studying the spectrum of a one-dimensional semi-classical pseudo-differential operator. A complication is that the Weyl symbol of this operator is given by matrix elements of another operator which depends on parameters. This requires an analysis of the reduced operator which is the subject of Sect. 4. We complete the proof of Theorem 1.3 in section 5, and in Sect. 6 we obtain some inverse spectral results, assuming that we know the spectrum of $\widetilde{\mathcal{H}}(B)$ for all B. In the appendices, we review some technical results that are needed in the analysis of the reduced operator.

2. The Main Lemma

In this section, we will show that, to leading order, the moments of the spectral measures of the eigenvalue clusters can be computed by "averaging" the perturbation; see Lemma 2.2 below.

We follow closely the arguments in [17, Sect. 4]. For $q = 0, 1, \ldots$, let us denote by $\widetilde{P}_q(B)$ the orthogonal projector with range the eigenspace of the operator $\widetilde{\mathcal{H}}_0(B)$ with eigenvalue $\lambda_q(B) = \frac{B}{2}(2q+1)$. We begin by the following result which is actually Lemma 4.1 in [17], but with the dependence on the intensity B of the magnetic field made explicit:

Lemma 2.1. *Assume the potential V satisfies condition (1.4). Given $q = 0, 1, 2, \ldots$, consider the positively oriented circle Γ_q with center $\lambda_q(B)$ and radius $\frac{B}{2}$. Then, for all $z \in \Gamma_q$ and any integer $\ell > 1$, $\ell > 1/(\sigma - 1)$, we have*

$$\sup_{z\in\Gamma_q} \| \, |V|^{1/2}\widetilde{R}_0(z;B)|V|^{1/2} \, \|_\ell = B^{\frac{-\ell+1}{2\ell}} O(q^{\frac{-\ell+1}{2\ell}} \log q), \tag{2.1}$$

where $\widetilde{R}_0(z;B)$ denotes the resolvent operator $(\widetilde{\mathcal{H}}_0(B) - z)^{-1}$ and $\| \cdot \|_\ell$ denotes the norm in the Schatten ideal on $L^2(\mathbb{R}^2)$.

Proof. First, we write

$$\widetilde{R}_0(z;B) = \sum_{k=0}^{\infty} \frac{\widetilde{P}_k(B)}{\lambda_k(B) - z}. \tag{2.2}$$

From part (ii) of Theorem 1.6 in [17], we know that for $\ell > 1(\sigma - 1)$ and $B_0 > 0$ there exists $C = C(B_0, \ell)$ such that

$$\sup_{q\geq 0} \sup_{B\geq B_0} \lambda_q(B)^{(\ell-1)/(2\ell)} B^{-1} \|\widetilde{P}_q(B) V \widetilde{P}_q(B)\|_\ell \leq C \sup_{x\in\mathbb{R}^2} \left(1 + |x|^2\right)^{\sigma/2} |V(x)|. \tag{2.3}$$

Thus, for any integer $k \geq 0$, we have $\| \; |V|^{1/2}\widetilde{P}_k(B)|V|^{1/2} \; \|_\ell = CB\lambda_k(B)^{-(\ell-1)/(2\ell)}$. In the last equation and in the sequel, we denote different constants whose values are not relevant for our purposes by the same letter C. Defining $\nu \equiv (\ell - 1)/(2\ell)$, we obtain:

$$\begin{aligned}
\| \, |V|^{1/2}\widetilde{R}_0(z;B)|V|^{1/2} \, \|_\ell &\leq CB \sum_{k=0}^{\infty} \frac{\lambda_k(B)^{-\nu}}{|\lambda_k(B) - z|} \\
&\leq CB^{-\nu} \sum_{k=0}^{\infty} \frac{(k+1)^{-\nu}}{|\,(2k+1) - [(2q+1) + \exp(\imath\theta)]\,|} \\
&\leq CB^{-\nu} \left[\sum_{k=0}^{q-1} \frac{(k+1)^{-\nu}}{a - 2(k+1)} + (q+1)^{-\nu} + \sum_{k=q+1}^{\infty} \frac{(k+1)^{-\nu}}{2(k+1) - c} \right],
\end{aligned} \tag{2.4}$$

where $a = 2q+1$, $c = 2q+3$, $z = \frac{B}{2}(2q+1) + \frac{B}{2}\exp(\imath\theta)$, with $\theta \in [0, 2\pi]$. Let $f(x) = \frac{(x+1)^{-\nu}}{a-2(x+1)}$, $x \in [0, q-1]$. Note that $f(x)$ has a minimum at $x_0 = (\nu(a-2) - 2)/(2(\nu+1))$. Since $1/4 \leq \nu < 1/2$, then we have

$$\begin{aligned}
\sum_{k=0}^{q-1} \frac{(k+1)^{-\nu}}{a - 2(k+1)} &\leq f(0) + f(q-1) + \int_0^{q-1} f(x)\mathrm{d}x \\
&= O(q^{-1}) + O(q^{-\nu}) + \int_0^{q-1} f(x)\mathrm{d}x = O(q^{-\nu}) + \int_0^{q-1} f(x)\mathrm{d}x.
\end{aligned} \tag{2.5}$$

The integral in the last equation can be estimated as follows:

$$\begin{aligned}\int_0^{q-1} f(x)\mathrm{d}x &\le \frac{1}{a-2(x_0+1)}\int_0^{x_0}(x+1)^{-\nu}\mathrm{d}x+(x_0+1)^{-\nu}\int_{x_0}^{q-1}\frac{1}{a-2(x+1)}\mathrm{d}x\\ &= O(q^{-1})O(q^{-\nu+1})+O(q^{-\nu})O(\log(q))=O(q^{-\nu})O(\log(q)),\end{aligned} \tag{2.6}$$

where we have used that $x_0=O(q)$. Thus, the first sum in (2.4) is $O(q^{-\nu})O(\log(q))$.

Now, let $g(x)=\frac{(x+1)^{-\nu}}{2(x+1)-c}$, $x\in[q+1,\infty)$. Since g is a decreasing function,

$$\begin{aligned}\sum_{k=q+1}^{\infty}\frac{(k+1)^{-\nu}}{2(k+1)-c} &\le g(q+1)+\int_{q+1}^{\infty}g(x)\mathrm{d}x=O(q^{-\nu})+\int_{q+2}^{\infty}\frac{y^{-\nu}}{2y-c}\mathrm{d}y\\ &= O(q^{-\nu})+\frac{1}{2}\left(\frac{c}{2}\right)^{-\nu}\int_{\frac{2q+4}{2q+3}}^{\infty}\frac{w^{-\nu}}{w-1}\mathrm{d}w\\ &= O(q^{-\nu})+\frac{1}{2}\left(\frac{c}{2}\right)^{-\nu}\left[\int_{\frac{2q+4}{2q+3}}^{2}\frac{w^{-\nu}}{w-1}\mathrm{d}w+\int_{2}^{\infty}\frac{w^{-\nu}}{w-1}\mathrm{d}w\right]\\ &\le O(q^{-\nu})+\frac{1}{2}\left(\frac{c}{2}\right)^{-\nu}\left[\int_{\frac{2q+4}{2q+3}}^{2}\frac{1}{w-1}\mathrm{d}w+\tilde{C}]\right]\\ &= O(q^{-\nu})+\frac{1}{2}\left(\frac{c}{2}\right)^{-\nu}O(\log(q))=O(q^{-\nu}\log(q)),\end{aligned} \tag{2.7}$$

where $\tilde{C}$ is the constant $\int_2^{\infty}\frac{w^{-\nu}}{w-1}\mathrm{d}w$. This concludes the proof.

□

Now, we are ready to establish a crucial "averaging lemma," which will allow us to compute asymptotically the moments of the eigenvalue clusters of the operator $\mathcal{H}(\hbar)$.

For $n=0,1,\ldots$, denote by $P_n=P_n(\hbar)$ the orthogonal projector with range the eigenspace of the operator $\mathcal{H}_0(\hbar)$ with eigenvalue $\mathcal{E}=\hbar(2n+1)$. We have the following:

Lemma 2.2. *Fix $\mathcal{E}>0$, and let $Q_{\mathcal{E},\hbar}$ denote the projector of $\mathcal{H}(\hbar)$ associated with its cluster of eigenvalues in the interval $(\mathcal{E}-h\,,\,\mathcal{E}+\hbar)$. Then, for each $\ell>1/(\sigma-1)$ we have*

$$\operatorname{Tr}\left[(\mathcal{H}(\hbar)-\mathcal{E}I)^{\ell}\,Q_{\mathcal{E},\hbar}\right]=\hbar^{2\ell}\operatorname{Tr}\left[(\;P_n(\hbar)\;V\;P_n(\hbar)\;)^{\ell}\right]+o(\hbar^{2\ell-1}), \tag{2.8}$$

as $\hbar\to0$ and $n\to\infty$ in such a way that $\hbar(2n+1)=\mathcal{E}$.

Remark 2.3. As we will see below, (2.15), $\operatorname{Tr}\left[(\;P_n(\hbar)\;V\;P_n(\hbar)\;)^{\ell}\right]=O(\hbar^{-1})$, so the remainder term in (2.8) is indeed smaller than the first term.

Remark 2.4. It is not hard to check that, in the context of the previous lemma, for all sufficiently large n the left-hand side of (2.8) equals

$$\operatorname{Tr}\left[(\mathcal{H}(\hbar)-\mathcal{E}I)^{\ell}\;\;Q_{\mathcal{E},\hbar}\right]=\sum_j\tau_{n,j}^{\ell}. \tag{2.9}$$

Proof. The proof follows the corresponding proof of Lemma 1.5 in reference [17], but using the estimate provided by Lemma 2.1. Throughout the proof, we will assume the following identities:

$$B = \frac{2}{\hbar} \quad \text{and} \quad \hbar = \mathcal{E}/(2n+1). \tag{2.10}$$

Let us denote by $R(\eta;\hbar) = \left(\mathcal{H}(\hbar) - \eta I\right)^{-1}$ the resolvent operator associated with the operator $\mathcal{H}(\hbar)$ at the point $\eta \in \mathbb{C}$, whenever it is well defined.

If $\mathcal{C}_{\mathcal{E}}$ denotes the positively oriented circle with center $\mathcal{E}$ and radius $\hbar$, we can write:

$$\left(\mathcal{H}(\hbar) - \mathcal{E}I\right)^{\ell} Q_{\mathcal{E},\hbar} = \frac{-1}{2\pi\imath} \int_{\mathcal{C}_{\mathcal{E}}} \left(\eta - \mathcal{E}\right)^{\ell} R(\eta;\hbar)\, \mathrm{d}\eta. \tag{2.11}$$

Keeping in mind (2.10), notice that

$$R(\eta;\hbar) = \hbar^{-2} \widetilde{R}\left(z;B\right), \quad \widetilde{R}(z;B) := (\widetilde{\mathcal{H}}(B) - zI)^{-1} \tag{2.12}$$

provided

$$z = \lambda_n\left(B\right) + \frac{B}{2}\exp(\imath\theta).$$

Therefore, Eq. (2.11) can be written as

$$\left(\mathcal{H}(\hbar) - \mathcal{E}I\right)^{\ell} Q_{\mathcal{E},\hbar} = \frac{-\hbar^{2\ell}}{2\pi\imath} \int_{\Gamma_n} \left(z - \lambda_n\left(B\right)\right)^{\ell} \widetilde{R}\left(z,B\right)\, \mathrm{d}z, \tag{2.13}$$

where Γ_n denotes the positively oriented circle with center $\lambda_n\left(B\right)$ and radius $B/2$.

Since $\widetilde{R}\left(z,B\right) = \widetilde{R}_0\left(z,B\right)\left[I + V\widetilde{R}_0\left(z,B\right)\right]^{-1}$ and $\|V\widetilde{R}_0\left(z,B\right)\| \leq 2 \times \|V\|/B < 1$ (taking n sufficiently large), then we have the following series expansion convergent in the operator norm:

$$\begin{aligned} \left(\mathcal{H}(\hbar) - \mathcal{E}I\right)^{\ell} Q_{\mathcal{E},\hbar} &= -\hbar^{2\ell} \sum_{j=0}^{\infty} \frac{1}{2\pi\imath} \int_{\Gamma_n} \left(z - \lambda_n\left(B\right)\right)^{\ell} \widetilde{R}_0\left(z,B\right) \\ &\quad \times \left(V\widetilde{R}_0\left(z,B\right)\right)^{j}\, \mathrm{d}z. \end{aligned} \tag{2.14}$$

For $j < \ell$, the integrand is analytic which implies that the series in Eq. (2.14) actually goes from $j = \ell$ to infinity. Using Eq. (2.2), we can see that the $j = \ell$ term is equal to $\left(\widetilde{P}_n(B)V\widetilde{P}_k(B)\right)^{\ell} = \left(\, P_n(\hbar)\ V\ P_n(\hbar)\,\right)^{\ell}$ where we are using that $\widetilde{P}_n(B) = P_n(\hbar)$ are actually the same operator, always assuming (2.10). From Eq. (2.3), we have that $\|P_n(\hbar)\ V\ P_n(\hbar)\|_{\ell} = O(\hbar^{-1/\ell})$ which in turn implies by using the Hölder inequality with $\frac{1}{\ell} + \frac{1}{\ell} + \ldots + \frac{1}{\ell} = 1$ (ℓ terms)

$$\left\| \left(\, P_n(\hbar)\ V\ P_n(\hbar)\,\right)^{\ell} \right\|_1 = O(\hbar^{-1}). \tag{2.15}$$

The series $\sum_{j=\ell+1}^{\infty} \frac{1}{2\pi\imath} \int_{\Gamma_n} \left(z - \lambda_n\left(B\right)\right)^{\ell} \widetilde{R}_0\left(z,B\right) \left(V\widetilde{R}_0\left(z,B\right)\right)^{j}\, \mathrm{d}z$ has been studied in section 4.3 of reference [17] where, in particular, it is shown that such a series is convergent in the trace norm. Thus, $\left(\mathcal{H}(\hbar)-\mathcal{E}I\right)^{\ell} Q_{\mathcal{E},\hbar}$ is a trace class operator and the following expansion holds:

$$\begin{aligned}\operatorname{Tr}\left[\left(\mathcal{H}(\hbar)-\mathcal{E}I\right)^{\ell} \mathbb{1}_{(\mathcal{E}-\hbar\,,\,\mathcal{E}+\hbar)}(\mathcal{H}(\hbar))\right] &= \hbar^{2\ell} \operatorname{Tr}\left[\left(\, P_n(\hbar)\; V\; P_n(\hbar)\,\right)^{\ell}\right]\\ &+\frac{\ell\hbar^{2\ell}}{2\pi\imath}\sum_{j=\ell+1}^{\infty}\frac{(-1)^j}{j}\int_{\Gamma}\left(z-\lambda_n\left(B\right)\right)^{\ell-1}\operatorname{Tr}\left[\left(V\widetilde{R}_0\left(z,B\right)\right)^j\right]\,\mathrm{d}z,\end{aligned} \tag{2.16}$$

where we have used integration by parts.

As in reference [17], let us write $V=|V|^{1/2}\operatorname{sign}(V)|V|^{1/2}$. Then, we have for $j\geq \ell+1$:

$$\begin{aligned}&\left|\operatorname{Tr}\left[\left(V\widetilde{R}_0(z,B)]\right)^j\right]\right| = \left|\operatorname{Tr}\left[\left(\operatorname{sign}(V)|V|^{1/2}\widetilde{R}_0(z,B)|V|^{1/2}\right)^j\right]\right|\\ &\leq \left\|\left(\operatorname{sign}(V)|V|^{1/2}\widetilde{R}_0(z,B)|V|^{1/2}\right)^j\right\|_1 \leq \left\|\operatorname{sign}(V)|V|^{1/2}\widetilde{R}_0(z,B)|V|^{1/2}\right\|_j^j\\ &\leq \left\||V|^{1/2}\widetilde{R}_0(z,B)|V|^{1/2}\right\|_j^j \leq \left\||V|^{1/2}\widetilde{R}_0(z,B)|V|^{1/2}\right\|_{\ell+1}^j,\end{aligned} \tag{2.17}$$

where we have used the Hölder inequality.

Using the last inequality and Lemma 2.1 with $B=\frac{2}{\hbar}$ and $q=n$, we can show that the series on the right-hand side of Eq. (2.16) can be estimated by

$$\begin{aligned}&C_1\hbar^{2\ell}\sum_{j=\ell+1}^{\infty}\hbar^{1-\ell}\left(C_2\hbar^{\frac{\ell}{\ell+1}}\log(\hbar^{-1})\right)^j\hbar^{-1}\\ &\quad\leq C_1\hbar^{\ell}\left(C_2\hbar^{\frac{\ell}{\ell+1}}\log(\hbar^{-1})\right)^{\ell+1}\sum_{j=0}^{\infty}\left(C_2\hbar^{\frac{\ell}{\ell+1}}\log(\hbar^{-1})\right)^j\\ &\quad\leq C_1 S\hbar^{2\ell-1}\left(C_2\hbar^{\frac{1}{(\ell+1)}}\log(\hbar^{-1})\right)^{\ell+1} = o(\hbar^{2\ell-1}),\ \text{as}\ \hbar\to 0,\end{aligned} \tag{2.18}$$

where S denotes the infinite sum $\sum_{j=0}^{\infty}\left(C_2\hbar^{\frac{\ell}{\ell+1}}\log(\hbar^{-1})\right)^j$ which is uniformly bounded taking $\hbar$ sufficiently small and $\hbar^{\frac{1}{(\ell+1)}}\log(\hbar^{-1})=o(1)$ as $\hbar\to 0$. Equation (2.8) follows.

□

3. Reduction to a One-Dimensional Pseudo-Differential Operator

As we will see in this section, the analysis of the asymptotics of the eigenvalue clusters in the regime that we are interested in amounts to analyzing the spectrum of an $\hbar$-pseudo-differential operator on the real line.

3.1. A Preliminary Rotation

We begin by conjugating the unperturbed operator $\mathcal{H}_0$ by a suitable unitary operator that separates variables and converts $\mathcal{H}_0$ into a one-dimensional harmonic oscillator tensored with the identity operator on $L^2(\mathbb{R})$.

Proposition 3.1. *Let $\mathcal{U} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be a metaplectic operator quantizing the linear canonical transformation $\mathcal{T} : T^*\mathbb{R}^2 \to T^*\mathbb{R}^2$ such that, if $X_j = x_j \circ \mathcal{T}$, $P_j = p_j \circ \mathcal{T}$, $j = 1, 2$ then*

$$\begin{cases} P_1 = \frac{1}{\sqrt{2}}(p_1 + x_2),\ X_1 = \frac{1}{\sqrt{2}}(x_1 - p_2), \\ P_2 = \frac{1}{\sqrt{2}}(p_2 + x_1),\ X_2 = \frac{1}{\sqrt{2}}(x_2 - p_1). \end{cases} \tag{3.1}$$

Then

$$\mathcal{U}^{-1} \circ \mathcal{H}_0 \circ \mathcal{U} = -\hbar^2 \frac{\partial^2}{\partial x_1^2} + x_1^2 =: \mathcal{H}_1. \tag{3.2}$$

Proof. It is known that for metaplectic operators the Egorov theorem is *exact*: For any symbol $a : T^*\mathbb{R}^2 \to \mathbb{C}$, if $\mathrm{Op}^W(a)$ denotes Weyl quantization of a,

$$\mathcal{U}^{-1} \circ \mathrm{Op}^W(a) \circ \mathcal{U} = \mathrm{Op}^W(a \circ \mathcal{T}). \tag{3.3}$$

Therefore, the full symbol of $\mathcal{U}^{-1} \circ \mathcal{H}_0 \circ \mathcal{U}$ is just $P_1^2 + X_1^2$.

□

It is now clear that the spectrum of $\mathcal{H}_0$, which is to say, the spectrum of $\mathcal{H}_1$, consists of the eigenvalues $\hbar(2n+1)$, $n = 0, 1, \ldots$ with infinite multiplicity. Let us denote by

$$\{e_n(x_1), n = 0, \ldots\}$$

an orthonormal eigenbasis of the one-dimensional quantum harmonic oscillator

$$\mathcal{Z} := -\hbar^2 \frac{\partial^2}{\partial x_1^2} + x_1^2. \tag{3.4}$$

Then, the n-th eigenspace of $\mathcal{H}_1$ is the infinite-dimensional space

$$\mathcal{L}_n = \{e_n(x_1) f(x_2)\ ;\ f \in L^2(\mathbb{R})\}. \tag{3.5}$$

Let us now take $V : \mathbb{R}^2 \to \mathbb{R}$ to be Schwartz. We will denote by

$$K := \mathcal{U}^{-1} \circ \mathrm{Op}^W(V) \circ \mathcal{U} = \mathrm{Op}^W(V \circ \mathcal{T}) \tag{3.6}$$

the conjugate by $\mathcal{U}$ of the operator of multiplication by V. On the right-hand side, we are abusing the notation and denoting again by V the pull-back of V to $T^*\mathbb{R}^2$. Partially inverting (3.1), one has

$$x_1 = \frac{1}{\sqrt{2}}(X_1 + P_2), \quad x_2 = \frac{1}{\sqrt{2}}(X_2 + P_1), \tag{3.7}$$

and therefore, the function $W := V \circ \mathcal{T}$ is

$$W(X, P) := (V \circ \mathcal{T})(X, P) = V\left(\frac{1}{\sqrt{2}}(X_1 + P_2), \frac{1}{\sqrt{2}}(X_2 + P_1)\right). \tag{3.8}$$

The Schwartz kernel of the operator K is

$$\mathcal{K}(X,Y) = \frac{1}{(2\pi\hbar)^2}\iint e^{i\hbar^{-1}(X-Y)\cdot P}\, V\left(\frac{1}{2\sqrt{2}}(X_1+Y_1+2P_2, X_2+Y_2+2P_1)\right) dP_1\, dP_2, \tag{3.9}$$

and

$$\mathcal{U}^{-1}\circ\left(\mathcal{H}_0+\hbar^2 V\right)\circ\mathcal{U} = \mathcal{H}_1+\hbar^2 K. \tag{3.10}$$

This is the operator we will analyze.

3.2. Averaging

For ease of notation, we will re-name the (X,P) variables back to (x,p). Let us consider the unitary 2π-periodic one-parameter group of operators

$$\mathcal{V}(t) := e^{-it\mathcal{H}_1/\hbar}. \tag{3.11}$$

For each t, this is a metaplectic operator associated with the graph of the linear canonical transformation

$$\phi_t : T^*\mathbb{R}^2 \to T^*\mathbb{R}^2 \qquad \phi_t(x_1,p_1\ ;\ x_2,p_2) = (\mathfrak{h}_t(x_1,p_1)\ ;\ x_2,p_2), \tag{3.12}$$

where $\mathfrak{h}_t : T^*\mathbb{R}\to T^*\mathbb{R}$ is the one-dimensional harmonic oscillator of period π (the Hamilton flow of $x_1^2+p_1^2$).

Let us define

$$K^{\mathrm{ave}} := \frac{1}{\pi}\int_0^\pi \mathcal{V}(-t)K\,\mathcal{V}(t)\,\mathrm{d}t. \tag{3.13}$$

For each $n=1,2,\dots$ denote by $\mathcal{L}_n$ the eigenspace of $\mathcal{H}_1$ of eigenvalue $E_n = \hbar(2n+1)$, and let

$$\Pi_n : L^2(\mathbb{R}^2)\to\mathcal{L}_n$$

be the orthogonal projector. Then, it is not hard to verify that $[K^{\mathrm{ave}},\Pi_n]=0$ and that

$$\Pi_n\, K^{\mathrm{ave}}\,\Pi_n = \Pi_n\, K\,\Pi_n. \tag{3.14}$$

Therefore,

$$\forall\ell=1,2,\dots\quad \left(\Pi_n K\Pi_n\right)^\ell = \Pi_n\left[K^{\mathrm{ave}}\right]^\ell\Pi_n. \tag{3.15}$$

Lemma 3.2. *K^{ave} is a pseudo-differential operator of order zero. In fact*

$$K^{ave} = Op^W(W^{ave}) \tag{3.16}$$

where W^{ave} is the function

$$W^{ave}(x,p) = \frac{1}{\pi}\int_0^\pi W(\phi_t(x,p))\,\mathrm{d}t. \tag{3.17}$$

Proof. This is once again due to the fact that $\mathcal{V}(t)$ is a metaplectic operator for each t, and for such operators Egorov's theorem is exact. □

For future reference, we compute W^{ave} in terms of V when $x_1=0$. This determines W^{ave}, by ϕ_t invariance. A trajectory of the flow ϕ_t is

$$x_1(t) = \sin(2t)p_1(0),\quad p_1(t)=\cos(2t)p_1(0).$$

The energy of the trajectory is $E = p_1(0)^2$. Then,

$$W^{\mathrm{ave}}(x_1 = 0, x_2, p_1(0), p_2) = \frac{1}{\pi} \int_0^\pi V\left(u(t), v(t)\right) \mathrm{d}t,$$

where

$$u(t) = \frac{1}{\sqrt{2}} \left(\sin(2t) p_1(0) + p_2\right), \quad v(t) = \frac{1}{\sqrt{2}} \left(x_2 + \cos(2t) p_1(0)\right)$$

is a parametrization of the circle

$$\mathcal{C}_{x_2,p_2,E} := \left\{ (u,v)\ ;\ \left(u - \frac{p_2}{\sqrt{2}}\right)^2 + \left(v - \frac{x_2}{\sqrt{2}}\right)^2 = \frac{1}{2}E \right\}. \tag{3.18}$$

We see that it is then natural to regard W^{ave} as a circular Radon transform of V. More precisely, let us define

$$\forall \xi \in \mathbb{R}^2,\ E > 0 \qquad \widetilde{V}(\xi; E) := \frac{1}{2\pi} \int_{S^1} V\left(\check{\xi} + \sqrt{E/2}\,\omega\right) \mathrm{d}s(\omega), \tag{3.19}$$

where s is arc length and $\check{\xi} = \frac{1}{\sqrt{2}}(p_2, x_2)$ if $\xi = (x_2, p_2)$. Then,

$$W^{\mathrm{ave}}(x,p) = \widetilde{V}(x_2, p_2; x_1^2 + p_1^2). \tag{3.20}$$

We now fix $\mathcal{E} > 0$, and let $\hbar$ tend to zero along the sequence such that

$$E_n = \hbar\,(2n+1) = \mathcal{E}, \qquad n = 1, 2, \ldots. \tag{3.21}$$

By Lemma 2.2, the moments of the shifted eigenvalue clusters around $\mathcal{E}$ of $\mathcal{H}_1 + \hbar^2 K$ are, to leading order, the same as the moments of the eigenvalues of the operator

$$\Pi_n\, K^{\mathrm{ave}}|_{\mathcal{L}_n} : \mathcal{L}_n \to \mathcal{L}_n.$$

Lemma 3.3. *For each $n = 1, 2, \ldots$ there is an operator $T_n : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that*

$$\forall f \in L^2(\mathbb{R}) \qquad \Pi_n\, K^{\mathrm{ave}}(e_n \otimes f) = e_n \otimes T_n(f). \tag{3.22}$$

It is clear that

$$T_n(f)(x_2) = \int \overline{e_n}(x_1)\, K^{\mathrm{ave}}(e_n \otimes f)(x_1, x_2)\, \mathrm{d}x_1. \tag{3.23}$$

Note that we also have that $\Pi_n\, K(e_n \otimes f) = e_n \otimes T_n(f)$, by (3.14).

Definition 3.4. We call the sequence of operators (T_n) the *reduction* in K at level $\mathcal{E}$.

We emphasize that the interest of the operator T_n is that, by the previous considerations and by Lemma 2.2,

$$\mathrm{Tr}\left[\left(\mathcal{H}(\hbar) - \mathcal{E} I\right)^\ell Q_{\mathcal{E},\hbar} \right] = \hbar^{2\ell}\, \mathrm{Tr}(T_n^\ell) + o(\hbar^{2\ell - 1}), \quad \ell > 1/(\sigma - 1), \tag{3.24}$$

as $\hbar \to 0$ along the values (3.21), where we have used that

$$\mathrm{Tr}\left(P_n V P_n\right)^\ell = \mathrm{Tr}(T_n)^\ell \tag{3.25}$$

for $\ell \geq 1$.

4. Analysis of the Reduced Operator

Our goal in this section is to show that, for our purposes, T_n can be replaced by a semi-classical pseudo-differential operator whose symbol is $\widetilde{V}(x_2, p_2; \mathcal{E})$.

From now on, the parameters $\hbar$ and n are assumed to be related by the condition (3.21). Throughout this section, we will also assume that V is a Schwartz function.

4.1. The Weyl Symbol of T_n

Since K^{ave} is the Weyl quantization of the function (3.20), namely

$$W^{\mathrm{ave}}(x,p) = \widetilde{V}(x_2, p_2; x_1^2 + p_1^2),$$

one has

$$\begin{aligned} &K^{\mathrm{ave}}(e_n \otimes f)(x_1, x_2) \\ &\quad = \frac{1}{(2\pi\hbar)^2} \int e^{i\hbar^{-1}(x-y)p}\, \widetilde{V}\left(\frac{x_2+y_2}{2}, p_2; \left(\frac{x_1+y_1}{2}\right)^2 + p_1^2\right) \\ &\quad\quad \times e_n(y_1)\, f(y_2)\, \mathrm{d}y\, \mathrm{d}p. \end{aligned}$$

Therefore, after changing the order of integration, we can rewrite (3.23) as

$$T_n(f)(x_2) = \frac{1}{(2\pi\hbar)} \int e^{i\hbar^{-1}(x_2-y_2)p_2}\, \Phi\left(\frac{x_2+y_2}{2}, p_2, n\right) f(y_2)\, \mathrm{d}y_2\, \mathrm{d}p_2, \quad (4.1)$$

where

$$\begin{aligned} \Phi(x_2, p_2, n) &= \frac{1}{(2\pi\hbar)} \int e^{i\hbar^{-1}(x_1-y_1)p_1}\, \widetilde{V}\Bigg(x_2, p_2; \left(\frac{x_1+y_1}{2}\right)^2 \\ &\quad + p_1^2\Bigg)\, \overline{e_n}(x_1)\, e_n(y_1)\, \mathrm{d}y_1\, \mathrm{d}p_1\, \mathrm{d}x_1. \end{aligned} \quad (4.2)$$

From this, we immediately obtain:

Lemma 4.1. *Let, for each $\xi := (x_2, p_2)$, B_ξ be the operator which is the Weyl quantization of the ($\hbar$-independent) function*

$$b_\xi(x_1, p_1) := \widetilde{V}\left(\xi; x_1^2 + p_1^2\right). \quad (4.3)$$

Then, the Weyl symbol of T_n is

$$\Phi(\xi, n) = \langle B_\xi(e_n), e_n\rangle. \quad (4.4)$$

Remark 4.2. The function $b_\xi(x_1, p_1)$ is Schwartz as a function of the variables (x_1, p_1), with estimates uniform as ξ ranges on compact sets.

As a function of (x_1, p_1), the function $b_\xi(x_1, p_1)$ is radial, that is, it is a function of $x_1^2 + p_1^2$. We will make use of the following result on the Weyl quantization of a radial function on the plane. This result is in the literature, but for completeness we include a proof in Appendix A (see also Theorem 24.5 in [21]).

Proposition 4.3 ([3, Proposition 4.1]). *Let $a \in C^\infty(\mathbb{R}^2)$ be a (Schwartz) radial function, that is*

$$a(x,p) = \rho(r), \quad r = \sqrt{x^2+p^2},$$

and let $A := a^W(x,\hbar D)$ be its Weyl quantization. Then, $\forall n$ e_n is an eigenfunction of A with eigenvalue

$$\lambda_n = \frac{(-1)^n}{\hbar}\int_0^\infty \rho(\sqrt{u})e^{-u/\hbar}\, L_n(2u/\hbar)\,\mathrm{d}u, \tag{4.5}$$

where L_n is the normalized n-th Laguerre polynomial.

From this, we get the following explicit expression for the Weyl symbol of T_n:

$$\Phi(\xi,n) = \frac{(-1)^n}{\hbar}\int_0^\infty \widetilde{V}(\xi,u)e^{-u/\hbar}L_n(2u/\hbar)\mathrm{d}u. \tag{4.6}$$

Proposition 4.4. *For each $\hbar$ (and therefore n), the function Φ is Schwartz if V is.*

Proof. In view of (4.6), since n and $\hbar$ are fixed, it suffices to prove that the function

$$f(\xi) = \int_0^\infty \widetilde{V}(\xi,u)e^{-u/\hbar}u^m\mathrm{d}u$$

is Schwartz for any positive power m. Split the integral defining f in the form $f(\xi) = \int_0^{|\xi|/2}\widetilde{V}(\xi,u)e^{-u/\hbar}u^m du + \int_{|\xi|/2}^\infty \widetilde{V}(\xi,u)e^{-u/\hbar}u^m\mathrm{d}u.$

Since V is Schwartz, then $|V(y)| \lesssim \langle y\rangle^{-M}$ for any M. Therefore, by the definition of the Radon transform (3.19),

$$\left|\int_0^{|\xi|/2}\widetilde{V}(\xi,u)e^{-u/\hbar}u^m\mathrm{d}u\right| \lesssim \langle\xi\rangle^{-M}.$$

On the other hand

$$\left|\int_{|\xi|/2}^\infty \widetilde{V}(\xi,u)e^{-u/\hbar}u^m\mathrm{d}u\right|\mathrm{d}u \le \|V\|_\infty \int_{|\xi|/2}^\infty e^{-u/\hbar}u^m\mathrm{d}u = O(\langle\xi\rangle^{-\infty}).$$

Since $\partial_\xi\widetilde{V}(\xi,u) = \widetilde{\partial_\xi V}(\xi,u)$, we can repeat the argument on all derivatives of $\widetilde{V}$ and conclude that $\Phi(\cdot,n) \in \mathcal{S}$.

□

4.2. Localization

In this section, we cut Φ (and therefore T) into two pieces, and show that one can neglect one of the pieces. Let $M > \mathcal{E}$ and $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $[0,M]$ and $\chi(t) \equiv 0$ for $t > 2M$, and for each $\xi \in \mathbb{R}^2$ let

$$f_\xi(t) := \widetilde{V}(\xi;t)\,\chi(t). \tag{4.7}$$

Let us now define

$$\Phi_1(\xi,n) = \langle F_\xi(e_n), e_n\rangle, \tag{4.8}$$

where F_ξ is the Weyl quantization of the function

$$(x_1, p_1) \mapsto f_\xi(x_1^2 + p_1^2), \tag{4.9}$$

and let

$$\Phi_2(\xi, n) = \Phi(\xi, n) - \Phi_1(\xi, n). \tag{4.10}$$

We denote by $T_n^{(i)}$ the Weyl quantization of $\Phi_i(\cdot, n)$, $i = 1, 2$. These functions are Schwartz for each n (by the same proof that Φ is Schwartz), and $T_n = T_n^{(1)} + T_n^{(2)}$.

Next, we show that $T_n^{(2)}$ is negligible.

Theorem 4.5. *Let $V \in \mathcal{S}$. Then, there exists $M > \mathcal{E}$ such that if the support of the cut-off χ above satisfies supp$(\chi) \subset [0, 2M]$, then $\|T_n^{(2)}\|_{\mathcal{L}^1} = O(\hbar^\infty)$ provided $\hbar(2n+1) = \mathcal{E}$.*

Proof. Using (4.6),

$$\Phi_2(\xi, n) = \frac{(-1)^n}{\hbar} \int_0^\infty \widetilde{V}(\xi, u)(1 - \chi(u)) e^{-u/\hbar} L_n(2u/\hbar) \mathrm{d}u.$$

We want to apply the known trace-norm estimate

$$\|\mathrm{Op}^W(\Phi_2)\|_{\mathcal{L}^1} \leq \frac{C}{\hbar} \max_{|\beta| \leq 3} \int_{\mathbb{R}^2} |\partial_\xi^\beta \Phi_2(\xi)| \mathrm{d}\xi$$

(see [4] chapter 2, Theorem 5).

First notice that, from the definition of the Radon transform (3.19),

$$\partial_\xi^\beta \widetilde{V}(\xi, u) = 2^{-|\beta|/2} (\widetilde{\partial^{\beta^T} V})(\xi, u),$$

where $\beta^T = (\beta_2, \beta_1)$ if $\beta = (\beta_1, \beta_2)$. Therefore,

$$\int_{\mathbb{R}^2} \left| \partial_\xi^\beta \widetilde{V}(\xi, u) \right| \mathrm{d}\xi = 2^{-|\beta|/2} \, \|\partial^{\beta^T} V\|_{L^1}$$

for all $u > 0$. Since supp$(\chi) \subset [0, M]$, it follows that

$$\int_{\mathbb{R}^2} |\partial_\xi^\beta \Phi_2(\xi)| d\xi \leq \frac{1}{2^{|\beta|/2} \hbar} \|\partial^{\beta^T} V\|_{L^1} \int_M^\infty e^{-u/\hbar} |L_n(2u/\hbar)| \mathrm{d}u. \tag{4.11}$$

Next, we will use the representation of the Laguerre polynomials as a residue, namely

$$L_n(t) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{e^{-t \frac{z}{1-z}}}{(1-z) z^n} dz,$$

which holds for $0 < r < 1$. Since $\Re \frac{z}{1-z} = \frac{r\cos(\theta) - r^2}{|1-z|^2}$ where $z = re^{i\theta}$,

$$\begin{aligned} \int_M^\infty e^{-u/\hbar} |L_n(2u/\hbar)| du &\leq \frac{r}{2\pi(1-r) r^n} \int_M^\infty e^{-u/\hbar} \int_0^{2\pi} e^{-\frac{2u}{\hbar} \frac{r\cos(\theta) - r^2}{(1-r)^2}} \mathrm{d}\theta \, \mathrm{d}u \\ &\leq \frac{r}{(1-r) r^n} \int_M^\infty e^{-u/2\hbar} \mathrm{d}u, \end{aligned}$$

if r is small enough.

Now, since $\hbar(2n+1)=\mathcal{E}$

$$e^{-u/2\hbar}/r^n = r^{1/2}e^{-u/2\hbar+\log(1/r)\frac{\mathcal{E}}{2\hbar}} \lesssim e^{-u/4\hbar}$$

provided $u \geq M = -2\log(r)\mathcal{E} > \mathcal{E}$. Thus, for this choice of M,

$$\int_M^\infty e^{-u/\hbar}|L_n(2u/\hbar)|\mathrm{d}u = O(\hbar^\infty)$$

and the proof is complete.

□

4.3. Estimates on Φ_1

This section is devoted to the proof of the following

Theorem 4.6. *As $\hbar \to 0$ along the sequence (3.21),*

$$\Phi_1(\xi,n) = \widetilde{V}(\xi,\mathcal{E}) + \hbar^2\mathcal{R}(\xi,\hbar), \tag{4.12}$$

where $\mathcal{R}$ is a Schwartz function of ξ which is $O_{\mathcal{S}}(1)$, meaning that

$$\forall\alpha,\ \beta,\ \exists C,\ \hbar_0>0 \quad \textit{such that} \quad \forall\hbar\in(0,\hbar_0) \quad \sup_{\xi\in\mathbb{R}^2}\left|\xi^\alpha\partial_\xi^\beta\mathcal{R}(\xi,\hbar)\right| \leq C. \tag{4.13}$$

Proof. Recall that $\Phi_1(\xi,n)$ is defined by (4.8), where the operator F_ξ is the Weyl quantization of the radial function $f_\xi(x_1^2+p_1^2)$. Consider the first-order Taylor expansion of $f_\xi(t)$ at $t=\mathcal{E}$

$$f_\xi(t) = \widetilde{V}(\xi,\mathcal{E}) + (t-\mathcal{E})f'_\xi(\mathcal{E}) + (t-\mathcal{E})^2R_\xi(t)$$

where R_ξ is Schwartz, as follows from the explicit formula (see Appendix 6)

$$R_\xi(t) = \int_0^1\int_0^1 uf''_\xi(uv(t-\mathcal{E})+\mathcal{E})\,\mathrm{d}u\,\mathrm{d}v. \tag{4.14}$$

Denote the Weyl quantization of a function a as a^W, and let

$$\mathcal{I} = x_1^2+p_1^2.$$

Since $\langle(\mathcal{I}-\mathcal{E})^W(e_n),e_n\rangle = 0$, (4.12) holds with

$$\mathcal{R}(\xi,\hbar) = \frac{1}{\hbar^2}\langle\left[(\mathcal{I}-\mathcal{E})^2R_\xi(\mathcal{I})\right]^W(e_n),e_n\rangle. \tag{4.15}$$

Consider now the triple Moyal product with remainder

$$(\mathcal{I}-\mathcal{E})\#(\mathcal{I}-\mathcal{E})\#R_\xi(\mathcal{I}) = (\mathcal{I}-\mathcal{E})^2R_\xi(\mathcal{I}) + \hbar^2S_\xi(\mathcal{I},\hbar).$$

Since $\langle[(\mathcal{I}-\mathcal{E})\#(\mathcal{I}-\mathcal{E})\#R_\xi(\mathcal{I})]^W(e_n),e_n\rangle = 0$, we obtain that (4.15) equals

$$\mathcal{R}(\xi,\hbar) = -\langle S_\xi(\mathcal{I},\hbar)^W(e_n),e_n\rangle.$$

Claim: Every partial derivative $\partial^\alpha_{(x_1,p_1)}$ of $S_\xi(\mathcal{I},\hbar)$ is $O(\langle\xi\rangle^{-N})$ for any N, uniformly in ξ and $\hbar \leq \hbar_0$.

To see this, we use the following fact (see in [14] Theorem 2.7.4 and its proof): If $a \in S(m)$ and $b \in S(m')$, then the Moyal product $a\#b$ is in $S(m+m')$

and its asymptotic expansion is uniform in $S(m+m')$ (here $f \in S(m)$ if and only if $\|\langle\xi\rangle^{-m}\partial^\alpha(\xi)\| \le C_\alpha$ for every α). More precisely, if

$$a\#b \sim \sum_j \hbar^j c_j,$$

then for every j.

$$\langle x\rangle^{-(m+m')} |\partial^\alpha c_j(x)| \le C_{j,\alpha}.$$

$C_{j,\alpha}$ depends only on

$$\sup_{x\in\mathbb{R}^n, |\beta|\le M} \langle x\rangle^{-m} \left|\partial^\beta a(x)\right| \quad \text{and} \quad \sup_{x\in\mathbb{R}^n, |\beta|\le M} \langle x\rangle^{-m'} \left|\partial^\beta b(x)\right|,$$

where $M = M(\alpha, j)$. As a consequence of the stationary phase method, the same is true for each remainder of the asymptotic expansion of $a\#b$. The claim follows by applying this argument to combinations of $\mathcal{I} - \mathcal{E}$ and $R_\xi(\mathcal{I})$ and using that for every α

$$|\partial^\alpha_{(x_1,p_1)} R_\xi(\mathcal{I})| \le \frac{C_{\alpha,N}}{\langle\xi\rangle^N}.$$

Next, we use the estimate ([4, Ch. 2, Th. 4])

$$\|a^W\| \lesssim \sup_{|\alpha|\le 5} \|\partial^\alpha a\|_\infty \tag{4.16}$$

to conclude that

$$|\mathcal{R}(\xi,\hbar)| \le C_N \langle\xi\rangle^{-N}.$$

Finally, to estimate the derivatives $\partial^\alpha \mathcal{R}(\xi,\hbar)$ we simply notice that $\partial^\alpha\mathcal{R}(\xi,\hbar)$ replaces $\mathcal{R}(\xi,\hbar)$ when we study the Landau problem with the potential $\partial^\alpha V$. With the same calculations, we conclude that

$$|\partial^\alpha\mathcal{R}(\xi,\hbar)| \le C_{N,\alpha}\langle\xi\rangle^{-N},$$

that is, $\mathcal{R}(\cdot,\hbar) = O(1)$ in $\mathcal{S}(\mathbb{R}^2)$ for $\hbar \le \hbar_0$.

□

Remark 4.7. Using again that (see [4, Ch. 2, Th.5])

$$\|a^W\|_{\mathcal{L}_1} \lesssim \frac{1}{\hbar} \sum_{|\gamma|\le 5} \|\partial^\gamma a\|_{L^1(\mathbb{R}^2)},$$

we have by Theorem 4.6 and (4.16) that

$$\|\mathcal{R}(\cdot,\hbar)^W\|_{\mathcal{L}_1} \le \frac{C}{\hbar} \quad \text{and} \quad \|\mathcal{R}(\cdot,\hbar)^W\| \le C \tag{4.17}$$

for $\hbar \le \hbar_0$, and also

$$\|T_n^{(1)}\|_{\mathcal{L}_1} \le \frac{C}{\hbar} \quad \text{and} \quad \|T_n^{(1)}\| \le C. \tag{4.18}$$

It follows that

$$\|T_n\|_{\mathcal{L}_1} \le \frac{C}{\hbar}, \quad \|T_n\| \le C. \tag{4.19}$$

The previous Theorem and the symbol calculus imply:

Corollary 4.8. *For any $\ell = 1, 2, \ldots,$ as $n \to \infty$ and with $\hbar(2n+1) = \mathcal{E}$,*

$$\operatorname{Tr}\left(T_n^{(1)}\right)^\ell = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} \widetilde{V}(\xi, \mathcal{E})^\ell \, \mathrm{d}\xi + O(1). \tag{4.20}$$

Proof. $\left(T_n^{(1)}\right)^\ell = \left(\widetilde{V}(\cdot, \mathcal{E})^W + \hbar^2 \mathcal{R}(\cdot, \hbar)^W\right)^\ell$, hence

$$\operatorname{Tr}\left(T_n^{(1)}\right)^\ell = \operatorname{Tr}\left[\left(\widetilde{V}(\cdot, \mathcal{E})^W\right)^\ell\right] + \hbar^2 \operatorname{Tr} G_\hbar, \tag{4.21}$$

where $G_\hbar$ is a finite sum of terms each consisting of the product of a non-negative power of $\hbar$ and an operator of the form $S_1 S_2 \cdots S_m$, with $S_i \in \{\widetilde{V}(\ldots, \mathcal{E})^W, \mathcal{R}(\cdot, \hbar)^W\}$. Using that

$$\|AB\|_{\mathcal{L}_1} \leq \|A\| \|B\|_{\mathcal{L}_1}, \tag{4.22}$$

we conclude using (4.17) that

$$\|\hbar^2 \operatorname{Tr} G_\hbar\|_{\mathcal{L}_1} = O(\hbar). \tag{4.23}$$

Therefore, by the symbol calculus

$$\operatorname{Tr}\left(T_n^{(1)}\right)^\ell = \operatorname{Tr}\left[\left(\widetilde{V}(\cdot, \mathcal{E})^W\right)^\ell\right] + O(\hbar) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} \widetilde{V}(\xi, \mathcal{E})^\ell \, \mathrm{d}\xi + O(1).$$

□

5. Proof of Theorem 1.2

We first establish a Szegő-type theorem which is interesting on its own and where we consider the class of potentials V in the Banach space X_σ, $\sigma > 1$, defined by

$$X_\sigma = \left\{V : \mathbb{R}^2 \to \mathbb{R} \,|\, V \text{ is continuous and } \exists\, C > 0 \text{ s.t. } |V(x)| \langle x \rangle^\sigma \leq C, x \in \mathbb{R}^2\right\}.$$

Following [17], we endow X_σ with the norm $\|V\|_{X_\sigma} = \sup\left\{|V(x)| \langle x \rangle^\sigma,\ x \in \mathbb{R}^2\right\}$. Then, using such a Szegő-type theorem, we prove Theorem 1.2 using the Weierstrass approximation theorem.

Theorem 5.1. *Let $\sigma > 1$ and $V \in X_\sigma$. Then, for any integer $\ell > 1/(\sigma - 1)$, we have*

$$\lim_{n\to\infty} 2\pi\hbar \operatorname{Tr}(P_n V P_n)^\ell = \int_{\mathbb{R}^2} \left(\widetilde{V}(x, p; \mathcal{E})\right)^\ell \mathrm{d}x \, \mathrm{d}p, \tag{5.1}$$

where $\hbar = \mathcal{E}/(2n+1)$, $n = 0, 1, \ldots$.

Proof. We divide our proof into two parts.

Part A. We first prove the theorem for V a Schwartz function. In this case, σ can be taken as any number greater than one, so we establish Eq. (5.1) for any $\ell \geq 1$. Then, by (3.25) we reduce our analysis to the study of $\operatorname{Tr}(T_n)^\ell$.

When $\ell = 1$, the result follows from Theorem 4.5 and (4.20). If $\ell \geq 2$, we have that $(T_n)^\ell = (T_n^{(1)})^\ell + S$ where the operator S is a finite sum of operators of the form $S_1 S_2 \cdots S_\ell$, with $S_i \in \{T_n^{(1)}, T_n^{(2)}\}$ and where at least one factor S_{i_0} is equal to $T_n^{(2)}$. Hence, from Theorem 4.5,

$$\begin{aligned}|\operatorname{Tr}(S_1 S_2 \cdots S_\ell)| &= \left|\operatorname{Tr}\left(\prod_{i_0 \le i \le m} S_i \prod_{1 \le j < i_0} S_j\right)\right| \\ &\le C\|T_n^{(2)}\|/\hbar^m = O(\hbar^\infty)\end{aligned}$$

for some power $m > 0$, where we have used several times (4.22).

We conclude that

$$\hbar \operatorname{Tr}(T_n)^\ell = \hbar \operatorname{Tr}(T_n^{(1)})^\ell + O(\hbar^\infty), \tag{5.2}$$

and from (4.20),

$$2\pi\hbar \operatorname{Tr}(T_n)^\ell = \int_{\mathbb{R}^2} \left(\widetilde{V}(x,p;E)\right)^\ell \mathrm{d}x\,\mathrm{d}p + O(\hbar),$$

which implies Eq. (5.1) when V is a Schwartz function.

Part B. Our proof for the general case $V \in X_\sigma$, $\sigma > 1$, follows very closely the strategy indicated in the corresponding proof in [17]. Namely, for $\ell > 1/(\sigma - 1)$ fixed, we can always take $1 < \sigma' < \sigma$ such that $\ell > 1/(\sigma' - 1)$. Then, by using a continuity argument, we prove Eq. (5.1) for V actually in the closure $X_{\sigma'}^0$ of the subspace of Schwartz functions in $X_{\sigma'}$ (with respect to the norm $\|\cdot\|_{X_{\sigma'}}$). Finally, using that $X_\sigma \subset X_{\sigma'}^0$, we conclude our proof.

For $\ell > 1/(\sigma - 1)$, consider the functions γ_ℓ, Δ_ℓ, δ_ℓ: $X_{\sigma'} \to \mathbb{R}$ defined as follows:

$$\gamma_\ell(V) = \int_{\mathbb{R}^2} \left(\widetilde{V}(x,p;\mathcal{E})\right)^\ell \mathrm{d}x\,\mathrm{d}p, V \in X_{\sigma'}, \tag{5.3}$$

$$\Delta_\ell(V) = \limsup_{n\to\infty} 2\pi\hbar \operatorname{Tr}(P_n V P_n)^\ell, V \in X_{\sigma'}, \tag{5.4}$$

$$\delta_\ell(V) = \liminf_{n\to\infty} 2\pi\hbar \operatorname{Tr}(P_n V P_n)^\ell, V \in X_{\sigma'}. \tag{5.5}$$

The fact that the functions γ_ℓ, Δ_ℓ, δ_ℓ are well-defined is a consequence of the following two estimates:

(a) For $V \in X_{\sigma'}$, we have:

$$\left|\widetilde{V}(x,p;\mathcal{E})\right| \le C(\mathcal{E})\|V\|_{X_{\sigma'}} \frac{1}{\left\langle (p,x)/\sqrt{2}\right\rangle^{\sigma'}}, \tag{5.6}$$

where $C(\mathcal{E})$ is a constant independent of V. Estimate (5.6) can be shown by using Peetre's inequality: for all $\tilde{x}, \tilde{y} \in \mathbb{R}^n$, we have $\langle \tilde{x}\rangle / \langle \tilde{y}\rangle \le \sqrt{2}\,\langle \tilde{x} - \tilde{y}\rangle$. One can check that for any integer ℓ satisfying $\ell > 1/(\sigma' - 1)$ the function $1/\left\langle (p,x)/\sqrt{2}\right\rangle^{\ell\sigma'}$ is in $L^1(\mathbb{R}^2)$, and therefore, γ_ℓ is well defined.

(b) Using inequality (2.3) with $q = n$, $n = 1, 2, \ldots$, $\tilde{P}_n(B) = P_n$, $B = 2/\hbar$, we have for $V \in X_{\sigma'}$, $\ell > 1/(\sigma' - 1)$ that $P_n V P_n$ is in the Schatten class and

$$\hbar^{1/\ell}\|P_n V P_n\|_\ell \le C(\ell)\|V\|_{X_{\sigma'}}. \tag{5.7}$$

Thus, we have using Hölder's inequality that

$$\left|\hbar\,\mathrm{Tr}\left(P_nVP_n\right)^{\ell}\right| \le C(\ell)\|V\|_{X_{\sigma'}}^{\ell}, \tag{5.8}$$

which implies that both Δ_ℓ and δ_ℓ are well-defined on $X_{\sigma'}$.

Next, we want to study the continuity of γ_ℓ on $X_{\sigma'}$. We use the following identity, which is valid for both cases when $A_j \in \mathbb{R}$ and A_j is a bounded operator with $j = 1, 2$:

$$A_1^{\ell} - A_2^{\ell} = \sum_{j=0}^{\ell-1} A_1^{\ell-j-1}\left(A_1 - A_2\right)A_2^{j}. \tag{5.9}$$

Then, using (5.9) we have that for $V_1, V_2 \in X_{\sigma'}$ and $\ell > 1/(\sigma-1)$:

$$\begin{aligned} &|\gamma_\ell(V_2) - \gamma_\ell(V_1)| \\ &\le \sum_{j=0}^{\ell-1}\int_{(x,p)\in\mathbb{R}^2}\left|\widetilde{V_2}(x,p;\mathcal{E})\right|^{\ell-j-1}\left|\widetilde{V_2}(x,p;\mathcal{E}) - \widetilde{V_1}(x,p;\mathcal{E})\right|\left|\widetilde{V_1}(x,p;\mathcal{E})\right|^{j}\mathrm{d}x\mathrm{d}p \\ &\le C(\mathcal{E},\ell)\max\left\{\|V_2\|_{X_{\sigma'}}^{\ell-1}, \|V_1\|_{X_{\sigma'}}^{\ell-1}\right\}\|V_2 - V_1\|_{X_{\sigma'}} \ , \end{aligned} \tag{5.10}$$

where we have used (5.6) and the fact $1/\left\langle (p,x)/\sqrt{2}\right\rangle^{\ell\sigma'}$ is in $L^1(\mathbb{R}^2)$. From Eq. (5.10), we conclude the continuity of the function γ_ℓ on $X_{\sigma'}$.

Using again (5.9), we have for $V_1, V_2 \in X_{\sigma'}$ and $\ell > 1/(\sigma-1)$:

$$\begin{aligned} &\left|\hbar\left[\mathrm{Tr}\left(P_nV_2P_n\right)^{\ell} - \mathrm{Tr}\left(P_nV_1P_n\right)^{\ell}\right]\right| \\ &\le \sum_{j=0}^{\ell-1}\hbar\left|\mathrm{Tr}\left[\left(P_nV_2P_n\right)^{\ell-j-1}P_n(V_2-V_1)P_n\left(P_nV_1P_n\right)^{j}\right]\right| \\ &\le \sum_{j=0}^{\ell-1}\hbar\,\|P_nV_2P_n\ \|_{\ell}^{\ell-j-1}\,\|P_n(V_2-V_1)P_n\|_{\ell}\,\|P_nV_1P_n\ \|_{\ell}^{j} \\ &\le C(\ell)\max\left\{\|V_2\|_{X_{\sigma'}}^{\ell-1}, \|V_1\|_{X_{\sigma'}}^{\ell-1}\right\}\|V_2 - V_1\|_{X_{\sigma'}}, \end{aligned} \tag{5.11}$$

where we have used Hölder's inequality in the third row and Eq. (5.7) in the fourth one.

Now, take V in $X_{\sigma'}^0$. Thus, for $\epsilon > 0$ given, there exists V_ϵ a Schwartz function such that $\|V - V_\epsilon\|_{X_{\sigma'}} < \epsilon$. Then,

$$\begin{aligned} |\Delta_\ell(V) - \Delta_\ell(V_\epsilon)| &= \left|\limsup_{n\to\infty}\hbar\,\mathrm{Tr}\left(P_nVP_n\right)^{\ell} - \lim_{n\to\infty}\hbar\,\mathrm{Tr}\left(P_nV_\epsilon P_n\right)^{\ell}\right| \\ &= \left|\limsup_{n\to\infty}\hbar\left[\mathrm{Tr}\left(P_nVP_n\right)^{\ell} - \mathrm{Tr}\left(P_nV_\epsilon P_n\right)^{\ell}\ \right]\right| \\ &\le C(\ell)\max\left\{\|V\|_{X_{\sigma'}}^{\ell-1}, \|V_\epsilon\|_{X_{\sigma'}}^{\ell-1}\right\}\|V \\ &\quad - V_\epsilon\|_{X_{\sigma'}} \le C(V,\ell)\epsilon \ , \end{aligned} \tag{5.12}$$

where we have used $\limsup_{n\to\infty}\hbar\,\mathrm{Tr}\left(P_nV_\epsilon P_n\right)^{\ell} = \lim_{n\to\infty}\hbar\,\mathrm{Tr}\left(P_nV_\epsilon P_n\right)^{\ell}$, and Eq. (5.11). Equation (5.12) implies the continuity of Δ_ℓ on $X_{\sigma'}^0$.

Similarly, we can show the continuity of δ_ℓ on $X^0_{\sigma'}$ through the following inequality:

$$|\delta_\ell(V) - \delta_\ell(V_\epsilon)| \leq C(V,\ell)\epsilon. \tag{5.13}$$

Since $\Delta_\ell(V_\epsilon) = \delta_\ell(V_\epsilon)$ (a consequence of part (a) of this proof), then using Eqs. (5.12) and (5.13), we conclude that $\lim_{n\to\infty} \hbar \operatorname{Tr}(P_n V P_n)^\ell$ exists and is equal to $\Delta_\ell(V) = \delta_\ell(V)$.

Finally, using the continuity of Δ_ℓ on $X^0_{\sigma'}$ and that for all Schwartz functions V_ϵ the equality $\Delta_\ell(V_\epsilon) = \delta_\ell(V_\epsilon) = \gamma_\ell(V\epsilon)$ holds (use part (a) of our proof), we conclude Eq. (5.1) for $V \in X^0_{\sigma'}$. Using $X_\sigma \subset X^0_{\sigma'}$, we conclude the proof of Theorem 5.1 for $V \in X_\sigma$.

□

Combining this Theorem with (3.24), we can conclude that Theorem 1.2 is valid for polynomials:

Corollary 5.2. *For any polynomial $q:\mathbb{R}\to\mathbb{R}$, if $p(t) = t^\beta q(t)$, then*

$$\lim_{\substack{q,B\to\infty\\ \frac{4q+2}{B}=\mathcal{E}}} B^{-1}\sum_j p(\tau_{q,j}(B)) = \frac{1}{4\pi}\int_{\mathbb{R}^2} p\left(\widetilde{V}(x,p;\mathcal{E})\right) \mathrm{d}x\,\mathrm{d}p. \tag{5.14}$$

Proof of Theorem 1.2: For n a given positive integer, we are taking $\hbar = \mathcal{E}/(2n+1)$. The spectrum of the corresponding operator $\mathcal{H}(\hbar) = \mathcal{H}_0(\hbar) + \hbar^2 V$ is the set of eigenvalues $\lambda_{q,j} := \hbar(2q+1) + \hbar^2\,\tau_{q,j} \quad q,j = 0,1,\ldots$. Using basic perturbation theory, one can show that the spectral shifts $\tau_{q,j}$ are uniformly bounded: $|\tau_{q,j}| \leq \|V\|_\infty \;\; q,j = 0,1,\ldots$. Since the support of the test function ρ is bounded, we have that for all $\hbar$ sufficiently small (i.e. n large enough) the eigenvalues $(\lambda_{q,j}-\mathcal{E})/\hbar^2$ with $q \neq n$ lie outside of the support of ρ. Thus, for $\hbar$ sufficiently small $\rho\left(\frac{\mathcal{H}(\hbar)-\mathcal{E}}{\hbar^2}\right)$ is trace-class if and only if $\rho\left(\frac{\mathcal{H}(\hbar)-\mathcal{E}}{\hbar^2}Q_{\mathcal{E},\hbar}\right)$ is trace class, in which case $\operatorname{Tr}\left(\rho\left(\frac{\mathcal{H}(\hbar)-\mathcal{E}}{\hbar^2}\right)\right) = \operatorname{Tr}\left(\rho\left(\frac{\mathcal{H}(\hbar)-\mathcal{E}}{\hbar^2}Q_{\mathcal{E},\hbar}\right)\right)$.

Let K be the set $K := \operatorname{supp}(\rho) \cup [-\|V\|_\infty, \|V\|_\infty]$ where $\operatorname{supp}(\rho)$ denotes the support of the test function ρ.

Let $\epsilon > 0$. Recall that the test function ρ is of the form $\rho(t) = t^\beta g(t)$ with g continuous and β even. As a consequence of the Weierstrass approximation theorem, there exist polynomials $q_\pm$ such that

$$\forall t \in K \qquad q_-(t) \leq g(t) \leq q_+(t) \quad \text{and} \quad q_+(t) - q_-(t) \leq \epsilon. \tag{5.15}$$

Then, since β is even,

$$4\pi B^{-1}\sum_j \tau^\beta_{n,j}\, q_-(\tau_{n,j}) \leq 4\pi B^{-1}\sum_j \rho(\tau_{n,j}) \leq 4\pi B^{-1}\sum_j \tau^\beta_{n,j}\, q_+(\tau_{n,j}), \tag{5.16}$$

where we have omitted the B-dependence of the $\tau_{n,j}$ for convenience. In what follows, we let $B, q \to \infty$ in the desired manner. By Corollary 5.2,

$$\lim 4\pi B^{-1}\sum_j \tau^\beta_{n,j}\, q_\pm(\tau_{n,j}) = \int_{\mathbb{R}^2} \widetilde{V}^\beta q_\pm\left(\widetilde{V}\right) \mathrm{d}x\,\mathrm{d}p,$$

(omitting the variables in $\widetilde{V}$ for simplicity), and using (5.16) we obtain

$$\int_{\mathbb{R}^2} \widetilde{V}^\beta q_-\left(\widetilde{V}\right) \mathrm{d}x\,\mathrm{d}p \le \liminf 4\pi B^{-1} \sum_j \rho(\tau_{n,j}) \tag{5.17}$$

$$\le \limsup 4\pi B^{-1} \sum_j \rho(\tau_{n,j}) \le \int_{\mathbb{R}^2} \widetilde{V}^\beta q_+\left(\widetilde{V}\right) \mathrm{d}x\,\mathrm{d}p.$$

On the other hand,

$$\int_{\mathbb{R}^2} \widetilde{V}^\beta q_-\left(\widetilde{V}\right) \mathrm{d}x\,\mathrm{d}p \le \int_{\mathbb{R}^2} \rho\left(\widetilde{V}\right) \mathrm{d}x\,\mathrm{d}p \le \int_{\mathbb{R}^2} \widetilde{V}^\beta q_+\left(\widetilde{V}\right) \mathrm{d}x\,\mathrm{d}p$$

and

$$\int_{\mathbb{R}^2} \widetilde{V}^\beta q_+\left(\widetilde{V}\right) \mathrm{d}x\,\mathrm{d}p - \int_{\mathbb{R}^2} \widetilde{V}^\beta q_-\left(\widetilde{V}\right) \mathrm{d}x\,\mathrm{d}p \le \epsilon \|\widetilde{V}^\beta\|_{L^1(\mathbb{R}^2)}.$$

Since $\epsilon > 0$ was arbitrary, the theorem is proved. □

6. An Inverse Spectral Result

Let us assume, we know the spectrum of $\widetilde{\mathcal{H}}_0(B) + V$ with $V \in \mathcal{S}(\mathbb{R}^2)$, for all B in a neighborhood of infinity. What can we say about V? In this section, we prove:

Theorem 6.1. *If V and V' are two isospectral potentials (in the sense above) in the Schwartz class, then $\forall s \in \mathbb{R}$ their Sobolev s-norms are equal:*

$$\|V\|_s = \|V'\|_s.$$

We will proceed as in [10] and use that, by Theorem 1.2, the spectral data above determine the function

$$I(r) = (2\pi)^2 \int_{\mathbb{R}^2} \mathcal{R}_r(V)^2(y)\,\mathrm{d}y \tag{6.1}$$

for all $r > 0$, where $\mathcal{R}_r(V)(y)$ is the Radon transform of V, namely the integral transform that averages V over the circle of radius r and center y (hence, in the notation of Sect. 3, $\tilde{V}(y, \mathcal{E}) = \mathcal{R}_{\sqrt{\mathcal{E}/2}}(V)(\check{y})$, $\check{y} = \frac{1}{\sqrt{2}}(p, x)$ if $y = (x, p)$).

Lemma 6.2. *Let J_0 denote the zeroth Bessel function. Then,*

$$\mathcal{R}_r(V)(y) = \frac{1}{(2\pi)^2} \int e^{iy\cdot\xi}\, J_0(r|\xi|)\, \widehat{V}(\xi)\,\mathrm{d}\xi \tag{6.2}$$

where $\widehat{V}$ is the Fourier transform of V.

Proof. By the Fourier inversion formula, it suffices to compute

$$\mathcal{R}_r(e^{-ix\cdot\xi})(y) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(-i[\xi_1(y_1 + r\cos(\theta)) + \xi_2(y_2 + r\sin(\theta))]\right)\,\mathrm{d}\theta$$

$$= \frac{e^{-iy\cdot\xi}}{2\pi} \int_0^{2\pi} \exp\left(-ir[\xi_1\cos(\theta) + \xi_2\sin(\theta)]\right)\,\mathrm{d}\theta.$$

Let us now introduce polar coordinates for ξ,

$$\xi = |\xi|\, u, \quad u = \langle \sin(\phi)\,,\, \cos(\phi)\rangle.$$

Then,

$$\xi_1 \cos(\theta) + \xi_2 \sin(\theta) = |\xi|\, \sin(\theta + \phi),$$

and therefore,

$$\mathcal{R}_r(e^{-ix\cdot\xi})(y) = \frac{e^{-iy\cdot\xi}}{2\pi} \int_0^{2\pi} e^{-ir|\xi|\, \sin(\theta+\phi)}\, \mathrm{d}\theta = \frac{e^{-iy\cdot\xi}}{2\pi} \int_0^{2\pi} e^{-ir|\xi|\, \sin(\theta)}\, \mathrm{d}\theta. \tag{6.3}$$

However, it is known that

$$J_0(s) = \frac{1}{2\pi} \int_0^{2\pi} e^{-is\sin(\theta)}\, \mathrm{d}\theta,$$

so we obtain

$$\mathcal{R}_r(e^{-ix\cdot\xi})(y) = e^{-iy\cdot\xi}\, J_0(r|\xi|). \tag{6.4}$$

□

Using Parseval's theorem, we immediately obtain:

Corollary 6.3.

$$I(r) = \int_{\mathbb{R}^2} J_0(r|\xi|)^2 \left|\widehat{V}(\xi)\right|^2 \mathrm{d}\xi. \tag{6.5}$$

Let us now introduce polar coordinates (ρ, ϕ) on the ξ plane, and let us define

$$W(\rho) := \rho^2 \int_0^{2\pi} \left|\widehat{V}(\rho^{-1}\cos(\phi),\, \rho^{-1}\sin(\phi))\right|^2 \mathrm{d}\phi \tag{6.6}$$

and

$$K(s) = J_0(s)^2.$$

Then, (6.5) reads

$$I(r) = \int_0^{\infty} K(r\rho)\, W(\rho^{-1})\, \frac{\mathrm{d}\rho}{\rho}. \tag{6.7}$$

In other words, $I(r)$ is the convolution of K and W in the multiplicative group $(\mathbb{R}^+, \times)$.

Corollary 6.4. *For each $\rho > 0$, the integral*

$$\int_0^{2\pi} \left|\widehat{V}(\rho\cos(\phi),\, \rho\sin(\phi))\right|^2 \mathrm{d}\phi \tag{6.8}$$

of $|\widehat{V}|^2$ over the circle centered at the origin and of radius ρ is a spectral invariant of V.

Proof. By (6.7), the Mellin transform of I is the product of the Mellin transforms of K and W. Since K and its Mellin transform are analytic, and the Mellin transform of W is continuous, this determines the Mellin transform of W, and hence determines W.

□

Theorem 6.1 follows from this, as

$$\|V\|_s^2 = \int_{\mathbb{R}^2} (1+|\xi|^2)^{s/2} |\hat{V}(\xi)|^2 \mathrm{d}\xi.$$

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A. The Weyl Quantization of Radial Functions

For the benefit of the reader, we include here some results on the Weyl quantization of radial functions that shed light on the material in Sect. 4. The result in Eq. (A.13) has been originally shown in [3] [Proposition 4.1]; we include a derivation here for completeness. See also [21], §4.

If $a(x,p)$ is a symbol in $\mathbb{R}^{2n}$, its Weyl quantization is the operator $a^W(x, \hbar D)$ with kernel

$$\mathcal{K}_a(x,y) = \frac{1}{(2\pi\hbar)^n} \int e^{i\hbar^{-1}(x-y)\cdot p} a\left(\frac{x+y}{2}, p\right) \mathrm{d}p.$$

The corresponding bilinear form $Q_a(f,g) = \langle a^W(x,\hbar D)(f), \overline{g}\rangle$ is

$$\forall f,\, g \in \mathcal{S}(\mathbb{R}^n) \qquad Q_a(f,g) = \frac{1}{(2\pi\hbar)^n} \iiint e^{i\hbar^{-1}(x-y)\cdot p} a\left(\frac{x+y}{2}, p\right) \times f(y)\, g(x) dp\, dx\, dy. \tag{A.1}$$

It is not hard to see that

$$Q_a(f,g) = \iint a(u,p)\, \mathcal{G}(f,g)(u,p)\, du\, dp, \tag{A.2}$$

where

$$\mathcal{G}(f,g)(u,p) = \frac{1}{(\pi\hbar)^n} \int e^{2i\hbar^{-1} v\cdot p} f(u-v)\, g(u+v)\, \mathrm{d}v. \tag{A.3}$$

Let $a \in \mathcal{S}(\mathbb{R}^2)$ be a radial function, that is

$$a(x,p) = \rho(r), \quad r = \sqrt{x^2+p^2}.$$

To simplify notation let $A := a^W(x,\hbar D)$. By the equivariance of Weyl quantization with respect to the action of the symplectic (metaplectic) group, A commutes with the quantum harmonic oscillator $\mathcal{Z} = -\hbar^2 d^2/dx^2 + x^2$ and, by simplicity of the eigenvalues of the latter, the eigenfunctions e_n of $\mathcal{Z}$ are also eigenfunctions of A. Our goal is to compute the corresponding eigenvalues. We follow the argument in [5].

One can show (starting with section 13.1 of [1], for example) that if one defines the functions $g_n(x)$ by the generating function

$$\pi^{-1/4}\, e^{-x^2/2+xt-t^2/4} = \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{2^n n!}}\, g_n(x) \tag{A.4}$$

then the g_n are orthonormal in $L^2(\mathbb{R})$ and satisfy

$$-g_n''(x) + x^2 g_n(x) = (2n+1)\, g_n(x).$$

For our problem, we need the eigenfunctions of $\mathcal{Z}$, so we need to re-scale the variable x. Define

$$e_n(x) := \hbar^{-1/4}\, g_n(x/\sqrt{\hbar}). \tag{A.5}$$

Then, for each $\hbar$, e_n is L^2-normalized and

$$\begin{aligned} \hbar^{1/4}\left[-\hbar^2 \tfrac{d^2}{dx^2} e_n(x) + x^2 e_n(x)\right] &= -\hbar\, g_n''(x/\sqrt{\hbar}) + \hbar\left(\tfrac{x}{\hbar}\right)^2 g_n(x/\sqrt{\hbar}) \\ &= \hbar(2n+1) g_n(x/\sqrt{\hbar}). \end{aligned} \tag{A.6}$$

In other words, *the normalized eigenfunctions* e_n *are given by the generating function*

$$G_t(x) := (\pi\hbar)^{-1/4}\, e^{-x^2/2\hbar + xt/\sqrt{\hbar} - t^2/4} = \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{2^n n!}}\, e_n(x,\hbar), \tag{A.7}$$

where the notation emphasizes that e_n also depends on $\hbar$.

We now use this generating function to compute the eigenvalues of A. Note that

$$Q_a(G_t, G_t) = \langle A(G_t), G_t\rangle = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!}\, \lambda_n,$$

where $\lambda_n = \langle A(e_n), e_n\rangle$ is the eigenvalue of A corresponding to e_n. Computing using (A.2) and (A.3):

$$\begin{aligned} \mathcal{G}(G_t, G_t)(x,p) &= \frac{e^{-t^2/2}}{(\pi\hbar)^{3/2}} \int e^{2\hbar^{-1} ivp}\, e^{-x^2/\hbar}\, e^{-v^2/\hbar}\, e^{2xt/\sqrt{\hbar}}\, \mathrm{d}v \\ &= \frac{e^{-t^2/2 - x^2/\hbar + 2xt/\sqrt{\hbar}}}{(\pi\hbar)^{3/2}} \int e^{2\hbar^{-1} ivp}\, e^{-v^2/\hbar}\, \mathrm{d}v \\ &= \frac{1}{\pi\hbar}\, e^{-t^2/2 - (x^2+p^2)/\hbar + 2xt/\sqrt{\hbar}}, \end{aligned}$$

and therefore,

$$\sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!}\, \lambda_n = \frac{e^{-t^2/2}}{\pi\hbar} \int_{\mathbb{R}^2} a(x,p)\, e^{-(x^2+p^2)/\hbar + 2xt/\sqrt{\hbar}}\, \mathrm{d}x\, \mathrm{d}p. \tag{A.8}$$

Next, we use that a is radial and integrate in polar coordinates. The key integral is

$$\int_0^{2\pi} e^{2tr\cos(\theta)/\sqrt{\hbar}}\,\mathrm{d}\theta = 2\pi I_0(2tr/\sqrt{\hbar}), \tag{A.9}$$

where I_0 is the modified Bessel function of order zero. At this point, we can conclude that

$$\sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!}\,\lambda_n = \frac{2e^{-t^2/2}}{\hbar} \int_0^{\infty} \rho(r) e^{-r^2/\hbar}\, I_0(2tr/\sqrt{\hbar})\, r\mathrm{d}r. \tag{A.10}$$

Now, it is known that, for any $u \in \mathbb{R}$,

$$I_0(s) = e^u \sum_{k=0}^{\infty} \frac{(-u)^k}{k!}\, L_k(s^2/4u), \tag{A.11}$$

where the L_k are the Laguerre polynomials (in particular the right-hand side is independent of u). If we take $u = t^2/2$, (A.11) gives us that

$$I_0(s) = e^{t^2/2} \sum_{k=0}^{\infty} \frac{(-t^2/2)^k}{k!} L_k(s^2/2t^2).$$

Substituting back into (A.10), we obtain

$$\sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!}\,\lambda_n = \frac{2}{\hbar} \sum_{k\geq 0} \int_0^{\infty} \rho(r) e^{-r^2/\hbar}\, \frac{(-t^2)^k}{2^k k!} L_k(2r^2/\hbar)\, r\mathrm{d}r. \tag{A.12}$$

Equating coefficients of like powers of t we conclude that $\forall n$

$$\lambda_n = \frac{(-1)^k 2}{\hbar} \int_0^{\infty} \rho(r) e^{-r^2/\hbar}\, L_n(2r^2/\hbar)\, r\mathrm{d}r.$$

If we now let $u = r^2$, we finally get

$$\boxed{\lambda_n = \frac{(-1)^n}{\hbar} \int_0^{\infty} \rho(\sqrt{u}) e^{-u/\hbar}\, L_n(2u/\hbar)\, \mathrm{d}u.} \tag{A.13}$$

Although we do not need it for the proof of our main theorem, we note the following:

Theorem A.1. *Let (as in the main body of the paper)*

$$\hbar = \frac{\mathcal{E}}{2n+1}, \quad \mathcal{E} \text{ fixed}, \quad n = 1, 2, \ldots,$$

Then, maintaining the previous notation, as $n \to \infty$

$$\lambda_n = \rho(\sqrt{\mathcal{E}}) + O(\hbar).$$

Proof. By the functional calculus the operator $\rho(\mathcal{Z}^{1/2})$ is an $\hbar$ pseudo-differential operator with principal symbol $\rho(r)$, that is, with the same principal symbol as a^W. Therefore,

$$\lambda_n = \langle a^W(e_n), e_n\rangle = \langle \rho(\mathcal{Z}^{1/2})(e_n), e_n\rangle + O(\hbar) = \rho(\sqrt{\mathcal{E}}) + O(\hbar).$$

□

In view of (A.13), we immediately obtain:

Corollary A.2. *Let*

$$\psi_n(u) := \frac{(-1)^n}{\hbar} e^{-u/\hbar} L_n\left(\frac{2u}{\hbar}\right)$$

so that $\lambda_n = \int_0^\infty \rho(\sqrt{u})\psi_n(u)\,\mathrm{d}u$. *Then, if* $\hbar$ *and* n *are related as above, the sequence* (ψ_n) *tends weakly to the delta function at* $\mathcal{E}$.

It is instructive to consider directly the behavior of the functions ψ_n. As we will see, there is an oscillatory and a decaying region of ψ_n (similar to the Airy function). For a fixed n, ψ_n has n zeros. As n increases, where do the zeros concentrate? According to [8], the zeros of L_n are real and simple.

Let us denote by $\lambda_{n,k}$ the zeros of L_n. According to [7] (restricting to the case $\alpha = 0$), the zeros $\lambda_{n,k}$ are in the oscillatory region

$$0 < x < \nu := 4n + 2$$

and satisfy the following inequalities and asymptotic approximation:

Theorem A.3 ([7]). *The first zero* $\lambda_{n,1}$ *satisfies*

$$0 < \lambda_{n,1} \leq \frac{3}{2n+1}, n = 1, 2, \ldots.$$

Theorem A.4 ([7]). *For a fixed* m, *the zeros of* L_n *satisfy*

$$\lambda_{n,n-m+1} = \nu + 2^{1/3} a_m \nu^{1/3} + \frac{1}{5} 2^{4/3} a_m^2 \nu^{-1/3} + O(n^{-1}), \text{ as } n \to \infty,$$

where a_m *is the* m*-th negative zero of the Airy function, in decreasing order.*

Let us now denote by $\mu_{n,k}$ the zeros of $\psi_n(u)$, so that $\mu_{n,k} = \frac{\hbar}{2}\lambda_{n,k}$. Substituting $\hbar = \mathcal{E}/(2n+1)$, Theorem A.3 implies that the first zero satisfies

$$\mu_{n,1} \leq \frac{3}{2\mathcal{E}}\hbar^2.$$

On the other hand, the last zero satisfies

$$\mu_{n,n} = \mathcal{E} + (\mathcal{E}/2)^{1/3} a_1 \hbar^{2/3} + \frac{(\mathcal{E})^{-1/3} a_1^2}{5}\hbar^{4/3} + O(\hbar^2), \text{ as } \hbar \to 0.$$

This implies that the first zero is close to 0 while the last one is close to $\mathcal{E}$ as $\hbar \to 0$. In fact, if we define

$$N_n(x) = \left| \{k \in \{1, 2, \ldots, n\} | \lambda_{n,k} \leq x\} \right|, x \in \mathbb{R},$$

it can be shown ([8]) that

$$\lim_{n\to\infty} \frac{1}{n} N_n(4nx) = \frac{2}{\pi}\int_0^x t^{-1/2}(1-t)^{1/2}\mathrm{d}t, \quad \text{for } 0 \leq x \leq 1.$$

We note that $\lambda_{n,k} \leq 4nx$ if and only if $\mu_{n,k} \leq \mathcal{E}x\left(1 - \frac{1}{2n+1}\right)$. This implies that

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k : \mu_{n,k} \leq z\left(1 - \frac{1}{2n+1}\right)\right\}\right|$$
$$= \frac{2}{\pi}\int_0^{z/\mathcal{E}} t^{-1/2}(1-t)^{1/2}\mathrm{d}t, \;\; 0 \leq z \leq \mathcal{E}.$$

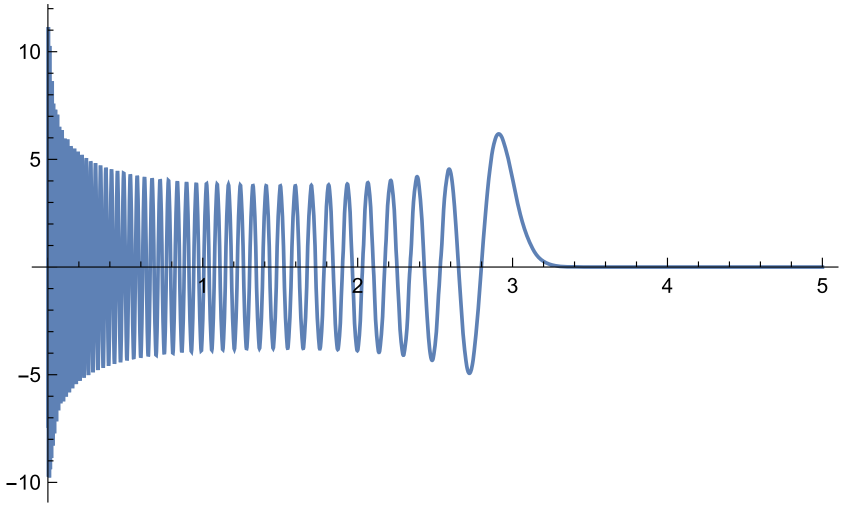

FIGURE 2. Graph of ψ_n in the interval $[0,5]$. Here, $n=100$, $\mathcal{E}=3$

We note that the integral on the right-hand side is equal to one for $z=\mathcal{E}$. In particular, this shows that the zeros of ψ_n "cover" the entire oscillatory region $[0,\mathcal{E}]$, asymptotically for n large.

Choosing $n = 100$ and $\mathcal{E} = 3$, the corresponding graph of ψ_n in the interval $[0,5]$ is shown in Fig. 2. We can corroborate numerically that the zeros of ψ_n are located in the oscillatory region $[0,\mathcal{E}]$. We can easily see that L_n is always locally decreasing near the origin and locally increasing/decreasing around the last zero for n even/odd. As a result, the last critical point of ψ_n is always a local maximum.

B. The Remainder in Taylor's Theorem

For completeness, we include here the elementary derivation of the expression for the remainder in Taylor's theorem that we used in the proof of Theorem 4.6. Let us start with a smooth one-variable function f and write

$$\begin{aligned} f(t) &= f(\mathcal{E}) + \int_0^1 \frac{d}{\mathrm{d}u} f(ut+(1-u)\mathcal{E})\,\mathrm{d}u \\ &= f(\mathcal{E}) + (t-\mathcal{E})\int_0^1 f'(ut+(1-u)\mathcal{E})\,\mathrm{d}u. \end{aligned}$$

So if we let

$$g(t) := \int_0^1 f'(ut+(1-u)\mathcal{E})\,\mathrm{d}u, \tag{B.1}$$

then g is smooth and $f(t)=f(\mathcal{E})+(t-\mathcal{E})g(t)$. Repeating the argument with f replaced by g, we obtain that

$$g(t) = g(\mathcal{E}) + (t-\mathcal{E})R(t),$$

where

$$R(t) = \int_0^1 g'(vt + (1-v)\mathcal{E})\,\mathrm{d}v.$$

Since $g(\mathcal{E}) = f'(\mathcal{E})$, substituting we obtain $f(t) = f(\mathcal{E}) + (t-\mathcal{E})f'(\mathcal{E}) + (t-\mathcal{E})^2 R(t)$, as desired. Finally, we compute the remainder $R(t)$. Using (B.1),

$$g'(x) = \int_0^1 uf''(ux + (1-u)\mathcal{E})\,\mathrm{d}u,$$

and therefore

$$R(t) = \int_0^1 \int_0^1 uf''[u(vt + (1-v)\mathcal{E}) + (1-u)\mathcal{E}]\mathrm{d}u\,\mathrm{d}v.$$

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