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Global well-posedness of a model for precipitating convection with hydrostatic pressure under fast autoconversion and rain evaporation conditions

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A B S T R A C T

Moist convection in the tropical atmosphere involves the interaction of deep precipitating clouds that can be organized into different patterns and propagate long distances. Their modeling to predict their evolution can be done with PDE-based models but it requires the parametrization of different cloud micro-physical processes. Simplified models can maintain a balance between complexity and precision, and can still capture qualitative observations made in nature. In this work, we prove global existence and uniqueness of a model for precipitating turbulent convection. The model considers a moisture dynamics with phase changes. It assumes fast auto-conversion and rain evaporation together with a hydrostatic pressure approximation. We also show positivity of the variables associated to moisture and equivalent potential temperature. The existing literature is discussed and the new challenges in this model are highlighted.

1. Introduction

A variety of natural phenomena in the atmosphere can be successfully studied and predicted with the aid of PDE-based models. Geophysical fluid flows are complex and in many cases the mathematical models involved in their analysis present a variety of limitations. The complexity arises in part due to the turbulent motions and the big range of time- and length-scales entailed in their evolution. Consequently, sophisticated models that can accurately capture features of the phenomenon under consideration may inherit theoretical complications and its implementation may face numerical and computational challenges.

Moist convection in the atmosphere involves the evolution of water in its different phases, vapor, liquid and ice [7,29]. In contrast with dry convection, the addition of water and the corresponding phase changes bring

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into play several cloud microphysical processes. Mesoscale convective systems, appearing as individual cloud systems in horizontal scales of roughly 100 km and 1 h to 1 day, form an important aspect of organized tropical convection [15]. Theoretical understanding of the physical processes involved can improve their numerical modeling and predictions [30,23,10]. Instead of modeling the detailed cloud microphysics involving individual droplets, bulk cloud physics with closures involving mixing ratios is often a favorable alternative. See for instance [11,16,17,29] and references therein. Moist convection can be modeled via the anelastic equations [1,19,25] but its implementation could be computationally expensive due to its complexity, and simplifications resulting in reduced models are often a more attractive path. The thermodynamic component of the model can, for instance, follow the Boussinesq approximations valid for flows in which the depth of the fluid motion is small compared to the density scale height [28,31], or the primitive equations where the pressure follows the hydrostatic approximation [24]. Regardless of the approximation, those models are aimed at understanding the formation and evolution of different natural geophysical phenomena. For instance, cloud resolving models have shown to be very useful in providing details for convective organizations that are often not available from observational data.

The above approaches are some examples of different techniques and approximations, resulting in mathematical PDE-based models that can be used to study geophysical flows. Many of those models consider a dry atmosphere or treat moisture in an implicit fashion. Nevertheless, the impact of moisture in atmospheric dynamics is crucial. Some models are more comprehensive than others and can provide more detailed information to improve their predictions. Such needed complexity can sometimes result in less accessible systems that are computationally expensive but also difficult to be treated theoretically. Models with a good balance between complexity and precision can provide useful insight into the mechanisms behind different physical processes. A variety of phenomena can be replicated with simplified models by focusing on specific physical processes while still being able to be examined conceptually. Well-posedness and regularity are fundamental theoretical aspects of any mathematical model that one must assess when possible. Mathematical analysis of some of those models involving moisture dynamics explicitly can be found in [3,18,20,32–34]. Global well-posedness can be proved for a model that couples the primitive equations to moisture dynamics where phase changes are allowed between water vapor, cloud water and rain. First, it was done in [13] for passively transported nonlinear dynamics, where the velocity field is known. Later, in [14], the proof was extended to the case where the velocity field evolves as part of the solution.

A minimal model for precipitating turbulent convection was derived and numerically analyzed in [12]. The model was able to capture different cloud regimes in response to different background wind shear profiles. It was shown to be able to reproduce squall lines in the presence of strong wind shear at low altitudes and scattered convection in the absence of it. The model agreed with qualitative observations made in nature when squall lines are formed, such as their tilted profile, speed and direction of propagation and the formation of cold pools right beneath it. Despite limitations needed for a minimal and idealized model, the numerical results showed that the above squall line features observed in nature were also reproduced by the model. The minimal model used a Boussinesq system together with a moisture dynamics. One assumption in this model is fast auto-conversion, where rainwater forms instantaneously when the atmosphere saturates. Fast auto-conversion was also used in [21,6] where squall lines and cyclogenesis were investigated. The minimal model in [12] further assumes fast rain evaporation, where rainwater evaporates when it falls to unsaturated regions at a timescale faster than the dynamical scales of interest. As a result, cloud water is ignored and one only retains water vapor and rainwater. The minimal model, which is known as FARE for its Fast Autoconversion and Rain Evaporation assumptions, is written in terms of conservation laws for momentum, energy, moist entropy and total water.

In this work, we present a mathematical analysis of a model that assumes the same moisture dynamics based on fast auto-conversion and rain evaporation from [12]. Instead of a Boussinesq approximation, we use the primitive equations where a hydrostatic balance occurs between the vertical pressure gradient and the buoyancy force of the system. Our main contribution consists of proving the local and global existence of solutions of the PDE that dictates the time evolution in our model. We also guarantee global uniqueness and positivity of solution variables such as total water and equivalent potential temperature. For local existence, we follow ideas found in [13], where velocity is known (passive transport) but water vapor and rainwater evolve in time according to their own equations of motion where condensation and evaporation processes are parameterized using piece-wise defined stiff terms that activate when the total water exceeds a threshold (water vapor at saturation). On the contrary, in the present work only the total water evolves in time while the water vapor and rainwater are computed diagnostically from it. In a nonlinear manner, it is done via a transition smooth function, which avoids stiff terms and possible theoretical complications. See for instance, the work in [34], where the Authors deal with nonlinear and discontinuous terms coming from phase changes when water vapor reaches saturation, making the well-posedness proofs more challenging. In particular, a variational inequality is included in order to represent the Heaviside graph as a subdifferential of a convex functional.

The use of a smooth transition function requires certain bounding assumptions and the application of known results from Bochner spaces. In order to guarantee existence and global uniqueness, we also need to introduce Lipschitz continuity conditions on the transition function. This is a property that our transition function satisfies, but it could be easily generalized to other Lipschitz continuous functions. For well-posedness, we also follow ideas of [14], which considers time evolving velocities to be solved as part of the solution. Under the FARE assumptions, some care is required with the estimations that involve the buoyant force. This is because the estimations of the errors depend on the velocity, for which it is necessary to redefine a total error and the choice of certain appropriate parameters coming from Young's inequality. In this paper, we present a detailed analysis for the local and global existence and uniqueness, keeping track of the coefficients involved in the bounds required in the proofs. One has to solve for the horizontal velocity in a smaller space and the vertical velocity obtained via the continuity constraint is shown to belong into a bigger one. Likewise, the positivity of total water and equivalent potential temperature requires the incorporation of a new integration space where the time derivative of the transition function makes sense. Careful estimations are also needed for the hydrostatic pressure, buoyancy and precipitation terms in order to show local and global existence, positivity and global uniqueness. All the constants involved in our analysis are presented explicitly for both local and global existence, which may allow for a better understanding of the behavior of the estimates. In order to make it more clear, technical results in Appendix A that are shown in references [13,14] do not include the proofs and are clearly identified. Throughout the text, we also specify when we follow the techniques in [13,14] and highlight the main differences. For instance, here we do not use pressure coordinates and the uniqueness proof does not decompose the velocity into its baroclinic and barotropic components. Instead, logarithmic type anisotropic Sobolev embedding inequalities from [4] are employed. In the proof for existence, the hydrostatic pressure is integrated in a special way to guarantee the rigid lid assumptions. The pressure estimates require additional steps that are explained in Lemmas 11 and 12 in Appendix A.

Although the techniques in this work heavily rely on the works found in [13,14], the PDE-based model in this paper has fundamental differences. Our FARE assumptions remove one of the phases (cloud water) and rain forms according to a transition function as soon as the moisture exceeds a saturation threshold. Such threshold appears in the estimates and requires a special treatment. Although we impose a specific structure for the transition function to be consistent with the FARE assumptions, in general one only requires certain mild conditions. In a case where rain is bounded (which is physically consistent), one can assume a transition function of compact support with important implications in the proofs, and one can show that the square of the L^2 norm of the velocity field grows at most linearly in time. The rest of this work is organized as follows. Section 2 contains information about the variables involved and explains the corresponding physical interpretations. Section 3 starts with the local and global well-posedness, followed by uniqueness. Some estimates are left to Appendix A.

2. The hydrostatic-FARE model

The Fast Auto-conversion and Rain Evaporation (FARE) model presented in [12] has demonstrated to be able to capture observations made in nature such as the response to scattered convection versus squall line formation in the absence or presence of strong windshear near the bottom surface in the troposphere [12]. The model was minimal in the sense that it considers bulk cloud physics in a simplified way, assuming that any excess of water from saturation levels is instantaneously converted into rain. Moisture dynamics is then dictated by one evolution equation for total water from which vapor and rain can be computed. In deriving such a simplified model that captures qualitative features of turbulent convection, one can achieve the goal of analyzing the model in a more theoretical way. The purpose of this work is to go in that direction for a model with similar assumptions to that in the FARE model but adopting hydrostatic pressure conditions. This section shows details of the model and introduces notation.

The model is given by the following system of PDEs:

$$\frac{D\mathbf{u}_h}{Dt} + f_0(\hat{z}\times\mathbf{u})_h = -\nabla_h \mathrm{p} + \nu_1\Delta_h\mathbf{u}_h + \nu_2\partial_z^2\mathbf{u}_h, \tag{1a}$$

$$\partial_z \mathrm{p} = b(q_t, \theta_e; z), \tag{1b}$$

$$\frac{D\theta_e}{Dt} = \kappa_1\Delta_h\theta_e + \kappa_2\partial_z^2\theta_e, \tag{1c}$$

$$\frac{Dq_t}{Dt} - V_T\partial_z q_r = \mu_1\Delta_h q_t + \mu_2\partial_z^2 q_t, \tag{1d}$$

$$\nabla_h\cdot\mathbf{u}_h + \partial_z w = 0. \tag{1e}$$

In the above description and throughout the paper, ∇_h and Δ_h are the horizontal gradient and horizontal Laplacian respectively. The material derivative is defined as

$$\frac{D}{Dt} = \partial_t + \mathbf{u}_h\cdot\nabla_h + w\partial_z. \tag{2}$$

Furthermore, the velocity field is denoted as $\mathbf{u} = (u, v, w) = (\mathbf{u}_h, w)$ and is separated into horizontal and vertical component; f_0 is the Coriolis parameter (set to be a constant); p is the rescaled pressure; ν_1 and ν_2 are the horizontal and vertical kinematic viscosity coefficients respectively; and $\kappa_1, \kappa_2, \mu_1$ and μ_2 are diffusion coefficients. The total water mixing ratio $q_t = q_v + q_r$ accounts for the water vapor q_v (gas phase) and rain q_r (liquid phase) components of water in warm clouds. As mixing ratios, those quantities are measured in units of density of the phase by density of the total fluid's parcel. The equivalent potential temperature θ_e describes the temperature that a parcel of air would reach if all the water vapor in the parcel is condensed and the parcel was brought adiabatically to a reference pressure. This quantity is conserved under adiabatic processes even if water condenses, releasing its latent heat. The ordinary potential temperature θ is related to the equivalent potential temperature through a thermodynamic linearized relation given by $\theta_e = \theta + \frac{L}{c_p}q_v$, where $L = 2.5\times10^6$ J kg^{-1} is the latent heat release, and $c_p = 10^3$ J kg^{-1} K^{-1} is the specific heat at constant pressure. The rainfall speed is denoted by V_T, and it is assumed to be constant for simplicity [12].

Equation (1a) describes conservation of horizontal momentum. Equation (1b) corresponds to the hydrostatic pressure assumption, where $b(q_t, \theta_e; z)$ is the buoyancy force given by

$$b(q_t, \theta_e; z) := g\left(\frac{\theta_e}{\theta_0} - \left(\frac{L}{c_p\theta_0} - \epsilon_0\right)q_v - q_r\right). \tag{3}$$

Here θ_0 is a reference value for the potential temperature, g is the gravitational constant, and $R_v/R_d = \epsilon_0+1$ is the ratio of gas constants. Conservation of equivalent potential temperature is dictated by equation (1c),

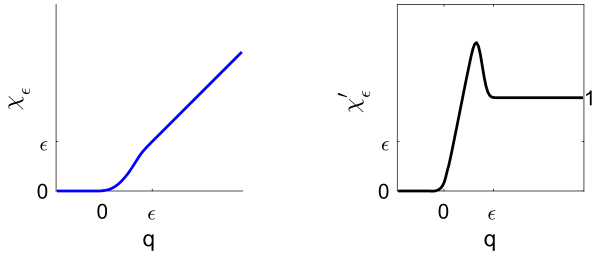

Fig. 1. Schematic graphs of χ_ε (left) and χ'_ε (right).

indicating that it is a material invariant in the absence of viscosity. Equation (1d) describes conservation of total moisture plus precipitation of rainwater. Finally, the last equation (1e) corresponds to the incompressibility condition. We will call the above model HFARE, where H stands for the hydrostatic pressure condition.

The evolution of water vapor and rainwater usually involves parameterizations of condensation and evaporation processes [21]. Such parameterizations activate when the atmosphere saturates, typically inserting stiff terms. In [12], the fast autoconversion and rain evaporation constraints led to a phase change process in which the excess of water from a threshold (water vapor at saturation q_{vs}) will automatically be considered rain. As a consequence, one evolution equation for the total water dictates the moisture dynamics while the water vapor and rain components are obtained diagnostically. However, in this work q_v and q_r are smooth functions of q_t and q_{vs} in a phase transition from vapor to liquid. This is in contrasts to what it was done in [12], where such dependence consists of piece-wise defined expressions. Our approach avoids dealing with stiff terms that might not be differentiable and allows for a progressive transition from water vapor to rain. Specifically, rain is computed as

$$q_r := \chi_\varepsilon(q_t - q_{vs}(z)),$$

where $q_{vs}(z)$ is the water vapor at saturation which is in turn a given decreasing function of height, ε is a small positive parameter, and $\chi_\varepsilon \in C^\infty(\mathbb{R})$ is a smooth transition function with bounded derivative, satisfying

$$\chi_\varepsilon(\zeta) = \begin{cases} 0 & \text{if} \quad \zeta \leq 0, \\ \zeta & \text{if} \quad \zeta > \varepsilon. \end{cases} \tag{4}$$

Since the derivative of χ_ε is bounded and $\chi_\varepsilon(0) = 0$, then there exist positive constants c_χ and c_ε such that

$$|\chi_\varepsilon(\zeta)| \leq c_\chi|\zeta| \qquad \text{and} \qquad |\chi'_\varepsilon(\zeta)| \leq c_\varepsilon. \tag{5}$$

Moreover, due to the mean value theorem, χ_ε and its derivative are Lipschitz–continuous, i.e., there exists L_1 and L_2, such that

$$|\chi_\varepsilon(q_1) - \chi_\varepsilon(q_2)| \leq L_1|q_1 - q_2| \qquad \text{and} \qquad |\chi'_\varepsilon(q_1) - \chi'_\varepsilon(q_2)| \leq L_2|q_1 - q_2|. \tag{6}$$

See Fig. 1 for an schematic of χ_ε and its derivative.

We note that the buoyancy can be explicitly computed as

$$b(q_t, \theta_e; z) = g\left(\frac{\theta_e}{\theta_0} - \left(\frac{L}{c_p\theta_0} - \epsilon_0\right) q_t + \left(\frac{L}{c_p\theta_0} - \epsilon_0 - 1\right) \chi_\varepsilon(q_t - q_{vs}(z))\right). \tag{7}$$

2.1. Domain, initial and boundary conditions

We seek for solutions to system (1) with spatial domain of cylindrical type: $\mathcal{M} = \mathcal{M}' \times [z_0, z_1]$, where $\mathcal{M}'$ is a smooth bounded domain in $\mathbb{R}^2$ and $z_0 < z_1$. The boundary is given by

$$\begin{aligned}
\Gamma_0 &:= \{(x,y,z) \in \overline{\mathcal{M}} : z = z_0\},\\
\Gamma_1 &:= \{(x,y,z) \in \overline{\mathcal{M}} : z = z_1\},\\
\Gamma_s &= \{(x,y,z) \in \overline{\mathcal{M}} : (x,y) \in \partial\mathcal{M}', \quad z_0 \le z \le z_1\}.
\end{aligned}$$

Under the definition of the material derivative (see equation (2)), we can rewrite the HFARE model (1) as:

$$\partial_t \mathbf{u}_h + (\mathbf{u}_h \cdot \nabla_h)\mathbf{u}_h + w\partial_z \mathbf{u}_h + f_0(\hat{z} \times \mathbf{u})_h = -\nabla_h \mathrm{p} + \nu_1 \Delta_h \mathbf{u}_h + \nu_2 \partial_z^2 \mathbf{u}_h, \tag{8a}$$

$$\partial_z \mathrm{p} = b(q_t, \theta_e; z), \tag{8b}$$

$$\partial_t q_t + \mathbf{u}_h \cdot \nabla_h q_t + w\partial_z q_t = V_T \partial_z \chi_\varepsilon(q_t - q_{vs}) + \kappa_1 \Delta_h q_t + \kappa_2 \partial_z^2 q_t, \tag{8c}$$

$$\partial_t \theta_e + \mathbf{u}_h \cdot \nabla_h \theta_e + w\partial_z \theta_e = \mu_1 \Delta_h \theta_e + \mu_2 \partial_z^2 \theta_e, \tag{8d}$$

$$\nabla_h \cdot \mathbf{u}_h + \partial_z w = 0, \tag{8e}$$

with initial data

$$\mathbf{u}_h(\cdot,0) = \mathbf{u}_{h0}\ , \quad w(\cdot,0) = w_0\ , \quad q_t(\cdot,0) = q_{t0}\ , \text{ and } \theta_e(\cdot,0) = \theta_{e0}, \tag{9}$$

subject to boundary conditions

$$\Gamma_0 : \quad \partial_z \mathbf{u}_h = 0, \qquad w = 0, \qquad \partial_z \theta_e = \alpha_{\theta 0}\theta_e, \qquad \partial_z q_t = \alpha_{q0} q_t, \tag{10a}$$

$$\Gamma_1 : \quad \partial_z \mathbf{u}_h = 0, \qquad w = 0, \qquad \partial_z \theta_e = -\alpha_{\theta 1}\theta_e, \qquad \partial_z q_t = -\alpha_{q1} q_t, \tag{10b}$$

$$\Gamma_s : \quad \partial_{\boldsymbol{n}} \mathbf{u}_h = 0, \qquad \partial_n \theta_e = -\alpha_{\theta s}\theta_e. \qquad \partial_n q_t = -\alpha_{qs} q_t, \tag{10c}$$

where $\alpha_{\star 0}, \alpha_{\star 1}$ and $\alpha_{\star s}$ with $\star \in \{q, \theta\}$, are non–negatives scalars. That is, we require Robin boundary conditions for the equivalent potential temperature and total water. The normal velocity at the lateral boundaries vanishes. Zero Neumann boundary conditions are required for the horizontal velocity at the top and bottom. This will be particularly helpful in guaranteeing that the vertical velocity vanishes at those boundaries, under rigid lid assumptions. More details will be provided in Section 3.

2.2. Spaces of functions

We introduce the following spaces of functions using standard notation and terminology from Sobolev space theory. In particular, if $\Omega \subset \mathbb{R}^d$ is a domain with $d \in \mathbb{N}$, Γ is an open or closed Lipschitz curve, and $s \in \mathbb{R}$. We define $\boldsymbol{H}^s(\Omega) = [H^s(\Omega)]^d$ and $\boldsymbol{H}^s(\Gamma) := [H^s(\Gamma)]^d$, whereas for the case $s = 0$ we will simply write $\boldsymbol{H}^0(\Omega) := \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}^0(\Gamma) := \boldsymbol{L}^2(\Gamma)$. The associated norms will be denoted by $\|\cdot\|_{\boldsymbol{H}^s(\Omega)}$, $\|\cdot\|_{\boldsymbol{H}^s(\Gamma)}$, $\|\cdot\|_{\boldsymbol{L}^2(\Omega)}$ and $\|\cdot\|_{\boldsymbol{L}^2(\Gamma)}$. In addition, X will denote a Banach space with norm $\|\cdot\|_X$. We also define the Bochner space $L^p(0,T,X) := \{\mathrm{f} : [0,T] \to X : \|\mathrm{f}\|_{L^p(0,T,X)} < +\infty\}$, with its corresponding norm

$$\|\mathrm{f}\|_{L^p(0,T,X)} := \left(\int_0^T \|\mathrm{f}\|_X^p \, dt \right)^{1/p},$$

for $1 \le p < \infty$ and

$$\|\mathrm{f}\|_{L^\infty(0,T,X)} := \operatorname*{ess\,sup}_{0\le t\le T} \|\mathrm{f}\|_X.$$

Similarly, $C([0,T];X)$ denotes the Banach's space that includes all continuous functions $\mathrm{f}:[0,T]\to X$ with associated norm

$$\|\mathrm{f}\|_{C([0,T];X)} := \max_{0\le t\le T} \|\mathrm{f}\|_X < +\infty.$$

3. Well-posedness

Our main goal in this work is to prove the local and global existence of the hydrostatic-FARE model described in (8) subject to boundary conditions (10). The details are provided in this section.

3.1. Local well-posedness

The local existence is proved based on the following known results for linear parabolic equations, subject to the Robin boundary conditions on cylindrical-type domains.

Lemma 1. *Let the initial data* $(u_0, v_0, q_{t0}, \theta_{e0}) \in \boldsymbol{H}^2(\mathcal{M})$, $w_0 \in H^1(\mathcal{M})$ *be such that the total water and equivalent potential temperature are non negative; the initial velocity field is divergence-free; and satisfy the boundary conditions* (10). *Then there exists a unique local solution* (u, v, w, q_t, θ_e), *which depends continuously on the initial data, in some short time interval* $(0, \mathcal{T}_0)$, *to system* (8) *subject to the boundary conditions* (10), *satisfying*

$$\begin{aligned} u, v, q_t, \theta_e, \in C([0,\mathcal{T}_0]; H^2(\mathcal{M})), w \in C([0,\mathcal{T}_0]; H^1(\mathcal{M}))\\ \partial_t u, \partial_t v, \partial_t q_t, \partial_t \theta_e, \in L^2(0,\mathcal{T}_0, L^2(\mathcal{M}))\end{aligned}$$

Proof. The proof consists of constructing a sequence of vector fields given by $\{(u^n, v^n, q_t^n, \theta_e^n)\}_{n=0}^\infty$ that converges to the strong solution. We will denote the solution vector by $\boldsymbol{S} := (u, v, q_t, \theta_e)$ and let $\boldsymbol{S}^{n+1} := (u^{n+1}, v^{n+1}, q_t^{n+1}, \theta_e^{n+1})$, with $n = 0, 1, \cdots$ be the unique strong solution to the linear parabolic system

$$\partial_t \mathbf{u}_h^{n+1} - \nu_1 \Delta_h \mathbf{u}_h^{n+1} - \nu_2 \partial_z^2 \mathbf{u}_h^{n+1} = -(\mathbf{u}_h^n \cdot \nabla_h)\mathbf{u}_h^n - w^n \partial_z \mathbf{u}_h^n - \nabla_h \mathrm{p}^n - f_0(\hat{z} \times \mathbf{u}^n)_h, \tag{11a}$$

$$\partial_t q_t^{n+1} - \kappa_1 \Delta_h q_t^{n+1} - \kappa_2 \partial_z^2 q_t^{n+1} = -\mathbf{u}_h^n \cdot \nabla_h q_t^n - w^n \partial_z q_t^n + V_T \partial_z \chi_\varepsilon (q_t^n - q_{vs}(z)), \tag{11b}$$

$$\partial_t \theta_e^{n+1} - \mu_1 \Delta_h \theta_e^{n+1} - \mu_2 \partial_z^2 \theta_e^{n+1} = -\mathbf{u}_h^n \cdot \nabla_h \theta_e^n - w^n \partial_z \theta_e^n, \tag{11c}$$

$$\partial_z \mathrm{p}^n = b(q_t^n, \theta_e^n; z), \tag{11d}$$

$$\partial_z w^n = -\nabla_h \cdot \mathbf{u}_h^n. \tag{11e}$$

We note that w is not an element in the sequence $\boldsymbol{S}$. It is computed via the continuity equation and we can only show that it belongs to $C([0,\mathcal{T}_0]; H^1(\mathcal{M}))$ instead of $C([0,\mathcal{T}_0]; H^2(\mathcal{M}))$. In addition, we consider the initial data

$$\boldsymbol{S}^{n+1}\big|_{t=0} := (u^{n+1}, v^{n+1}, q_t^{n+1}, \theta_e^{n+1})\big|_{t=0} := (u_0, v_0, q_{t0}, \theta_{e0}) = \boldsymbol{S}^0. \tag{12}$$

The incompressibility condition allows us to write the vertical velocity as in integral that can easily satisfy one zero Dirichlet boundary condition either at the top or at the bottom boundaries. However, it is not clear that such integral will satisfy the same boundary condition on the other side. The hydrostatic pressure needs to be integrated in a specific way in order to guarantee the rigid lid conditions in system (11) for the sequence $\boldsymbol{S}^n$ for $n = 0, 1, \cdots$. This is shown on the following lemma.

We note that the condition $\int_{z_0}^{z_1} \nabla_h \cdot \mathbf{u}_h^n \, dz = 0$, consistent with the incompressibility of the fluid, is related to the boundary condition $w^n_{z=z_0,z_1} = 0$. We directly compute w^n according to equation (11e) and the integral condition is proved in the next lemma.

Lemma 2. *The sequence* $\boldsymbol{S}^{n+1} = (u^{n+1}, v^{n+1}, q_t^{n+1}, \theta_e^{n+1})$ *satisfies the rigid lid assumptions for* w^{n+1} *in* z_0 *and* z_1*, that is* $w^{n+1}(z_0) = w^{n+1}(z_1) = 0$ *for all* n *provided that the flow is initially incompressible* ($n = 1$)*, and the hydrostatic pressure is given by*

$$\mathrm{p}^n = -\int_z^{z_1} b(q_t^n, \theta_e^n; \sigma) d\sigma + \bar{\mathrm{p}}^n(x, y, t),$$

where the pressure at the top surface $\bar{\mathrm{p}}^n(x, y, t)$ *satisfies the Poisson equation*

$$\begin{aligned} \frac{\Delta_h \bar{\mathrm{p}}^n(x, y, t)}{z_1 - z_0} = &-\int_{z_0}^{z_1} \nabla_h \cdot ((\mathbf{u}^n \cdot \nabla)\mathbf{u}_h^n) dz + \int_{z_0}^{z_1} \left(\int_{z_0}^{z} \Delta_h b(q_t^n, \theta_e^n; \sigma) d\sigma \right) dz \\ &- f_0 \int_{z_0}^{z_1} (\partial_x v^n - \partial_y u^n) dz. \end{aligned} \tag{13}$$

So, if the rigid-lid assumption is satisfied at step n*, condition* (13) *provides the pressure expression for the rigid-lid assumption to be satisfied at step* $n + 1$*.*

Proof. The condition $w(z_1) = 0$ implies that

$$w^n = \int_z^{z_1} \nabla_h \cdot \mathbf{u}_h^n \, dz.$$

In order for the vertical velocity to vanish at the bottom boundary, we still need to show that

$$\int_{z_0}^{z_1} \nabla_h \cdot \mathbf{u}_h^n \, dz = 0 \qquad \forall \, n.$$

Differentiating with respect to x and y, respectively the horizontal velocity components in equation (11a), we obtain

$$\begin{aligned} \partial_t \partial_x u^{n+1} - \nu_1 \partial_x \Delta_h u^{n+1} - \nu_2 \partial_x \partial_z^2 u^{n+1} = &- \partial_x \left((\mathbf{u}_h^n \cdot \nabla_h) u^n + w^n \partial_z u^n \right) \\ &- \partial_x^2 \mathrm{p}^n - f_0 \partial_x v^n, \\ \partial_t \partial_y v^{n+1} - \nu_1 \partial_y \Delta_h v^{n+1} - \nu_2 \partial_y \partial_z^2 v^{n+1} = &- \partial_y ((\mathbf{u}_h^n \cdot \nabla_h) v^n + w^n \partial_z v^n) \\ &- \partial_y^2 \mathrm{p}^n + f_0 \partial_y u^n. \end{aligned}$$

Then, by the incompresibility condition (8e), we get

$$\begin{aligned} &\partial_t(-\partial_z w^{n+1}) - \nu_1 \Delta_h(-\partial_z w^{n+1}) - \nu_2 \partial_z^2(-\partial_z w^{n+1}) \\ &\quad = -\nabla_h \cdot ((\mathbf{u}^n \cdot \nabla)\mathbf{u}_h^n) - \Delta_h \mathrm{p}^n - f_0(\partial_x v^n - \partial_y u^n). \end{aligned} \tag{14}$$

The pressure terms in the momentum equations ensure incompressibility. From (11d) and (11e), it follows that

$$\mathrm{p}^n = -\int_z^{z_1} b(q_t^n, \theta_e^n; \sigma)d\sigma + \bar{\mathrm{p}}^n(x,y,t), \tag{15}$$

where $\bar{\mathrm{p}}^n(x,y,t)$ is a function that is independent of height. Replacing (15) into (14), is not difficult to see that

$$\begin{aligned}
&\partial_t \left(\int_{z_0}^{z_1}(\nabla_h \cdot \mathbf{u}_h^{n+1})dz\right) - \nu_1\Delta_h\left(\int_{z_0}^{z_1}(\nabla_h \cdot \mathbf{u}_h^{n+1})dz\right) - \nu_2\nabla_h\cdot\left(\partial_z \mathbf{u}_h^{n+1}\right)\Big|_{z_0}^{z_1} \\
&= -\int_{z_0}^{z_1}\nabla_h\cdot(\mathbf{u}^n\cdot\nabla\mathbf{u}_h^n)dz + \int_{z_0}^{z_1}\Delta_h\left(\int_z^{z_1} b(q_t^n,\theta_e^n;\sigma)d\sigma + \bar{\mathrm{p}}^n(x,y,t)\right)dz \\
&\quad -f_0\int_{z_0}^{z_1}(\partial_x v^n - \partial_y u^n)dz.
\end{aligned} \tag{16}$$

We note that the right hand side vanishes if the pressure at the top surface $\bar{p}(x,y,t)$ is chosen according to equation (13). The horizontal Laplacian can be inverted for functions that do not depend on z. Integrating equation (14), and using the fact that $w(z_1) = 0$ it follows from (16), that

$$\partial_t w^{n+1}(x,y,z_o,t) - \nu_1\Delta_h w^{n+1}(x,y,z_0,t) - \nu_2\nabla_h\cdot\left(\partial_z\mathbf{u}_h^{n+1}\Big|_{z_0}^{z_1}\right) = 0.$$

Notice that, if we impose the following boundary condition

$$\partial_z\mathbf{u}_h^{n+1}\Big|_{z_0}^{z_1} = 0, \tag{17}$$

we get

$$(\partial_t - \nu_1\Delta_h)w^{n+1}(x,y,z_0,t) = 0.$$

Since the initial condition for the above heat equation is $w^{n+1}(x,t,z_0,t = 0) = 0$, we finally get $w^{n+1}(x,t,z_0,t) = 0$ for all t. □

Now that we have verified the rigid lid conditions at any time n, we are ready to apply Lemma 9, with details provided in Appendix A. We will first show that the right-hand side terms involved in system (11) are bounded with the norm in $L^2(Q_{\mathcal{T}_0})$, where $Q_{\mathcal{T}_0} := (0,\mathcal{T}_0)\times\mathcal{M}$ for $0 < \mathcal{T}_0 \le 1$. For the sake of simplicity, we define the spaces $Z_{\mathcal{T}_0} := C([0,\mathcal{T}_0]; H^2(\mathcal{M}))$.

In the following two results, we will denote by S_j^n the j-th component of the vector associated to the solution $\boldsymbol{S}^n = (u^n, v^n, q_t^n, \theta_e^n)$ defined previously.

Bound for $(\mathbf{u}_h^n\cdot\nabla_h)S_j^n$*:* From Hölder's inequality, it follows that

$$\begin{aligned}
\|(\mathbf{u}_h^n\cdot\nabla_h)S_j^n\|_{L^2(Q_{\mathcal{T}_0})} &= \left(\int_0^{\mathcal{T}_0}\|(\mathbf{u}_h^n\cdot\nabla_h)S_j^n\|^2_{L^2(\mathcal{M})}\right)^{1/2} \\
&\le \left(\int_0^{\mathcal{T}_0}\left(\|\mathbf{u}_h^n\|_{\boldsymbol{L}^4(\mathcal{M})}\|\nabla_h S_j^n\|_{\boldsymbol{L}^4(\mathcal{M})}\right)^2\right)^{1/2}
\end{aligned}$$

$$\leq \left(\int_0^{\mathcal{T}_0} \left(\|\mathbf{u}_h^n\|_{\boldsymbol{L}^4(\mathcal{M})} \right)^8 \right)^{1/8} \left(\int_0^{\mathcal{T}_0} \left(\|\nabla_h S_j^n\|_{\boldsymbol{L}^4(\mathcal{M})} \right)^{8/3} \right)^{3/8}$$

$$\leq \left(\int_0^{\mathcal{T}_0} \left(\|\mathbf{u}_h^n\|_{\boldsymbol{L}^4(\mathcal{M})} \right)^8 \right)^{1/8} \left(\int_0^{\mathcal{T}_0} \|\nabla_h S_j^n\|_{\boldsymbol{L}^2(\mathcal{M})}^{2/3} \|\nabla_h S_j^n\|_{\boldsymbol{L}^6(\mathcal{M})}^2 \right)^{3/8}.$$

Now, we use the continuous injection $\boldsymbol{i}_4$ of $\boldsymbol{H}^1(\Omega)$ into $\boldsymbol{L}^4(\Omega)$ (see [26, Theorem 1.3.4] to obtain

$$\begin{aligned}
&\|(\mathbf{u}_h^n \cdot \nabla_h) S_j^n\|_{L^2(Q_{\mathcal{T}_0})} \\
&\quad \leq \|\mathbf{u}_h^n\|_{L^8(0,\mathcal{T}_0,\boldsymbol{L}^4(\mathcal{M}))} \|\nabla_h S_j^n\|_{L^\infty(0,\mathcal{T}_0,\boldsymbol{L}^2(\mathcal{M}))}^{\frac{1}{4}} \|\nabla_h S_j^n\|_{L^2(0,\mathcal{T}_0,\boldsymbol{L}^6(\mathcal{M}))}^{\frac{3}{4}} \\
&\quad \leq |\mathcal{T}_0|^{\frac{1}{8}} \|\mathbf{u}_h^n\|_{L^\infty(0,\mathcal{T}_0,\boldsymbol{L}^4(\mathcal{M}))} \|\nabla_h S_j^n\|_{L^\infty(0,\mathcal{T}_0,\boldsymbol{L}^2(\mathcal{M}))}^{\frac{1}{4}} \|\nabla_h S_j^n\|_{L^2(0,\mathcal{T}_0,\boldsymbol{L}^6(\mathcal{M}))}^{\frac{1}{4}} \\
&\quad \leq C_{\text{sob}} |\mathcal{T}_0|^{\frac{1}{8}} \|\mathbf{u}_h^n\|_{L^\infty(0,\mathcal{T}_0,\boldsymbol{H}^1(\mathcal{M}))} \|S_j^n\|_{L^\infty(0,\mathcal{T}_0,\boldsymbol{H}^1(\mathcal{M}))}^{\frac{1}{4}} \|S_j^n\|_{L^2(0,\mathcal{T}_0,\boldsymbol{H}^2(\mathcal{M}))}^{\frac{3}{4}} \\
&\quad \leq C_{\text{sob}} |\mathcal{T}_0|^{\frac{5}{8}} \|\boldsymbol{S}^n\|_{L^\infty(0,\mathcal{T}_0,\boldsymbol{H}^1(\mathcal{M}))} \|\boldsymbol{S}^n\|_{Z_{\mathcal{T}_0}},
\end{aligned}$$

where C_{sob} is the Sobolev constant associated to $\boldsymbol{i}_4$. Thus,

$$\|(\mathbf{u}_h^n \cdot \nabla_h) S_j^n\|_{L^2(Q_{\mathcal{T}_0})} \leq C_{\text{sob}} |\mathcal{T}_0|^{\frac{5}{8}} \|\boldsymbol{S}^n\|_{Z_{\mathcal{T}_0}}^2. \tag{18}$$

Bound for $w^n \cdot \partial_z S_j^n$: Under the same arguments as in the previous estimation, together with the Lemma 13 (see Appendix A), we arrive at

$$\begin{aligned}
&\|w^n \cdot \partial_z S_j^n\|_{L^2(Q_{\mathcal{T}_0})} \\
&\leq \|w^n\|_{L^8(0,\mathcal{T}_0,L^4(\mathcal{M}))} \|\partial_z S_j^n\|_{L^\infty(0,\mathcal{T}_0,L^2(\mathcal{M}))}^{\frac{1}{4}} \|\partial_z S_j^n\|_{L^2(0,\mathcal{T}_0,L^6(\mathcal{M}))}^{\frac{3}{4}} \\
&\leq |\mathcal{T}_0|^{\frac{1}{8}} \|w^n\|_{L^\infty(0,\mathcal{T}_0,L^4(\mathcal{M}))} \|\partial_z S_j^n\|_{L^\infty(0,\mathcal{T}_0,L^2(\mathcal{M}))}^{\frac{1}{4}} \|\partial_z S_j^n\|_{L^2(0,\mathcal{T}_0,L^6(\mathcal{M}))}^{\frac{1}{4}} \\
&\leq C_{\text{sob}} |z_1 - z_0| |\mathcal{T}_0|^{\frac{1}{8}} \|\nabla_h \cdot \mathbf{u}_h^n\|_{L^\infty(0,\mathcal{T}_0,H^1(\mathcal{M}))} \|S_j^n\|_{L^\infty(0,\mathcal{T}_0,H^1(\mathcal{M}))}^{\frac{1}{4}} \|S_j^n\|_{L^2(0,\mathcal{T}_0,H^2(\mathcal{M}))}^{\frac{3}{4}} \\
&\leq C_{\text{sob}} |z_1 - z_0| |\mathcal{T}_0|^{\frac{5}{8}} \|\mathbf{u}_h^n\|_{L^\infty(0,\mathcal{T}_0,\boldsymbol{H}^2(\mathcal{M}))} \|\boldsymbol{S}^n\|_{Z_{\mathcal{T}_0}} \\
&\leq C_{\text{sob}} |z_1 - z_0| |\mathcal{T}_0|^{\frac{5}{8}} \|\boldsymbol{S}^n\|_{L^\infty(0,\mathcal{T}_0,\boldsymbol{H}^2(\mathcal{M}))} \|\boldsymbol{S}^n\|_{Z_{\mathcal{T}_0}},
\end{aligned}$$

which concludes that

$$\|w^n \cdot \partial_z S_j^n\|_{L^2(Q_{\mathcal{T}_0})} \leq |z_1 - z_0| C_{\text{sob}} |\mathcal{T}_0|^{\frac{5}{8}} \|\boldsymbol{S}^n\|_{Z_{\mathcal{T}_0}}^2. \tag{19}$$

Remark 1. In [5], the velocity field is assumed to be given and they are assumed to be in the same space. In this work, the velocity field is evolving according to the equations of motion. The continuity constraint with appropriate boundary conditions gives us the vertical velocity in terms of the horizontal components. However, the vertical and horizontal components do not necessarily belong to the same space. This is reflected in the application of continuous inclusion $\boldsymbol{i}_p$ of $\boldsymbol{H}^1(\cdot) \to \boldsymbol{L}^p(\cdot)$, with $p \in \{4, 6\}$ that we need to make for the terms $(\mathbf{u}_h^n \cdot \nabla_h) S_j^n$ and $w^n \cdot \partial_z S_j^n$.

Bound for $f_0(\hat{z} \times \mathbf{u})_h$: It is clear that

$$\|f_0(\hat{z} \times \mathbf{u})_h\|_{\boldsymbol{L}^2(Q_{\mathcal{T}_0})} = \left(\int_0^{\mathcal{T}_0} \|f_0(-v, u)\|_{\boldsymbol{L}^2(\mathcal{M})}^2 \right)^{1/2} \leq |f_0| \sqrt{\mathcal{T}_0} \|\boldsymbol{S}^n\|_{Z_{\mathcal{T}_0}}.$$

As noted in Fig. 1, the transition function χ_ϵ is introduced in order to smooth out the dependence of rainwater $q_r = \chi_\epsilon(q_t - q_{vs}(z))$ on total water q_t and water vapor at saturation $q_{vs}(z)$. It is defined in a way that rainwater is absent ($q_r = 0$) in unsaturated regions ($q_t < q_{vs}(z)$) while it consists of the excess of moisture from the water vapor at saturation ($q_r = q_t - q_{vs}(z)$) in fully saturated regions ($q_t > q_{vs}(z) + \epsilon$). It is required that the transition function is smooth with bounded derivative. The use of these conditions and the treatment of the function χ_ε is reflected in the estimates of terms $V_T\partial_z\chi_\varepsilon(q_t^n - q_{vs}(z))$ and p^n that are presented below.

Bound for $V_T\partial_z\chi_\varepsilon(q_t^n - q_{vs}(z))$*:* Differentiating the function χ_ε together with the assumption (5), and applying Cauchy–Schwarz inequality, it is not difficult to see that

$$\begin{aligned}
&\|V_T\partial_z\chi_\varepsilon(q_t^n - q_{vs}(z))\|^2_{L^2(\mathcal{M})} \\
&\quad = \int\limits_{\mathcal{M}} |V_T\partial_z\chi_\varepsilon(q_t^n - q_{vs}(z))|^2 d\mathcal{M} = \int\limits_{\mathcal{M}} |V_T\chi'_\varepsilon(q_t^n - q_{vs}(z))\partial_z(q_t^n - q_{vs}(z))|^2 d\mathcal{M} \\
&\quad \le 2|V_T|^2 c_\varepsilon^2 \int\limits_{\mathcal{M}} \left(|\partial_z q_t^n|^2 + |\partial_z q_{vs}(z)|^2\right) d\mathcal{M} \\
&\quad \le 2|V_T|^2 c_\varepsilon^2 \left(\|q_t^n\|^2_{H^1(\mathcal{M})} + \|\partial_z q_{vs}(z)\|^2_{L^2(\mathcal{M})}\right).
\end{aligned}$$

Integrating over $(0, \mathcal{T}_0)$, we deduce that

$$\begin{aligned}
\|V_T\partial_z\chi_\varepsilon(q_t^n - q_{vs}(z))\|_{L^2(Q_{\mathcal{T}_0})} &= \left(\int\limits_0^{\mathcal{T}_0} \|V_T\partial_z\chi_\varepsilon(q_t^n - q_{vs}(z))\|^2_{L^2(\mathcal{M})}\right)^{1/2} \\
&\le \sqrt{2}|V_T|c_\varepsilon \left(\|q_t^n\|_{L^2(0,\mathcal{T}_0,H^1(\mathcal{M}))} + \|q_{vs}(z)\|_{L^2(0,\mathcal{T}_0,L^2(\mathcal{M}))}\right) \\
&\le \sqrt{2}|V_T|c_\varepsilon \left(\sqrt{\mathcal{T}_0}\|q_t^n\|_{Z_{\mathcal{T}_0}} + \|q_{vs}(z)\|_{L^2(0,\mathcal{T}_0,L^2(\mathcal{M}))}\right).
\end{aligned}$$

So,

$$\|V_T\partial_z\chi_\varepsilon(q_t^n - q_{vs})\|_{L^2(Q_{\mathcal{T}_0})} \le \sqrt{2}|V_T|c_\varepsilon \left(\sqrt{\mathcal{T}_0}\|\boldsymbol{S}^n\|_{Z_{\mathcal{T}_0}} + \|q_{vs}(z)\|_{L^2(0,\mathcal{T}_0,L^2(\mathcal{M}))}\right). \tag{20}$$

Rain water is one of the solution variables in [5]. As a result, its estimate is equivalent to the direct estimate of q_t. Since rain here is a function of the total water and a threshold, the estimate involves the L^2 norm of that threshold. A similar situation occurs when estimating the pressure term. As we will see below in equation (24), an appropriate factorization combined with the Gronwall inequality directs us to the desired estimation.

Bound for $\nabla_h\mathrm{p}^n$*:* Thanks to the definition of the buoyancy force given in (3), it follows that

$$\begin{aligned}
\|\nabla_h\mathrm{p}^n\|^2_{\boldsymbol{L}^2(\mathcal{M})} &= \int\limits_{\mathcal{M}} |\nabla_h\mathrm{p}^n|^2 d\mathcal{M} \\
&\le \int\limits_{\mathcal{M}}\int\limits_{z_0}^{z_1} |\nabla_h b(q_t^n, \theta_e^n; z)|^2 dz d\mathcal{M} + \int\limits_{\mathcal{M}} |\nabla_h\bar{\mathrm{p}}^n|^2 d\mathcal{M} \\
&= \int\limits_{\mathcal{M}}\int\limits_{z_0}^{z_1} g\left|\left(\frac{\nabla_h\theta_e^n}{\theta_0} - \left(\frac{L}{c_p\theta_0} - \epsilon_0\right)\nabla_h q_t^n\right.\right.
\end{aligned}$$

$$+\left(\frac{L}{c_p\theta_0}-\epsilon_0-1\right)\nabla_h\chi_\varepsilon(q_t^n-q_{vs}(z))\Bigg)\Bigg|^2 dzd\mathcal{M}+\|\nabla_h\bar{\mathrm{p}}^n\|^2_{L^2(\mathcal{M})}.$$

To bound the horizontal gradient of $\bar{\mathrm{p}}(x,y,t)$, we apply Lemma 12 (see Appendix A), with which we obtain

$$\begin{aligned}\|\nabla_h\bar{\mathrm{p}}^n\|^2_{L^2(\mathcal{M})}&=\int_{\mathcal{M}'}\int_{z_0}^{z_1}|\nabla_h\bar{\mathrm{p}}|^2dzd\mathcal{M}'\\&=\int_{z_0}^{z_1}\int_{\mathcal{M}'}|\nabla_h\bar{\mathrm{p}}|^2d\mathcal{M}'dz\le(z_1-z_0)\bar{C}^2_{\bar{\mathrm{p}}}\|\boldsymbol{S}^n\|^2_{\boldsymbol{H}^2(\mathcal{M})}.\end{aligned}\tag{21}$$

Using the inequality $(a+b+c)^2\le 3(a^2+b^2+c^2)$ and the condition imposed on the function χ_ε given in (5) in combination with (21), we obtain

$$\begin{aligned}\|\nabla_h\mathrm{p}^n\|^2_{\boldsymbol{L}^2(\mathcal{M})}\le\;&3\frac{|g|^2}{|\theta_0|^2}(z_1-z_0)^2\left(\int_{\mathcal{M}}|\nabla_h\theta_e|^2d\mathcal{M}\right)\\&+3|g|^2(z_1-z_0)^2\left|\frac{L}{c_p\theta_0}-\epsilon_0\right|^2\int_{\mathcal{M}}|\nabla_h q_t^n|^2d\mathcal{M}\\&+3|g|^2(z_1-z_0)^2\left|\frac{L}{c_p\theta_0}-\epsilon_0-1\right|^2\int_{\mathcal{M}}|\nabla_h\chi_\varepsilon(q_t^n-q_{vs}(z))|^2d\mathcal{M}\\&+(z_1-z_0)\bar{C}^2_{\bar{\mathrm{p}}}\|\boldsymbol{S}^n\|^2_{\boldsymbol{H}^2(\mathcal{M})}\\\le\;&3|g|^2(z_1-z_0)^2C^2_{\theta_0,\epsilon_0}\left(\|\nabla_h\theta_e^n\|^2_{\boldsymbol{L}^2(\mathcal{M})}+2c_\varepsilon^2\|\nabla_h q_t^n\|^2_{\boldsymbol{L}^2(\mathcal{M})}\right)\\&+(z_1-z_0)\bar{C}^2_{\bar{\mathrm{p}}}\|\boldsymbol{S}^n\|^2_{\boldsymbol{H}^2(\mathcal{M})}\\\le\;&(6|g|^2(z_1-z_0)^2\max\{1,c_\varepsilon^2\}C^2_{\theta_0,\epsilon_0}+(z_1-z_0)\bar{C}^2_{\bar{\mathrm{p}}})\|\boldsymbol{S}^n\|^2_{\boldsymbol{H}^2(\mathcal{M})},\end{aligned}$$

where

$$C_{\theta_0,\epsilon_0}:=\max\left\{\frac{1}{\theta_0},\left|\frac{L}{c_p\theta_0}-\epsilon_0\right|,c_\chi\left|\frac{L}{c_p\theta_0}-\epsilon_0-1\right|\right\}.\tag{22}$$

Finally, calculating the norm in the whole space $L^2(Q_{\mathcal{T}_0})$, we arrive at

$$\begin{aligned}&\|\nabla_h\mathrm{p}^n\|_{\boldsymbol{L}^2(Q_{\mathcal{T}_0})}\\&\le\int_0^{\mathcal{T}_0}\left(\left(6|g|^2(z_1-z_0)^2 2\max\{1,c_\varepsilon^2\}C^2_{\theta_0,\epsilon_0}+(z_1-z_0)\bar{C}^2_{\bar{\mathrm{p}}}\right)\|\boldsymbol{S}^n\|^2_{\boldsymbol{H}^2(\mathcal{M})}\right)^{1/2}\\&=\left(\sqrt{6}|g|(z_1-z_0)\max\{1,c_\varepsilon\}C_{\theta_0,\epsilon_0}+(z_1-z_0)^{1/2}\bar{C}_{\bar{\mathrm{p}}}\right)\|\boldsymbol{S}^n\|_{L^2(0,\mathcal{T}_0,\boldsymbol{H}^2(\mathcal{M}))}\\&=\left(\sqrt{6}|g|(z_1-z_0)\max\{1,c_\varepsilon\}C_{\theta_0,\epsilon_0}+(z_1-z_0)^{1/2}\bar{C}_{\bar{\mathrm{p}}}\right)|\mathcal{T}_0|^{1/2}\|\boldsymbol{S}^n\|_{Z_{\mathcal{T}_0}}.\end{aligned}\tag{23}$$

With the previous bounds, we are ready to apply Lemma 9 (see Appendix A), giving us

$$\begin{aligned}&\|\boldsymbol{S}^{n+1}\|_{Z_{\mathcal{T}_0}}+\|\partial_t\boldsymbol{S}^{n+1}\|_{L^2(Q_{\mathcal{T}_0})}\\&\le\mathcal{C}\left(\sum_{j=1}^{4}\left(\|(\mathbf{u}_h^n\cdot\nabla_h)S_J^n\|_{L^2(Q_{\mathcal{T}_0})}+\|w^n\cdot\partial_z S_j^n\|_{L^2(Q_{\mathcal{T}_0})}\right)+\|\nabla_h\mathrm{p}^n\|_{\boldsymbol{L}^2(Q_{\mathcal{T}_0})}\right.\end{aligned}$$

$$+ \|V_T\partial_z\chi_\varepsilon(q_t^n - q_{vs}(z))\|_{L^2(Q_{\mathcal{T}_0})} + \|f_0(\hat{z}\times\mathbf{u})_h\|_{\boldsymbol{L}^2(Q_{\mathcal{T}_0})} + \|(u_0, v_0, q_{t0}, \theta_{e0}\|_{\boldsymbol{L}^2(\mathcal{M})}\Big)$$
$$\leq C^*(\mathcal{T}_0^{5/8}\|\boldsymbol{S}^n\|_{Z_{\mathcal{T}_0}} + 1).$$

Here C^* is a positive constant that depends only on $C_{\text{sob}}, V_T, c_\varepsilon, |f_0|, C_{\theta_0,\epsilon_0}, \bar{C}_{\bar{\text{p}}}$ and the initial data $u_0, v_0, q_{t0}, \theta_{e0}$. Then, taking $\mathcal{T}_0 := \min\{1, K^{-5/16}\}$ for some $K > 0$, one can easily show by induction that

$$\|\boldsymbol{S}^{n+1}\|_{Z_{\mathcal{T}_0}} + \|\partial_t\boldsymbol{S}^{n+1}\|_{\boldsymbol{L}^2(Q_{\mathcal{T}_0})} \leq K, \qquad n = 0, 1, \cdots. \tag{24}$$

Defining

$$\gamma^n(t) := \sum_{j=1}^{4}\left(\|S_j^{n-1}\|^2_{L^\infty(\mathcal{M})} + \|S_j^n\|^2_{L^\infty(\mathcal{M})}\right),$$

and applying the estimate (24) together with the Gagliardo—Nirenberg inequality (see [22] for more details), that is $\|f\|_{L^\infty} \leq C_{GN}\|f\|_{L^2}^{1/4}\|f\|_{H^2}^{3/4}$, we get

$$\begin{aligned}
\int_0^{\mathcal{T}_0}\gamma^n(t)dt &= \sum_{j=1}^{4}\int_0^{\mathcal{T}_0}\left(\|S_j^{n-1}\|^2_{L^\infty(\mathcal{M})} + \|S_j^n\|^2_{L^\infty(\mathcal{M})}\right)dt \\
&\leq \int_0^{\mathcal{T}_0}\left(\|\boldsymbol{S}^{n-1}\|^2_{\boldsymbol{L}^\infty(\mathcal{M})} + \|\boldsymbol{S}^n\|^2_{\boldsymbol{L}^\infty(\mathcal{M})}\right)dt \\
&\leq C_{GN}\int_0^{\mathcal{T}_0}\left(\|\boldsymbol{S}^{n-1}\|^{1/2}_{\boldsymbol{L}^2(\mathcal{M})}\|\boldsymbol{S}^{n-1}\|^{3/2}_{\boldsymbol{H}^2(\mathcal{M})} + \|\boldsymbol{S}^n\|^{1/2}_{\boldsymbol{L}^2(\mathcal{M})}\|\boldsymbol{S}^n\|^{3/2}_{\boldsymbol{H}^2(\mathcal{M})}\right)dt \\
&\leq C_{GN}\sum_{m=0}^{1}\left(\int_0^{\mathcal{T}_0}\|\boldsymbol{S}^{n-m}\|^2_{\boldsymbol{L}^2(\mathcal{M})}\right)^{1/4}\left(\int_0^{\mathcal{T}_0}\|\boldsymbol{S}^{n-m}\|^2_{\boldsymbol{H}^2(\mathcal{M})}\right)^{3/4} \\
&\leq C_{GN}|\mathcal{T}_0|^{1/4}K^{1/2}\sum_{m=0}^{1}\left(\int_0^{\mathcal{T}_0}\|\boldsymbol{S}^{n-m}\|^2_{\boldsymbol{H}^2(\mathcal{M})}dt\right)^{3/4} \leq 2C_{GN}K^2|\mathcal{T}_0|^{1/4}.
\end{aligned} \tag{25}$$

Next, we show that $\{\boldsymbol{S}^n\}_{n=1}^\infty$ is a Cauchy sequence in the space $C([0,\mathcal{T}_0]; L^2(\mathcal{M}))$. For this, let us define

$$J^n(t) := \|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2_{L^2(\mathcal{M})} + \|q_t^n - q_t^{n+1}\|^2_{L^2(\mathcal{M})} + \|\theta_e^n - \theta_e^{n+1}\|^2_{L^2(\mathcal{M})}.$$

Using the same arguments presented in Section 3.2.3 but substituting the terms $\mathbf{u}_{h1}, q_{t1}, \theta_{e1}$ by $\mathbf{u}_h^{n+1}, q_t^{n+1}$, θ_e^{n+1} and $\mathbf{u}_{h2}, q_{t2}, \theta_{e2}$ by $\mathbf{u}_h^n, q_t^n, \theta_e^n$ in the estimation (77), we arrive at

$$\frac{1}{2}\frac{d}{dt}J^{n+1}(t) + \frac{1}{2}(\min\{\mu_1,\mu_2\})\|\nabla J^{n+1}(t)\|^2_{L^2(\mathcal{M})} \leq C_{\text{tot}}(t)\|J^{n+1}(t)\|^2_{L^2(\mathcal{M})}.$$

Here, $C_{\text{tot}}(t)$ is a positive constant depending on $V_T, L_1, c_\varepsilon, c_\chi, \mu_1, \mu_2, \partial_z q_v s$ and the initial data q_{t0} (see Section 3.2.3 for more details). Thanks to the definitions of γ_n and J_n, it is not difficult to see that $J_{n+1}(t) \leq \gamma_n(t)J_n(t)$. Thus

$$\frac{d}{dt}J^{n+1}(t) \leq 2C_{\text{tot}}(t)\gamma_n(t)J_n(t),$$

$$J^{n+1}(t) \leq 2\int_0^t C_{\text{tot}}(s)\gamma_n(s)J_n(s)ds, \qquad \forall\, t \in [0,\mathcal{T}_0], \text{ and}$$

$$\sup_{0<t<\mathcal{T}_0} J^{n+1}(t) \leq 2\|C_{\text{tot}}(t)\|_{L^\infty([0,\mathcal{T}_0])} \left(\int_0^{\mathcal{T}_0} \gamma_n(t)dt\right) \sup_{0<t<\mathcal{T}_0} J_n(t).$$

Recalling (25), and choosing $\mathcal{T}_0 < \frac{1}{(8C_{GN}K^2\|C_{\text{tot}}(t)\|_{L^\infty([0,\mathcal{T}_0])})^4}$, we get

$$\sup_{0<t<\mathcal{T}_0} J^{n+1}(t) \leq \frac{1}{2} \sup_{0<t<\mathcal{T}_0} J^n(t) \Longleftrightarrow \sup_{0<t<\mathcal{T}_0} J^{n+1}(t) \leq \frac{1}{2^n} \xrightarrow{n\to\infty} 0,$$

which implies that $\{\boldsymbol{S}^n\}_{n=1}^\infty$ is a Cauchy sequence in the space $C([0,\mathcal{T}_0];\boldsymbol{L}^2(\mathcal{M}))$. Finally, by Aubin—Lions Lemma (see [27] in Appendix A), there is a vector $\boldsymbol{S} \in \mathcal{X}_{\mathcal{T}_0}$, with $\partial_t \boldsymbol{S} \in L^2(Q_{\mathcal{T}_0})$ such that

$$\boldsymbol{S}^n \to \boldsymbol{S} \ \text{ in } C([0,\mathcal{T}_0];L^2(\mathcal{M})) \cap L^2(0,\mathcal{T}_0,H^1(\mathcal{M})) \quad \text{and} \quad \partial_t \boldsymbol{S}^n \to \partial_t \boldsymbol{S} \ \text{ in } L^2(Q_{\mathcal{T}_0}). \quad \square$$

3.2. Global well-posedness

In this section, we present the global existence and uniqueness of solutions to the HFARE model (8). We also guarantee the positivity of the equivalent potential temperature θ_e and the total water mixing ratio q_t. The main result of this section is presented in the following theorem. The proof has been divided into three parts for better understanding: existence, uniqueness, and positivity.

Theorem 1. *Assume that $u_0, v_0, q_{t0}, \theta_{e0} \in H^2(\mathcal{M}) \cap \mathbf{L}^\infty(\mathcal{M}), w_0 \in H^1(\mathcal{M}) \cap L^\infty(\mathcal{M})$ with $\theta_{e0}, q_{t0} \geq 0$ in $\mathcal{M}$. Then, system (8), subject to initial (9) and boundary conditions (10), has a unique global in time solution $(\mathbf{u}, \theta_e, q_t)$ satisfying*

$$\begin{aligned} \theta_e, q_t \geq 0, \qquad \theta_e, q_t \in L^\infty(0,\mathcal{T}, L^\infty(\mathcal{M})), \\ \mathbf{u}_h, q_t, \theta_e \in L^\infty(0,\mathcal{T}, H^2(\mathcal{M})), w \in C([0,\mathcal{T}]; H^1(\mathcal{M})) \ \textit{and} \\ \partial \mathbf{u}_h, \partial_t q_t, \partial_t \theta_e \in L^2(0,\mathcal{T}, L^2(\mathcal{M})). \end{aligned}$$

3.2.1. Global existence

We will begin this section by finding the spaces where each of our unknowns $\mathbf{u}_h, q_t, \theta_e$ will be estimated. In the following result we establish that q_t belongs to the space $C([0,\mathcal{T}];L^2(\mathcal{M})) \cap L^2(0,\mathcal{T},H^1(\mathcal{M}))$.

Lemma 3. *Assume that the initial data satisfies $q_{t0} \in H^1(\mathcal{M})$. Suppose also that the boundary conditions (10) and (8e) hold. Then there exists a function $C_1(\mathcal{T})$, which depends on the initial and boundary data, continuous for all $t \geq 0$, such that the estimate*

$$\max_{0\leq t\leq \mathcal{T}} \|q_t\|_{L^2(\mathcal{M})} + \int_0^{\mathcal{T}} \|\nabla q_t\|^2_{L^2(\mathcal{M})} dt \leq \mathcal{C}_1(\mathcal{T}),$$

holds for any $\mathcal{T} \in (0,\mathcal{T}^*)$.

Proof. We have used arguments from Proposition 5 in [14]. Here, one requires to treat the nonlinear term associated to precipitation involving the function χ_ε.

We will begin multiplying equation (8c) by q_t, and integrating over $\mathcal{M}$. We then get

$$\begin{aligned}&\int_{\mathcal{M}} (\partial_t q_t) q_t \, d\mathcal{M} + \int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h q_t + w \partial_z q_t) q_t \, d\mathcal{M} \\ &\quad = \int_{\mathcal{M}} (V_T \partial_z \chi_\varepsilon (q_t - q_{vs}(z))) q_t \, d\mathcal{M} + \int_{\mathcal{M}} (\kappa_1 \Delta_h q_t + \kappa_2 \partial_z^2 q_t) q_t \, d\mathcal{M}.\end{aligned} \tag{26}$$

It follows from Lemma 14 and Lemma 15 (see Appendix A) that

$$\begin{aligned}&\int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h q_t + w \partial_z q_t) q_t \, d\mathcal{M} = 0 \text{ and} \\ &\int_{\mathcal{M}} (\kappa_1 \Delta_h q_t + \kappa_2 \partial_z^2 q_t) q_t \, d\mathcal{M} \leq -\kappa_1 \|\nabla_h q_t\|^2_{L^2(\mathcal{M})} - \kappa_2 \|\partial_z q_t\|^2_{L^2(\mathcal{M})}.\end{aligned} \tag{27}$$

Based on the definition of χ_ε given by equation (4), the chain rule to derive $\partial_z \chi_\varepsilon(\cdot)$, the condition (5) and applying Young's inequality (with constant δ), we can deduce that

$$\begin{aligned}&\int_{\mathcal{M}} (V_T \partial_z \chi_\varepsilon (q_t - q_{vs}(z))) q_t \, d\mathcal{M} = \int_{\mathcal{M}} (V_T \chi'_\varepsilon (q_t - q_{vs}) \partial_z (q_t - q_{vs}(z))) \, q_t \, d\mathcal{M} \\ &\quad = \int_{\mathcal{M}} (V_T \chi'_\varepsilon (q_t - q_{vs}(z)) \partial_z q_t) \, q_t \, d\mathcal{M} - \int_{\mathcal{M}} (V_T \chi'_\varepsilon (q_t - q_{vs}(z)) \partial_z q_{vs}(z)) \, q_t \, d\mathcal{M} \\ &\quad \leq V_T c_\varepsilon \|q_t\|_{L^2(\mathcal{M})} \|\partial_z q_t\|_{L^2(\mathcal{M})} + V_T c_\varepsilon \|q_t\|_{L^2(\mathcal{M})} \|\partial_z q_{vs}(z)\|_{L^2(\mathcal{M})} \\ &\quad \leq \frac{1}{\delta} V_T^2 c_\varepsilon^2 \|q_t\|^2_{L^2(\mathcal{M})} + \frac{\delta}{2} \|\partial_z q_t\|^2_{L^2(\mathcal{M})} + \frac{\delta}{2} \|\partial_z q_{vs}(z)\|^2_{L^2(\mathcal{M})}.\end{aligned}$$

Due to the fact that δ is a free positive parameter, we can choose it as $\delta = \kappa_2 > 0$, from which one obtains

$$\int_{\mathcal{M}} (V_T \partial_z \chi_\varepsilon (q_t - q_{vs}(z))) q_t \, d\mathcal{M} \leq \frac{1}{\kappa_2} V_T^2 c_\varepsilon^2 \|q_t\|^2_{L^2(\mathcal{M})} + \frac{\kappa_2}{2} \|\partial_z q_t\|^2_{L^2(\mathcal{M})} + \frac{\kappa_2}{2} \|\partial_z q_{vs}(z)\|^2_{L^2(\mathcal{M})}. \tag{28}$$

This choice of δ allows the coefficient of the term $\|\partial_z q_t\|^2_{L^2(\mathcal{M})}$ to be less than μ_2, which is necessary for the required estimate. So, replacing (27) and (28) into (26), we get

$$\frac{1}{2} \frac{d}{dt} \|q_t\|^2_{L^2(\mathcal{M})} + \kappa_1 \|\nabla_h q_t\|^2_{L^2(\mathcal{M})} + \frac{\kappa_2}{2} \|\partial_z q_t\|^2_{L^2(\mathcal{M})} \leq c_1 + c_2 \|q_t\|^2_{L^2(\mathcal{M})}, \tag{29}$$

where

$$c_1 := \frac{\kappa_2}{2} \|\partial_z q_{vs}(z)\|^2_{L^2(\mathcal{M})} \qquad \text{and} \qquad c_2 := \frac{1}{\kappa_2} V_T^2 c_\varepsilon^2.$$

From (29), it is deduced that

$$\|q_t\|^2_{L^2(\mathcal{M})} \leq \|q_{t0}\|^2_{L^2(\mathcal{M})} + 2c_1 t + \int_0^t 2c_2 \|q_t(s,\cdot)\|^2_{L^2(\mathcal{M})} ds.$$

It follows from the above, by virtue the Gronwall inequality (see [2, Lemma 1.1, eq. 1.76]), that

$$\|q_t\|^2_{L^2(\mathcal{M})} \leq c_1(t), \tag{30}$$

where

$$c_1(t) := \|q_{t0}\|^2_{L^2(\mathcal{M})} + \kappa_2 t \|\partial_z q_{vs}(z)\|^2_{L^2(\mathcal{M})}$$
$$+ \int_0^t \frac{2V_T^2 c_\varepsilon^2}{\kappa_2} \left(\|q_{t0}\|^2_{L^2(\mathcal{M})} + \kappa_2\, s \|\partial_z q_{vs}(z)\|^2_{L^2(\mathcal{M})} \right) e^{\frac{2V_T^2 c_\varepsilon^2}{\kappa_2}(t-s)} ds.$$

Finally, combining (29) and (30) results in the inequality

$$\int_0^{\mathcal{T}} \|\nabla q_t\|^2_{L^2(\mathcal{M})}\, dt \le \frac{1}{\min\{\kappa_1, \frac{\kappa_2}{2}\}} \int_0^{\mathcal{T}} (\kappa_1 \|\nabla_h q_t\|^2_{L^2(\mathcal{M})} + \frac{\kappa_2}{2} \|\partial_z q_t\|^2_{L^2(\mathcal{M})})\, dt$$
$$\le \frac{1}{\min\{\kappa_1, \frac{\kappa_2}{2}\}} \int_0^{\mathcal{T}} (c_1 + c_2 c_1(t))\, dt,$$

and from (30), we have

$$\max_{0 \le t \le \mathcal{T}} \|q_t(t)\|_{L^2(\mathcal{M})} \le \max_{0 \le t \le \mathcal{T}} (c_1(t))^{1/2}.$$

The proof is concluded by choosing $\mathcal{C}_1(\mathcal{T}) > 0$ as;

$$\mathcal{C}_1(\mathcal{T}) = \frac{1}{\min\{\kappa_1, \frac{\kappa_2}{2}\}} \int_0^{\mathcal{T}} (c_1 + c_2 c_1(t)) dt + \max_{0 \le t \le \mathcal{T}} (c_1(t))^{1/2}. \quad \square$$

In the following result we will show that $\theta_e \in C([0, \mathcal{T}]; L^2(\mathcal{M})) \cap L^2(0, \mathcal{T}, H^1(\mathcal{M}))$.

Lemma 4. *Assume that initial temperature satisfies $\theta_{e0} \in H^1(\mathcal{M})$. Suppose also that (8) and the boundary conditions (10) are satisfied. Then, for all $t \ge 0$, the following estimate*

$$\max_{0 \le t \le \mathcal{T}} \|\theta_e\|_{L^2(\mathcal{M})} + \int_0^{\mathcal{T}} \|\nabla \theta_e\|^2_{L^2(\mathcal{M})} dt \le \|\theta_{e0}\|_{L^2(\mathcal{M})},$$

holds for any $\mathcal{T} \in (0, \mathcal{T}^*)$.

Proof. Under the same arguments made for the proof of Lemma 3, one can arrive at

$$\frac{1}{2}\frac{d}{dt} \|\theta_e\|^2_{L^2(\mathcal{M})} + \mu_1 \|\nabla_h \theta_e\|^2_{L^2(\mathcal{M})} + \mu_2 \|\partial_z \theta_e\|^2_{L^2(\mathcal{M})} \le 0. \tag{31}$$

From (31), it is clear that $\|\theta_e\|^2_{L^2(\mathcal{M})}$ is a decreasing function for all t, and therefore

$$\|\theta_e\|^2_{L^2(\mathcal{M})} \le \|\theta_{e0}\|^2_{L^2(\mathcal{M})}. \tag{32}$$

Finally, combining (31) and (32), we get

$$\int_0^{\mathcal{T}} \|\nabla \theta_e\|^2_{L^2(\mathcal{M})} dt \le \frac{1}{\min\{\mu_1, \mu_2\}} \int_0^{\mathcal{T}} (\mu_1 \|\nabla_h \theta_e\|^2_{L^2(\mathcal{M})} + \mu_2 \|\partial_z \theta_e\|^2_{L^2(\mathcal{M})} dt \le 0,$$

and from (32), we have

$$\max_{0\leq t\leq \mathcal{T}} \|\theta_e(t)\|_{L^2(\mathcal{M})} \leq \max_{0\leq t\leq \mathcal{T}} \|\theta_{e0}\|_{L^2(\mathcal{M})} = \|\theta_{e0}\|_{L^2(\mathcal{M})},$$

which concludes the proof. □

Lemma 5. *Assume that the initial data satisfies* $\mathbf{u}_{h0} \in \boldsymbol{H}^1(\mathcal{M})$. *Suppose also that the boundary conditions* (10) *and* (8) *hold. Then there exists a function* $C_2(\mathcal{T})$, *which depends continuously on the initial and boundary data, such that the estimate*

$$\max_{0\leq t\leq \mathcal{T}} \|\mathbf{u}_h\|_{L^2(\mathcal{M})} + \int_0^{\mathcal{T}} \|\nabla_h \mathbf{u}_h\|^2_{L^2(\mathcal{M})} dt \leq \mathcal{C}_2(\mathcal{T})$$

holds for any $\mathcal{T} \in (0, \mathcal{T}^*)$.

Proof. We will start by multiplying equation (8a) by $\mathbf{u}_h$, and integrating over $\mathcal{M}$, getting

$$\int_{\mathcal{M}} (\partial_t \mathbf{u}_h) \cdot \mathbf{u}_h d\mathcal{M} + \int_{\mathcal{M}} ((\mathbf{u}_h \cdot \nabla_h)\mathbf{u}_h + w\partial_z \mathbf{u}_h) \cdot \mathbf{u}_h d\mathcal{M} + \int_{\mathcal{M}} f_0(\hat{z} \times \mathbf{u})_h \cdot \mathbf{u}_h d\mathcal{M}$$
$$= -\int_{\mathcal{M}} \nabla_h \mathrm{p} \cdot \mathbf{u}_h d\mathcal{M} + \int_{\mathcal{M}} (\nu_1 \Delta_h \mathbf{u}_h + \nu_2 \partial_z^2 \mathbf{u}_h) \cdot \mathbf{u}_h d\mathcal{M}.$$

Due to the fact that $f_0(\hat{z} \times \mathbf{u})_h \cdot \mathbf{u}_h = f_0(-v, u) \cdot (u, v) = 0$ and Lemma 14 (see Appendix A), the second and third integrals on the left hand side vanish, and therefore the above expression reduces to

$$\int_{\mathcal{M}} (\partial_t \mathbf{u}_h) \cdot \mathbf{u}_h d\mathcal{M} = -\int_{\mathcal{M}} \nabla_h \mathrm{p} \cdot \mathbf{u}_h d\mathcal{M} + \int_{\mathcal{M}} (\nu_1 \Delta_h \mathbf{u}_h + \nu_2 \partial_z^2 \mathbf{u}_h) \cdot \mathbf{u}_h d\mathcal{M}. \tag{33}$$

As a direct consequence of Lemma 15 (see Appendix A), we have that

$$\int_{\mathcal{M}} (\nu_1 \Delta_h \mathbf{u}_h + \nu_2 \partial_z^2 \mathbf{u}_h) \cdot \mathbf{u}_h d\mathcal{M} \leq -\nu_1 \|\nabla_h \mathbf{u}_h\|^2_{L^2(\mathcal{M})} - \nu_2 \|\partial_z^2 \mathbf{u}_h\|^2_{L^2(\mathcal{M})}. \tag{34}$$

It follows by integrating by parts, and using the boundary conditions $\partial_{\boldsymbol{n}} \mathbf{u} = 0$ and (8e), that

$$-\int_{\mathcal{M}} \nabla_h \mathrm{p} \cdot \mathbf{u}_h d\mathcal{M} = \int_{\mathcal{M}} \mathrm{p}(\nabla_h \cdot \mathbf{u}_h) d\mathcal{M} - \int_{\Gamma_s} \mathrm{p}\, \partial_{\boldsymbol{n}} \mathbf{u}_h d\Gamma_s = \int_{\mathcal{M}} \mathrm{p}(\nabla_h \cdot \mathbf{u}_h) d\mathcal{M}$$
$$= -\int_{\mathcal{M}} \mathrm{p}\, \partial_z w d\mathcal{M} = \int_{\mathcal{M}} w \partial_z \mathrm{p} d\mathcal{M} - \int_{\mathcal{M}'} \mathrm{p} w \Big|_{z_0}^{z_1} d\mathcal{M}'.$$

The vertical velocity vanishes at the top and bottom boundaries: $w = 0$ at $z = z_0$ and $z = z_1$. Together with equation (8b), we get

$$-\int_{\mathcal{M}} \nabla_h \mathrm{p} \cdot \mathbf{u}_h d\mathcal{M} = \int_{\mathcal{M}} w \partial_z \mathrm{p} d\mathcal{M} = \int_{\mathcal{M}} w b(q_t, \theta_e; z) d\mathcal{M}.$$

Now, using the definition of buoyancy b (see (3)) and applying Young's inequality together with the fact that $|\chi_\varepsilon(\zeta)| \leq c_\chi |\zeta|$ (see (5)), we have that

$$
\begin{aligned}
&\int_{\mathcal{M}} wb(q_t,\theta_e;z)d\mathcal{M} \\
&\quad = \int_{\mathcal{M}} \frac{g}{\theta_0}\theta_e w d\mathcal{M} - \int_{\mathcal{M}} g\left(\frac{L}{c_p\theta_0} - \epsilon_0\right) q_t w d\mathcal{M} \\
&\qquad + \int_{\mathcal{M}} g\left(\frac{L}{c_p\theta_0} - \epsilon_0 - 1\right)\chi_\varepsilon(q_t - q_{vs}(z))w d\mathcal{M} \\
&\quad \le \frac{g}{\theta_0}\|\theta_e\|_{L^2(\mathcal{M})}\|w\|_{L^2(\mathcal{M})} + g\left|\frac{L}{c_p\theta_0} - \epsilon_0\right| \|q_t\|_{L^2(\mathcal{M})}\|w\|_{L^2(\mathcal{M})} \\
&\qquad + gc_\chi\left|\frac{L}{c_p\theta_0} - \epsilon_0 - 1\right| \left(\|q_t\|_{L^2(\mathcal{M})} + \|q_{vs}(z)\|_{L^2(\mathcal{M})}\right)\|w\|_{L^2(\mathcal{M})} \\
&\quad \le gC_{\theta_0,\epsilon_0}\left(\|\theta_e\|_{L^2(\mathcal{M})} + 2\|q_t\|_{L^2(\mathcal{M})} + \|q_{vs}(z)\|_{L^2(\mathcal{M})}\right)\|w\|_{L^2(\mathcal{M})} \\
&\quad \le \frac{1}{2\delta}g^2C^2_{\theta_0,\epsilon_0}\left(\|\theta_e\|_{L^2(\mathcal{M})} + 2\|q_t\|_{L^2(\mathcal{M})} + \|q_{vs}(z)\|_{L^2(\mathcal{M})}\right)^2 + \frac{\delta}{2}\|w\|^2_{L^2(\mathcal{M})},
\end{aligned}
$$

where C_{θ_0,ϵ_0} was defined in (22). Then, thanks to the fact that $(a+2b+c)^2 \le 4a^2 + 8b^2 + 4c^2$ is fulfilled, and using the estimates for $\|q_t\|_{L^2(\mathcal{M})}$ (see (30)) and $\|\theta_e\|_{L^2(\mathcal{M})}$ (see (32)), we obtain

$$
\begin{aligned}
\int_{\mathcal{M}} b(q_t,\theta_e;z)w d\mathcal{M} &\le \frac{1}{\delta}g^2C^2_{\theta_0,\epsilon_0}\left(2\|\theta_e\|^2_{L^2(\mathcal{M})} + 4\|q_t\|^2_{L^2(\mathcal{M})} + 2\|q_{vs}(z)\|^2_{L^2(\mathcal{M})}\right) + \frac{\delta}{2}\|w\|^2_{L^2(\mathcal{M})} \\
&\le \frac{1}{\delta}g^2C^2_{\theta_0,\epsilon_0}\left(2\|\theta_{e0}\|^2_{L^2(\mathcal{M})} + 4c_1(t) + 2\|q_{vs}(z)\|^2_{L^2(\mathcal{M})}\right) + \frac{\delta}{2}\|w\|^2_{L^2(\mathcal{M})}.
\end{aligned}
$$

We note that in the case where q_r is known to be bounded, the transition function can be assumed of compact support. In such case, the energy estimate for the horizontal velocity $\mathbf{u}_h$ in equation (36) is uniformly bounded in time.

Taking $\delta := \frac{\nu_1}{(z_1-z_0)^2}$, and applying Lemma 13 with $r_1 = 2$ (see Appendix A), it is deduced that

$$
-\int_{\mathcal{M}} \nabla_h \mathrm{p}\cdot\mathbf{u}_h d\mathcal{M} = \int_{\mathcal{M}} b(q_t,\theta_e;z)w d\mathcal{M} \le c_2(t) + \frac{\nu_1}{2}\|\nabla_h\mathbf{u}_h\|^2_{L^2(\mathcal{M})}, \tag{35}
$$

where

$$
c_2(t) := \frac{(z_1-z_0)^2}{\mu_1}g^2C^2_{\theta_0,\epsilon_0}\left(2\|\theta_{e0}\|^2_{L^2(\mathcal{M})} + 4c_1(t) + 2\|q_{vs}(z)\|^2_{L^2(\mathcal{M})}\right).
$$

Substituting (35) and (34) into (33), one arrives at

$$
\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_h\|^2_{L^2(\mathcal{M})} + \frac{\nu_1}{2}\|\nabla_h\mathbf{u}_h\|^2_{L^2(\mathcal{M})} + \nu_2\|\partial_z\mathbf{u}_h\|^2_{L^2(\mathcal{M})} \le c_2(t). \tag{36}
$$

By the above arguments, we have that

$$
\|\mathbf{u}_h\|^2_{L^2(\mathcal{M})} \le \|\mathbf{u}_{h0}\|^2_{L^2(\mathcal{M})} + \int_0^t 2c_2(s)ds. \tag{37}
$$

It is easy to see that due to the estimates (36) and (37), we obtain

$$\int_0^{\mathcal{T}} \|\nabla \mathbf{u}_h\|^2_{L^2(\mathcal{M})} dt \leq \frac{1}{\min\{\frac{\nu_1}{2}, \nu_2\}} \int_0^{\mathcal{T}} \left(\frac{\nu_1}{2} \|\nabla_h \mathbf{u}_h\|^2_{L^2(\mathcal{M})} + \nu_2 \|\partial_z \mathbf{u}_h\|^2_{L^2(\mathcal{M})} \right) dt$$
$$\leq \frac{1}{\min\{\frac{\nu_1}{2}, \nu_2\}} \int_0^{\mathcal{T}} c_2(t) dt,$$

and

$$\max_{0 \leq t \leq \mathcal{T}} \|\mathbf{u}_h(t)\|_{L^2(\mathcal{M})} \leq \max_{0 \leq t \leq \mathcal{T}} \left(\|\mathbf{u}_{h0}\|^2_{L^2(\mathcal{M})} + \int_0^t 2c_2(s) ds \right)^{1/2}.$$

The proof is concluded by choosing $\mathcal{C}_2(\mathcal{T}) > 0$ as;

$$\mathcal{C}_2(\mathcal{T}) = \frac{1}{\min\{\frac{\nu_1}{2}, \nu_2\}} \int_0^{\mathcal{T}} c_2(t) dt + \max_{0 \leq t \leq \mathcal{T}} \left(\|\mathbf{u}_{h0}\|^2_{L^2(\mathcal{M})} + \int_0^t 2c_2(s) ds \right)^{1/2}. \quad \square$$

The next result is necessary to guarantee uniqueness in our system (8). The proof is based on [14] but we do not decompose the velocity onto its baroclinic and adiabatic components. Instead, we employ the logarithmic type anisotropic Sobolev embedding inequality used in [4] to control $\|\mathbf{u}\|_{L^\infty(\mathcal{M})}$.

Lemma 6. *Assume that the initial data satisfies $(u_0, v_0, q_{t0}, \theta_{e0}) \in \boldsymbol{H}^2(\mathcal{M})$. Suppose also that the boundary conditions (10) and (8e) hold. Then there exist functions $c_3(t)$ and $c_4(t)$, which depend on the initial and boundary data, and are continuous for all $t \geq 0$, such that the following estimates hold*

$$\|\nabla_h(\partial_z q_t)\|^2_{L^2(\mathcal{M})} + \|\partial_z^2 q_t\|^2_{L^2(\mathcal{M})} \leq c_3(t), \tag{38a}$$
$$\|\nabla_h(\partial_z \theta_e)\|^2_{L^2(\mathcal{M})} + \|\partial_z^2 \theta_e\|^2_{L^2(\mathcal{M})} \leq 0, \tag{38b}$$

and

$$\|\nabla_h(\partial_z \mathbf{u}_h)\|^2_{L^2(\mathcal{M})} + \|\partial_z^2 \mathbf{u}_h\|^2_{L^2(\mathcal{M})} \leq c_4(t). \tag{38c}$$

Proof. We will begin the proof by analyzing the estimation involving the water mixing ratio q_t. For this, we multiply (8c) by $-\partial_z^2 q_t$ and we integrate over $\mathcal{M}$ to obtain

$$\begin{aligned} &- \int_{\mathcal{M}} (\partial_t q_t) \partial_z^2 q_t d\mathcal{M} + \int_{\mathcal{M}} (\kappa_1 \Delta_h q_t + \kappa_2 \partial_z^2 q_t) \partial_z^2 q_t d\mathcal{M} \\ &\quad = \int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h q_t + w \partial_z q_t) \partial_z^2 q_t d\mathcal{M} - \int_{\mathcal{M}} (V_T \partial_z \chi_\varepsilon (q_t - q_{vs}(z))) \partial_z^2 q_t d\mathcal{M}. \end{aligned} \tag{39}$$

It follows from integration by parts and the boundary conditions (10) that

$$\begin{aligned} -\int_{\mathcal{M}} (\partial_t q_t) \partial_z^2 q_t d\mathcal{M} &= \int_{\mathcal{M}} \partial_z (\partial_t q_t) \partial_z q_t d\mathcal{M} - \int_{\mathcal{M}'} (\partial_t q_t \partial_z q_t) \Big|_{z_0}^{z_1} d\mathcal{M}' \\ &= \int_{\mathcal{M}} \partial_t (\partial_z q_t) \partial_z q_t d\mathcal{M} + \alpha_{q1} \int_{\mathcal{M}'} (\partial_t q_t) q_t \Big|_{z=z_1} d\mathcal{M}' + \alpha_{q0} \int_{\mathcal{M}'} (\partial_t q_t) q_t \Big|_{z=z_0} d\mathcal{M}' \end{aligned}$$

$$= \frac{1}{2}\frac{d}{dt}\|\partial_z q_t\|^2_{L^2(\mathcal{M})} + \alpha_{q1}\int\limits_{\mathcal{M}'} \frac{1}{2}\frac{d}{dt}(q_t^2)\bigg|_{z=z_1} d\mathcal{M}' + \alpha_{q0}\int\limits_{\mathcal{M}'} \frac{1}{2}\frac{d}{dt}(q_t^2)\bigg|_{z=z_0} d\mathcal{M}'.$$

Since $\alpha_{qj} > 0,\ j \in \{0,1\}$, the second integral on the right hand side is nonnegative, and thus

$$-\int\limits_{\mathcal{M}} (\partial_t q_t)\partial_z^2 q_t d\mathcal{M} \geq \frac{1}{2}\frac{d}{dt}\|\partial_z q_t\|^2_{L^2(\mathcal{M})}. \tag{40}$$

Now, using integration by parts and the boundary conditions for q_t (see (10)), we get

$$\begin{aligned}
&\int\limits_{\mathcal{M}} (\kappa_1\Delta_h q_t + \kappa_2\partial_z^2 q_t)\partial_z^2 q_t d\mathcal{M} \\
&= -\kappa_1\int\limits_{\mathcal{M}} \partial_z\Delta_h q_t\partial_z q_t d\mathcal{M} + \kappa_1\int\limits_{\mathcal{M}'} \Delta_h q_t\partial_z q_t\bigg|_{z_0}^{z_1} d\mathcal{M}' + \kappa_2\|\partial_z^2 q_t\|^2_{L^2(\mathcal{M})} \\
&= -\kappa_1\int\limits_{\mathcal{M}} \Delta_h(\partial_z q_t)\partial_z q_t d\mathcal{M} - \alpha_{q1}\kappa_1\int\limits_{\mathcal{M}'} (\Delta_h q_t)q_t\bigg|_{z=z_1} d\mathcal{M}' \\
&\quad - \alpha_{q0}\kappa_1\int\limits_{\mathcal{M}'} (\Delta_h q_t)q_t\bigg|_{z=z_0} d\mathcal{M}' + \kappa_2\|\partial_z^2 q_t\|^2_{L^2(\mathcal{M})} \\
&= \kappa_1\int\limits_{\mathcal{M}} \nabla_h(\partial_z q_t)\cdot\nabla_h(\partial_z q_t)d\mathcal{M} - \kappa_1\int\limits_{\Gamma_s} \partial_n(\partial_z q_t)\partial_z q_t\, d\Gamma_s \\
&\quad + \alpha_{q1}\kappa_1\int\limits_{\mathcal{M}'} \nabla_h q_t\cdot\nabla_h q_t\bigg|_{z=z_1} d\mathcal{M}' - \alpha_{q1}\kappa_1\int\limits_{\partial\mathcal{M}'} (\partial_n q_t)q_t\bigg|_{z=z_1} d(\partial\mathcal{M}') \\
&\quad + \alpha_{q0}\mu_1\int\limits_{\mathcal{M}'} \nabla_h q_t\cdot\nabla_h q_t\bigg|_{z=z_0} d\mathcal{M}' \\
&\quad - \alpha_{q0}\kappa_1\int\limits_{\partial\mathcal{M}'} (\partial_n q_t)q_t\bigg|_{z=z_0} d(\partial\mathcal{M}') + \kappa_2\|\partial_z^2 q_t\|^2_{L^2(\mathcal{M})} \\
&= \kappa_1\|\nabla_h(\partial_z q_t)\|^2_{L^2(\mathcal{M})} - \kappa_1\int\limits_{\Gamma_s} \partial_z(\partial_n q_t)\partial_z q_t d\Gamma_s + \alpha_{q1}\kappa_1\|(\nabla_h q_t)\big|_{z=z_1}\|^2_{L^2(\mathcal{M}')} \\
&\quad - \alpha_{q1}\kappa_1\int\limits_{\partial\mathcal{M}'} (\partial_n q_t)q_t\bigg|_{z=z_1} d(\partial\mathcal{M}') + \alpha_{q0}\kappa_1\|(\nabla_h q_t)\big|_{z=z_0}\|^2_{L^2(\mathcal{M}')} \\
&\quad - \alpha_{q0}\kappa_1\int\limits_{\partial\mathcal{M}'} (\partial_n q_t)q_t\bigg|_{z=z_0} d(\partial\mathcal{M}') + \kappa_2\|\partial_z^2 q_t\|^2_{L^2(\mathcal{M})}.
\end{aligned}$$

Note that, the boundary conditions on Γ_s are still fulfilled on $\partial\mathcal{M}'$, then $\partial_n q_t = -\alpha_{qs}q_t$ on $\partial\mathcal{M}'$, and

$$\begin{aligned}
&\int\limits_{\mathcal{M}} (\kappa_1\Delta_h q_t + \kappa_2\partial_z^2 q_t)\partial_z^2 q_t d\mathcal{M} \\
&= \kappa_1\|\nabla_h(\partial_z q_t)\|^2_{L^2(\mathcal{M})} + \alpha_{qs}\kappa_1\|\partial_z q_t\|^2_{L^2(\Gamma_s)} \\
&\quad + \alpha_{q1}\kappa_1\|(\nabla_h q_t)\big|_{z=z_1}\|^2_{L^2(\mathcal{M}')} + \alpha_{q1}\alpha_{qs}\kappa_1\|\ q_t\big|_{z=z_1}\|^2_{L^2(\partial\mathcal{M}')} \\
&\quad + \alpha_{q0}\kappa_1\|(\nabla_h q_t)\big|_{z=z_0}\|^2_{L^2(\mathcal{M}')} + \alpha_{q0}\alpha_{qs}\kappa_1\|\ q_t\big|_{z=z_0}\|^2_{L^2(\partial\mathcal{M}')} + \kappa_2\|\partial_z^2 q_t\|^2_{L^2(\mathcal{M})}.
\end{aligned}$$

Due to the nonnegativity of scalars $\kappa_1, \alpha_{q0}, \alpha_{q1}, \alpha_{qs}$ we deduce that

$$\begin{aligned}\int_{\mathcal{M}} (\kappa_1 \Delta_h q_t + \kappa_2 \partial_z^2 q_t)\partial_z^2 q_t d\mathcal{M} &\geq \kappa_1 \|\nabla_h(\partial_z q_t)\|_{L^2(\mathcal{M})}^2 + \kappa_2 \|\partial_z^2 q_t\|_{L^2(\mathcal{M})}^2 \\ &\geq \min\{\kappa_1, \kappa_2\} \left(\|\nabla_h(\partial_z q_t)\|_{L^2(\mathcal{M})}^2 + \|\partial_z^2 q_t\|_{L^2(\mathcal{M})}^2 \right).\end{aligned} \tag{41}$$

For the term involving advection, we add and subtract the term

$$\int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h \partial_z q_t)\partial_z q_t d\mathcal{M}.$$

Then, conveniently combining terms in order to be able to apply the Lemma 14, we arrive at

$$\begin{aligned}\int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h q_t + w\partial_z q_t)\partial_z^2 q_t d\mathcal{M} &= \int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h q_t)\partial_z^2 q_t d\mathcal{M} \\ &- \int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h \partial_z q_t)\partial_z q_t d\mathcal{M} + \int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h \partial_z q_t + w\partial_z(\partial_z q_t))\partial_z q_t d\mathcal{M}.\end{aligned}$$

The third integral on the right hand side vanishes thanks to Lemma 14, while for the first two integrals we apply the Cauchy-Schwarz inequality using the norm of u_h on $L^\infty(\mathcal{M})$. That is

$$\begin{aligned}&\int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h q_t + w\partial_z q_t)\partial_z^2 q_t d\mathcal{M} \\ &\quad\leq \|\mathbf{u}_h\|_{L^\infty(\mathcal{M})} \left(\|\nabla_h q_t\|_{L^2(\mathcal{M})} \|\partial_z^2 q_t\|_{L^2(\mathcal{M})} + \|\nabla_h \partial_z q_t\|_{L^2(\mathcal{M})} \|\partial_z q_t\|_{L^2(\mathcal{M})} \right) \\ &\quad\leq \frac{1}{\delta_1} \|\mathbf{u}_h\|_{L^\infty(\mathcal{M})}^2 \left(\|\nabla_h q_t\|_{L^2(\mathcal{M})}^2 + \|\partial_z q_t\|_{L^2(\mathcal{M})}^2 \right) \\ &\quad\quad + \frac{\delta_1}{2} \left(\|\partial_z^2 q_t\|_{L^2(\mathcal{M})}^2 + \|\nabla_h \partial_z q_t\|_{L^2(\mathcal{M})}^2 \right).\end{aligned} \tag{42}$$

In the last step, we have used Young's inequality with constant δ_1. Note that, in the above inequality we need to control $\|\mathbf{u}_h\|_{L^\infty(\mathcal{M})}$. For this, we apply the logarithmic type embedding inequality for anisotropic Sobolev spaces (see Lemma 18 in Appendix A)

$$\|\mathbf{u}_h\|_{L^\infty(\mathcal{M})} \leq 3\widehat{C}_\lambda \log^\lambda \left((\|\mathbf{u}_h\|_{L^2(\mathcal{M})} + \|\nabla \mathbf{u}_h\|_{L^2(\mathcal{M})}) + e \right), \tag{43}$$

where $\widehat{C}_\lambda$ is a constant that depends only on C_λ and the term $\max\{1, \sup(\cdot)\}$ that appears in Lemma 18. Note that the norms involved on the right-hand side are bounded by Lemma 5 (see estimates (36) and (37)), so that estimate (43) reduces to

$$\begin{aligned}&\|\mathbf{u}_h\|_{L^\infty(\mathcal{M})} \\ &\leq 3\widehat{C}_\lambda \log^\lambda \left(\|\mathbf{u}_{h0}\|_{L^2(\mathcal{M})} + \left(\int_0^t 2c_2(s)ds \right)^{1/2} + \left(\frac{c_2(t)}{\min\left\{\frac{\nu_1}{2}, \nu_2\right\}} \right)^{1/2} + e \right) = c_{\mathbf{u},\infty}(t).\end{aligned} \tag{44}$$

This way, thanks to estimate (44) and Lemma 3, which allows us to find a bound for q_t in H^1 (see estimates (29) and (30)), we rewrite (42) as

$$\begin{aligned}\int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h q_t + w\partial_z q_t)\partial_z^2 q_t d\mathcal{M} \leq \frac{1}{\delta_1} \frac{(c_1 + c_2 c_1(t)) c_{\mathbf{u},\infty}^2(t)}{\min\{\kappa_1, \frac{\kappa_2}{2}\}} \\ + \tfrac{\delta_1}{2}\left(\|\nabla_h \partial_z q_t\|_{L^2(\mathcal{M})}^2 + \|\partial_z^2 q_t\|_{L^2(\mathcal{M})}^2\right).\end{aligned} \tag{45}$$

Next, we will proceed to bound the last integral involved in (39). To do this, we use the chain rule and Young's inequality

$$\begin{aligned}&-\int_{\mathcal{M}} (V_T \partial_z \chi_\varepsilon(q_t - q_{vs}(z)))\partial_z^2 q_t d\mathcal{M} \\ &\quad = -\int_{\mathcal{M}} \left[V_T \chi_\varepsilon'(q_t - q_{vs}(z))\partial_z(q_t - q_{vs}(z))\right]\partial_z^2 q_t d\mathcal{M} \\ &\quad \leq V_T c_\varepsilon \|\partial_z(q_t - q_{vs}(z))\|_{L^2(\mathcal{M})} \|\partial_z^2 q_t\|_{L^2(\mathcal{M})} \\ &\quad \leq \sqrt{2} V_T c_\varepsilon \left(\|\partial_z q_t\|_{L^2(\mathcal{M})} + \|\partial_z q_{vs}(z)\|_{L^2(\mathcal{M})}\right)\|\partial_z^2 q_t\|_{L^2(\mathcal{M})} \\ &\quad \leq \frac{1}{2\delta_2}\left(\sqrt{2} V_T c_\varepsilon \left(\|\partial_z q_t\|_{L^2(\mathcal{M})} + \|\partial_z q_{vs}(z)\|_{L^2(\mathcal{M})}\right)\right)^2 + \frac{\delta_2}{2}\|\partial_z^2 q_t\|_{L^2(\mathcal{M})}^2.\end{aligned}$$

Using the bound found for $\|\partial_z q_t\|_{L^2(\mathcal{M})}$ in the estimation (29) of Lemma 3, the above can be rewritten as

$$\begin{aligned}&-\int_{\mathcal{M}} (V_T \partial_z \chi_\varepsilon(q_t - q_{vs}(z)))\partial_z^2 q_t d\mathcal{M} \\ &\quad \leq \frac{2V_T^2 c_\varepsilon^2}{\delta_2}\left(\frac{c_1 + c_2 c_1(t)}{\min\{\kappa_1, \frac{\kappa_2}{2}\}} + \|\partial_z q_{vs}(z)\|_{L^2(\mathcal{M})}^2\right) + \frac{\delta_2}{2}\|\partial_z^2 q_t\|_{L^2(\mathcal{M})}^2.\end{aligned} \tag{46}$$

Finally, substituting (40), (41), (45) and (46) into (39), one arrives to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_z q_t\|_{L^2(\mathcal{M})}^2 + \left(\min\{\kappa_1,\kappa_2\} - \frac{(\delta_1+\delta_2)}{2}\right)\left(\|\nabla_h(\partial_z q_t)\|_{L^2(\mathcal{M})}^2 + \|\partial_z^2 q_t\|_{L^2(\mathcal{M})}^2\right) \\ &\quad \leq \frac{2V_T^2 c_\varepsilon^2}{\delta_2}\|\partial_z q_{vs}(z)\|_{L^2(\mathcal{M})}^2 + \frac{c_1 + c_2 c_1(t)}{\min\{\kappa_1, \frac{\kappa_2}{2}\}}\left(\frac{c_{\mathbf{u},\infty}^2(t)}{\delta_1} + \frac{2V_T^2 c_\varepsilon^2}{\delta_2}\right),\end{aligned}$$

and taking $\delta_1 = \delta_2 = \frac{\min\{\kappa_1,\kappa_2\}}{2} > 0$, it turns out that

$$\begin{aligned}\tfrac{\min\{\kappa_1,\kappa_2\}}{2}\left(\|\nabla_h(\partial_z q_t)\|_{L^2(\mathcal{M})}^2 + \|\partial_z^2 q_t\|_{L^2(\mathcal{M})}^2\right) \leq \tfrac{8V_T^2 c_\varepsilon^2}{\min\{\kappa_1,\kappa_2\}}\|\partial_z q_{vs}(z)\|_{L^2(\mathcal{M})}^2 \\ + \tfrac{4(c_1+c_2c_1(t))(c_{\mathbf{u},\infty}^2(t)+2V_T^2c_\varepsilon^2)}{\min\{\kappa_1,\frac{\kappa_2}{2}\}\min\{\kappa_1,\kappa_2\}}.\end{aligned} \tag{47}$$

The estimate (38a) is concluded, choosing

$$c_3(t) := \frac{16V_T^2 c_\varepsilon^2}{(\min\{\kappa_1,\kappa_2\})^2}\|\partial_z q_{vs}(z)\|_{L^2(\mathcal{M})}^2 + \frac{8(c_1 + c_2 c_1(t))(c_{\mathbf{u},\infty}^2(t) + 2V_T^2 c_\varepsilon^2)}{\min\{\kappa_1, \frac{\kappa_2}{2}\}(\min\{\kappa_1,\kappa_2\})^2}.$$

For the estimate (38b), we multiply (8d) by $-\partial_z^2\theta_e$, and integrating over $\mathcal{M}$, we get

$$\begin{aligned}-\int_{\mathcal{M}} (\partial_t\theta_e)\partial_z^2\theta_e d\mathcal{M} + \int_{\mathcal{M}} (\mu_1\Delta_h\theta_e + \mu_2\partial_z^2\theta_e)\partial_z^2\theta_e d\mathcal{M} \\ = \int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h\theta_e + w\partial_z\theta_e)\partial_z^2\theta_e d\mathcal{M}.\end{aligned} \tag{48}$$

Following the same arguments made for estimation (38a), all the integrals involved in the previous equality are controlled in the same way as (40), (41) and (45). That is,

$$\begin{aligned} -\int_{\mathcal{M}} (\partial_t \theta_e)\partial_z^2 \theta_e d\mathcal{M} &\geq \frac{1}{2}\frac{d}{dt}\|\partial_z \theta_e\|^2_{L^2(\mathcal{M})}, \\ \int_{\mathcal{M}} (\mu_1 \Delta_h \theta_e + \mu_2 \partial_z^2 \theta_e)\partial_z^2 \theta_e d\mathcal{M} &\geq \mu_1 \|\nabla_h(\partial_z \theta_e)\|^2_{L^2(\mathcal{M})} + \mu_2 \|\partial_z^2 \theta_e\|^2_{L^2(\mathcal{M})}, \\ \int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h \theta_e + w\partial_z \theta_e)\partial_z^2 \theta_e d\mathcal{M} &\leq 0. \end{aligned} \tag{49}$$

Substituting (49) in (48), we arrive at the required estimate (38b).

Finally, for the estimation of the velocity, we multiply (8a) by $-\partial_z^2 \mathbf{u}_h$, and integrating over $\mathcal{M}$, we obtain

$$\begin{aligned} -\int_{\mathcal{M}} (\partial_t \mathbf{u}_h)\cdot \partial_z^2 \mathbf{u}_h d\mathcal{M} + \int_{\mathcal{M}} (\nu_1 \Delta_h \mathbf{u}_h + \nu_2 \partial_z^2 \mathbf{u}_h)\cdot \partial_z^2 \mathbf{u}_h d\mathcal{M} \\ = \int_{\mathcal{M}} ((\mathbf{u}_h \cdot \nabla_h)\mathbf{u}_h + w\partial_z \mathbf{u}_h)\cdot \partial_z^2 \mathbf{u}_h d\mathcal{M} \\ + \int_{\mathcal{M}} f_0(\hat{z}\times \mathbf{u})_h \cdot \partial_z^2 \mathbf{u}_h d\mathcal{M} + \int_{\mathcal{M}} \nabla_h \mathrm{p}\cdot \partial_z^2 \mathbf{u}_h d\mathcal{M}. \end{aligned} \tag{50}$$

Repeating again the steps we did for (38a) and (38b), we can infer that

$$\begin{aligned} -\int_{\mathcal{M}} (\partial_t \mathbf{u}_h)\partial_z^2 \mathbf{u}_h d\mathcal{M} &\geq \frac{1}{2}\frac{d}{dt}\|\partial_z \mathbf{u}_h\|^2_{L^2(\mathcal{M})}, \\ \int_{\mathcal{M}} (\nu_1 \Delta_h \mathbf{u}_h + \nu_2 \partial_z^2 \mathbf{u}_h)\partial_z^2 \mathbf{u}_h d\mathcal{M} &\geq \min\{\nu_1,\nu_2\}\left(\|\nabla_h(\partial_z \mathbf{u}_h)\|^2_{L^2(\mathcal{M})} + \|\partial_z^2 \mathbf{u}_h\|^2_{L^2(\mathcal{M})}\right), \\ \int_{\mathcal{M}} ((\mathbf{u}_h \cdot \nabla_h)\mathbf{u}_h + w\partial_z \mathbf{u}_h)\partial_z^2 \mathbf{u}_h d\mathcal{M} &\leq \frac{1}{\delta_3}\frac{c_2(t)c^2_{\mathbf{u},\infty}(t)}{\min\{\frac{\nu_2}{2},\nu_2\}} \\ &\quad + \tfrac{\delta_3}{2}\left(\|\nabla_h \partial_z \mathbf{u}_h\|^2_{L^2(\mathcal{M})} + \|\partial_z^2 \mathbf{u}_h\|^2_{L^2(\mathcal{M})}\right). \end{aligned} \tag{51}$$

For the term that contains the Coriolis parameter f_0, we apply Young's inequality and the identity $(\hat{z}\times\mathbf{u})_h = (-v,u)$. So

$$\begin{aligned} \int_{\mathcal{M}} f_0(\hat{z}\times \mathbf{u})_h \cdot \partial_z^2 \mathbf{u}_h d\mathcal{M} &= \int_{\mathcal{M}} f_0(-v,u)\cdot(\partial_z^2 u, \partial_z^2 v)d\mathcal{M} \\ &\leq |f_0|\|\mathbf{u}_h\|_{L^2(\mathcal{M})}\|\partial_z^2 \mathbf{u}_h\|_{L^2(\mathcal{M})} \\ &\leq \frac{1}{\delta_4}|f_0|^2\|\mathbf{u}_h\|^2_{L^2(\mathcal{M})} + \frac{\delta_4}{2}\|\partial_z^2 \mathbf{u}_h\|^2_{L^2(\mathcal{M})}. \end{aligned}$$

Substituting the bound for $\mathbf{u}_h$ given in (37), it follows that

$$\int_{\mathcal{M}} f_0(\hat{z}\times \mathbf{u})_h \cdot \partial_z^2 \mathbf{u}_h d\mathcal{M} \leq \frac{1}{\delta_4}|f_0|^2\left(\|\mathbf{u}_{h0}\|^2_{L^2(\mathcal{M})} + \int_0^t 2c_2(s)ds\right) + \frac{\delta_4}{2}\|\partial_z^2 \mathbf{u}_h\|^2_{L^2(\mathcal{M})}. \tag{52}$$

For the last integral of (50), we apply integration by parts and we utilize the equation (8b) together with condition $\partial_z \mathbf{u}_h = 0$ in both z_0 and z_1 (see (10))

$$\int_{\mathcal{M}} \nabla_h \mathrm{p} \cdot \partial_z^2 \mathbf{u}_h d\mathcal{M} = -\int_{\mathcal{M}} \partial_z(\nabla_h \mathrm{p}) \cdot \partial_z \mathbf{u}_h d\mathcal{M} + \int_{\mathcal{M}'} \nabla_h \mathrm{p} \cdot \partial_z \mathbf{u}_h \Big|_{z_0}^{z_1} d\mathcal{M}'$$
$$= -\int_{\mathcal{M}} \nabla_h(\partial_z \mathrm{p}) \cdot \partial_z \mathbf{u}_h d\mathcal{M} = -\int_{\mathcal{M}} \nabla_h b(q_t, \theta_e; z) \cdot \partial_z \mathbf{u}_h d\mathcal{M}.$$

Now, using the definition of buoyancy b (see (3)) and applying Young's inequality together with the chain rule to derive $\nabla_h \chi_\varepsilon(\cdot)$ we have that

$$\begin{aligned}
&-\int_{\mathcal{M}} \nabla_h b(q_t, \theta_e; z) \cdot \partial_z^2 \mathbf{u}_h d\mathcal{M} \\
&= -\int_{\mathcal{M}} \frac{g}{\theta_0} \nabla_h \theta_e \cdot \partial_z^2 \mathbf{u}_h d\mathcal{M} + \int_{\mathcal{M}} g \left(\frac{L}{c_p \theta_0} - \epsilon_0 \right) \nabla_h q_t \cdot \partial_z^2 \mathbf{u}_h d\mathcal{M} \\
&\quad - \int_{\mathcal{M}} g \left(\frac{L}{c_p \theta_0} - \epsilon_0 - 1 \right) \nabla_h \chi_\varepsilon(q_t - q_{vs}(z)) \cdot \partial_z^2 \mathbf{u}_h d\mathcal{M} \\
&\leq \frac{g}{\theta_0} \|\nabla_h \theta_e\|_{L^2(\mathcal{M})} \|\partial_z^2 \mathbf{u}_h\|_{L^2(\mathcal{M})} + g \left| \frac{L}{c_p \theta_0} - \epsilon_0 \right| \|\nabla_h q_t\|_{L^2(\mathcal{M})} \|\partial_z^2 \mathbf{u}_h\|_{L^2(\mathcal{M})} \\
&\quad + g c_\varepsilon \left| \frac{L}{c_p \theta_0} - \epsilon_0 - 1 \right| \|\nabla_h q_t\|_{L^2(\mathcal{M})} \|\partial_z^2 \mathbf{u}_h\|_{L^2(\mathcal{M})} \\
&\leq g C_{\theta_0, \epsilon_0} \left(\|\nabla_h \theta_e\|_{L^2(\mathcal{M})} + 2 \|\nabla_h q_t\|_{L^2(\mathcal{M})} \right) \|\partial_z^2 \mathbf{u}_h\|_{L^2(\mathcal{M})} \\
&\leq \frac{1}{2\delta_5} g^2 C_{\theta_0, \epsilon_0}^2 \left(\|\nabla_h \theta_e\|_{L^2(\mathcal{M})} + 2 \|\nabla_h q_t\|_{L^2(\mathcal{M})} \right)^2 + \frac{\delta_5}{2} \|\partial_z^2 \mathbf{u}_h\|_{L^2(\mathcal{M})}^2,
\end{aligned}$$

where C_{θ_0,ϵ_0} was defined in (22). Then, thanks the estimates (29), (30)) and (31), we can control the terms $\|\nabla_h q_t\|_{L^2(\mathcal{M})}$ and $\|\nabla_h \theta_e\|_{L^2(\mathcal{M})}$ respectively. Thus,

$$-\int_{\mathcal{M}} \nabla_h b(q_t, \theta_e; z) \cdot \partial_z^2 \mathbf{u}_h d\mathcal{M} \leq \frac{2}{\delta_5} g^2 C_{\theta_0, \epsilon_0}^2 \left(\frac{c_1 + c_2 c_1(t)}{\min\{\kappa_1, \frac{\kappa_2}{2}\}} \right) + \frac{\delta_5}{2} \|\partial_z^2 \mathbf{u}_h\|_{L^2(\mathcal{M})}^2. \tag{53}$$

Substituting (51), (52) and (53) in (50), it results in

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\partial_z \mathbf{u}_h\|_{L^2(\mathcal{M})}^2 \\
&\quad + \left(\min\{\nu_1, \nu_2\} - \frac{(\delta_3 + \delta_4 + \delta_5)}{2} \right) \left(\|\nabla_h(\partial_z \mathbf{u}_h)\|_{L^2(\mathcal{M})}^2 + \|\partial_z^2 \mathbf{u}_h\|_{L^2(\mathcal{M})}^2 \right) \\
&\quad \leq \frac{1}{\delta_3} \frac{c_2(t) c_{\mathbf{u},\infty}^2(t)}{\min\{\frac{\nu_2}{2}, \nu_2\}} + \frac{1}{\delta_4} |f_0|^2 \left(\|\mathbf{u}_{h0}\|_{L^2(\mathcal{M})}^2 + \int_0^t 2 c_2(s) ds \right) \\
&\quad\quad + \frac{2}{\delta_5} g^2 C_{\theta_0, \epsilon_0}^2 \left(\frac{c_1 + c_2 c_1(t)}{\min\{\kappa_1, \frac{\kappa_2}{2}\}} \right).
\end{aligned}$$

Taking $\delta_3 = \delta_4 = \delta_5 = \frac{\min\{\nu_1,\nu_2\}}{3} > 0$, it turns out that

$$\tfrac{\min\{\nu_1,\nu_2\}}{2} \left(\|\nabla_h(\partial_z \mathbf{u}_h)\|_{L^2(\mathcal{M})}^2 + \|\partial_z^2 q_t\|_{L^2(\mathcal{M})}^2 \right) \leq c_4(t), \tag{54}$$

where

$$c_4(t) := \frac{6c_2(t)c^2_{\mathbf{u},\infty}(t)}{(\min\{\nu_1,\nu_2\})^2\min\{\frac{\nu_2}{2},\nu_2\}} + \frac{6|f_0|^2}{(\min\{\nu_1,\nu_2\})^2}\left(\|\mathbf{u}_{h0}\|^2_{L^2(\mathcal{M})} + \int_0^t 2c_2(s)ds\right) + \frac{12g^2C^2_{\theta_0,\epsilon_0}}{(\min\{\nu_1,\nu_2\})^2}\left(\frac{c_1+c_2c_1(t)}{\min\{\kappa_1,\frac{\kappa_2}{2}\}}\right). \quad \square$$

3.2.2. Positivity

Since the estimations that will be presented in this subsection imply time derivatives of functions that involve maxima, we will first show that the positivity of θ_e and q_t is possible in a subspace of $\mathcal{M}$, called $\mathcal{M}_0$ and defined as $\mathcal{M}_0 := \{(x,y,z) \in \mathcal{M} : q_t(x,y,z,t) \neq 0\}$, then by additivity we will cover the entire space of $\mathcal{M}$. Now we define $f := f^+ - f^-$, where $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$ are the positive and negative parts, respectively. Under this definition it is easy to see that

$$(f^+)(f^-) = 0 \qquad (\partial_* f^-)f^+ = 0 \qquad (\partial_* f^+)f^- = 0, \tag{55}$$

where ∂_* refers to the time or space derivative.

The following lemma shows the positivity of q_t.

Lemma 7. *Assume that the initial data $q_{t0} > 0$ is positive with $q_{t0} \in H^1(\mathcal{M})$. Suppose also that the boundary conditions (10) and (8e) are satisfied. Then $q_t > 0$ is always positive.*

Proof. As in [13], we also decompose into positive and negative parts. Due to our FARE assumptions, we do not deal with extra source terms. On the other hand, we need to define the above space $\mathcal{M}_0$ in order for the derivatives inside some of the integrals to exist.

Let us multiply (8c) by $-q_t^-$. Integrating over $\mathcal{M}$, we get

$$\begin{aligned} -\int_{\mathcal{M}} (\partial_t q_t)q_t^- d\mathcal{M} + \int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h q_t + w\partial_z q_t)q_t^- d\mathcal{M} \\ = -\int_{\mathcal{M}} (V_T\partial_z \chi_\varepsilon(q_t - q_{vs}(z))q_t^- d\mathcal{M} - \int_{\mathcal{M}} (\kappa_1\Delta_h q_t + \kappa_2\partial_z^2 q_t)q_t^- d\mathcal{M}. \end{aligned} \tag{56}$$

By (55), we have that

$$\begin{aligned} -\int_{\mathcal{M}} (\partial_t q_t)q_t^-\, d\mathcal{M} &= -\int_{\mathcal{M}_0} [\partial_t(q_t^+ - q_t^-)]q_t^-\, d\mathcal{M}_0 = \int_{\mathcal{M}_0} (\partial_t q_t^-)q_t^- d\mathcal{M}_0 \\ &= \frac{1}{2}\frac{d}{dt}\|q_t^-\|^2_{L^2(\mathcal{M}_0)} = \frac{1}{2}\frac{d}{dt}\|q_t^-\|^2_{L^2(\mathcal{M})}. \end{aligned} \tag{57}$$

It follows from Lemma 14 (see Appendix A) and the identity (55), that

$$\begin{aligned} -\int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h q_t + w\partial_z q_t)q_t^- d\mathcal{M} \\ &= -\int_{\mathcal{M}_0} (\mathbf{u}_h \cdot \nabla_h q_t^+ + w\partial_z q_t^+)q_t^- d\mathcal{M}_0 \\ &\quad + \int_{\mathcal{M}_0} (\mathbf{u}_h \cdot \nabla_h q_t^- + w\partial_z q_t^-)q_t^- d\mathcal{M}_0 \\ &= \int_{\mathcal{M}_0} (\mathbf{u}_h \cdot \nabla_h q_t^- + w\partial_z q_t^-)q_t^- d\mathcal{M}_0 = 0. \end{aligned} \tag{58}$$

As a consequence of Lemma 15 (see Appendix A) and (55), we also obtain

$$\int_{\mathcal{M}} (\kappa_1 \Delta_h q_t + \kappa_2 \partial_z^2 q_t) q_t^- \, d\mathcal{M} = - \int_{\mathcal{M}_0} (\kappa_1 \Delta_h q_t^- + \kappa_2 \partial_z^2 q_t^-) q_t^- \, d\mathcal{M}_0 \\ \geq \kappa_1 \|\nabla_h q_t^-\|^2_{L^2(\mathcal{M}_0)} + \kappa_2 \|\partial_z q_t^-\|^2_{L^2(\mathcal{M}_0)}. \tag{59}$$

Notice that, due to the fact that $\partial_z q_{vs} \leq 0$, we get

$$-\partial_z (q_t - q_{vs}(z)) = -\partial_z q_t + \partial_z q_{vs}(z) \leq -\partial_z q_t. \tag{60}$$

Then, using the chain rule to derive $\partial_z \chi_\varepsilon(\cdot)$ and the inequality in (60), we deduce that

$$\begin{aligned} - \int_{\mathcal{M}} (V_T \partial_z \chi_\varepsilon (q_t - q_{vs}(z))) q_t^- \, d\mathcal{M} \\ &= - \int_{\mathcal{M}} \left(V_T \chi_\varepsilon' (q_t - q_{vs}(z)) \partial_z (q_t - q_{vs}(z))\right) q_t^- \, d\mathcal{M} \\ &= - \int_{\mathcal{M}} \left(V_T \chi_\varepsilon' (q_t - q_{vs}(z)) \partial_z q_t\right) q_t^- \, d\mathcal{M} \\ &= - \int_{\mathcal{M}_0} \left(V_T \chi_\varepsilon' (q_t - q_{vs}(z)) \partial_z q_t^+\right) q_t^- \, d\mathcal{M}_0 \\ &\quad + \int_{\mathcal{M}_0} \left(V_T \chi_\varepsilon' (q_t - q_{vs}(z)) \partial_z q_t^-\right) q_t^- \, d\mathcal{M}_0 \\ &= \int_{\mathcal{M}_0} \left(V_T \chi_\varepsilon' (q_t - q_{vs}(z)) \partial_z q_t^-\right) q_t^- \, d\mathcal{M}_0. \end{aligned}$$

From condition (5), and after applying Young's inequality, we deduce that

$$\begin{aligned} - \int_{\mathcal{M}} (V_T \partial_z \chi_\varepsilon (q_t - q_{vs}(z))) q_t^- \, d\mathcal{M} &\leq V_T c_\varepsilon \|\partial_z q_t^-\|_{L^2(\mathcal{M}_0)} \|q_t^-\|_{L^2(\mathcal{M}_0)} \\ &\leq \frac{1}{2\delta} V_T^2 c_\varepsilon^2 \|q_t^-\|^2_{L^2(\mathcal{M})} + \frac{\delta}{2} \|\partial_z q_t^-\|^2_{L^2(\mathcal{M}_0)}. \end{aligned}$$

Taking $\delta = \kappa_2$, we get

$$- \int_{\mathcal{M}} (V_T \partial_z \chi_\varepsilon (q_t - q_{vs})(z)) q_t^- \, d\mathcal{M} \leq \frac{1}{2\kappa_2} V_T^2 c_\varepsilon^2 \|q_t^-\|^2_{L^2(\mathcal{M})} + \frac{\kappa_2}{2} \|\partial_z q_t^-\|^2_{L^2(\mathcal{M}_0)}. \tag{61}$$

Now, substituting (57), (58), (59) and (61) into (56), the result is

$$\frac{1}{2} \frac{d}{dt} \|q_t^-\|^2_{L^2(\mathcal{M})} + \kappa_1 \|\nabla_h q_t^-\|^2_{L^2(\mathcal{M}_0)} + \frac{\kappa_2}{2} \|\partial_z q_t^-\|^2_{L^2(\mathcal{M}_0)} \leq \frac{1}{2\kappa_2} V_T^2 c_\varepsilon^2 \|q_t^-\|^2_{L^2(\mathcal{M})}, \tag{62}$$

which implies

$$\|q_t^-\|^2_{L^2(\mathcal{M})} \leq \|q_t^-(0)\|^2_{L^2(\mathcal{M})} + \int_0^t \frac{1}{\kappa_2} V_T^2 c_\varepsilon^2 \|q_t^-(s)\|^2_{L^2(\mathcal{M})} ds.$$

Finally, by virtue of the Gronwall inequality (see [2, Lemma 1.1, eq. (1.77)]), one arrives at

$$\|q_t^-\|^2_{L^2(\mathcal{M})} \le \|q_t^-(0)\|^2_{L^2(\mathcal{M})} \exp\left(\int_0^t \frac{1}{\kappa_2} V_T^2 c_\varepsilon^2 ds\right).$$

Thanks to the fact that $q_t(0) = q_{t0} > 0$, then $q_t^-(0) = \max\{-q_t(0), 0\} = 0$ and therefore $q_t^- \equiv 0$, or equivalently $q_t > 0$. $\square$

Lemma 8. *Assume that the initial data satisfies $\theta_{e0} > 0$ with $\theta_{e0} \in H^1(\mathcal{M})$. Furthermore, let us assume that the boundary conditions in equations (10) and (8e) hold. Then $\theta_e > 0$ is always positive.*

Proof. For the proof of this Lemma, we use the same arguments made in the proof of the positivity of q_t in the Lemma 7, for which we will omit details. $\square$

3.2.3. Uniqueness

Let $(\mathbf{u}_{hi}, \theta_{ei}, q_{ti})$; $i = 1, 2$, be two solutions of system (1). That is

$$\partial_t \mathbf{u}_{hi} + (\mathbf{u}_i \cdot \nabla_h)\mathbf{u}_{hi} + w_i \partial_z \mathbf{u}_i + f_0(\hat{z} \times \mathbf{u}_i)_h = -\nabla_h \mathrm{p}_i + \nu_1 \Delta_h \mathbf{u}_{hi} + \nu_2 \partial_z^2 \mathbf{u}_{hi}, \tag{63a}$$

$$\partial_z \mathrm{p}_i = b(q_{ti}, \theta_{ei}; z), \tag{63b}$$

$$\partial_t q_{ti} + \mathbf{u}_{hi} \cdot \nabla_h q_{ti} + w_i \partial_z q_{ti} = V_T \partial_z \chi_\varepsilon(q_{ti} - q_{vs}(z)) + \kappa_1 \Delta_h q_{ti} \tag{63c}$$

$$+ \kappa_2 \partial_z^2 q_{ti}, \tag{63d}$$

$$\partial_t \theta_{ei} + \mathbf{u}_{hi} \cdot \nabla_h \theta_{ei} + w_i \partial_z \theta_{ei} = \mu_1 \Delta_h \theta_{ei} + \mu_2 \partial_z^2 \theta_{ei}, \tag{63e}$$

$$\nabla_h \cdot \mathbf{u}_{hi} + \partial_z w_i = 0. \tag{63f}$$

We denote the total error as:

$$\|\varepsilon^{\mathrm{tot}}\|^2_{L^2(\mathcal{M})} := \|\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})} + \|\varepsilon^{q_t}\|^2_{L^2(\mathcal{M})} + \|\varepsilon^{\theta_e}\|^2_{L^2(\mathcal{M})}, \tag{64}$$

where

$$\varepsilon^{\theta_e} := \theta_{e1} - \theta_{e2}, \qquad \varepsilon^{q_t} := q_{t1} - q_{t2} \quad \text{and} \quad \varepsilon^{\mathbf{u}_h} := \mathbf{u}_{h1} - \mathbf{u}_{h2}.$$

Let us multiply equation (63d) for the difference $q_{t1} - q_{t2}$ by the error ε^{q_t} and integrate over $\mathcal{M}$. It gives us the following identity:

$$\begin{aligned} &\int_{\mathcal{M}} (\partial_t \varepsilon^{q_t})\varepsilon^{q_t} d\mathcal{M} + \int_{\mathcal{M}} (\mathbf{u}_{h1} \cdot \nabla_h q_{t1} + w_1 \partial_z q_{t1} - (\mathbf{u}_{h2} \cdot \nabla_h q_{t2} + w_2 \partial_z q_{t2})\varepsilon^{q_t} d\mathcal{M} \\ &\quad = \int_{\mathcal{M}} (\kappa_1 \Delta_h \varepsilon^{q_t} + \kappa_2 \partial_z^2 \varepsilon^{q_t})\varepsilon^{q_t} d\mathcal{M} + \int_{\mathcal{M}} V_T(\partial_z \chi_\varepsilon(q_{t1} - q_{vs}) \\ &\qquad - \partial_z \chi_\varepsilon(q_{t2} - q_{vs})(z))\varepsilon^{q_t} d\mathcal{M}. \end{aligned} \tag{65}$$

Note that the boundary conditions are still fulfilled for the error of q_t. That is,

$$\begin{aligned} \Gamma_0 : &\quad \partial_z \varepsilon^{q_t} = (\partial_z q_{t1} - \partial_z q_{t2}) = \alpha_{q0} q_{t1} - \alpha_{q0} q_{t2} = \alpha_{q0}, \varepsilon^{q_t} \\ \Gamma_1 : &\quad \partial_z \varepsilon^{q_t} = (\partial_z q_{t1} - \partial_z q_{t2}) = -\alpha_{q1} q_{t1} + \alpha_{q1} q_{t2} = -\alpha_{q1}, \varepsilon^{q_t} \\ \Gamma_s : &\quad \partial_n \varepsilon^{q_t} = \partial_n q_{t1} - \partial_n q_{t2} = -\alpha_{qs} q_{t1} + \alpha_{qs} q_{t2} = -\alpha_{qs} \varepsilon^{q_t}. \end{aligned}$$

Then, we can apply Lemmas 15 and 19, obtaining, respectively

$$\int_{\mathcal{M}} (\mu_1 \Delta_h \varepsilon^{q_t} + \partial_z^2 \varepsilon^{q_t}) \varepsilon^{q_t} d\mathcal{M} \leq -\kappa_1 \|\nabla_h \varepsilon^{q_t}\|_{L^2(\mathcal{M})}^2 - \kappa_2 \|\partial_z \varepsilon^{q_t}\|_{L^2(\mathcal{M})}^2, \tag{66}$$

and

$$\begin{aligned}
&\int_{\mathcal{M}} \left[(\mathbf{u}_{h1} \cdot \nabla_h q_{t1} + w_1 \partial_z q_{t1}) - (\mathbf{u}_{h2} \cdot \nabla_h q_{t2} + w_2 \partial_z q_{t2}) \right] \varepsilon^{q_t} d\mathcal{M} \\
&\quad \leq \left[\widetilde{C}_1 \left(1 + \frac{\widetilde{C}_1}{4\widetilde{\delta}_1}\right) + \frac{1}{\widetilde{\delta}_3} \left(1 + \frac{1}{2\widetilde{\delta}_2}\right)^2 \right] \|\varepsilon^{q_t}\|_{L^2(\mathcal{M})}^2 \\
&\qquad + \widetilde{C}_1 \left(1 + \tfrac{\widetilde{C}_1}{4\widetilde{\delta}_1}\right) \|\varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})}^2 \\
&\qquad + \left[\widetilde{\delta}_1 + \tfrac{\widetilde{\delta}_2^2}{4\widetilde{\delta}_3}\right] \|\nabla_h \varepsilon^{q_t}\|_{L^2(\mathcal{M})}^2 + \left[\widetilde{\delta}_1 + \tfrac{\widetilde{C}_2^2}{2} \widetilde{\delta}_3\right] \|\nabla_h \varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})}^2.
\end{aligned} \tag{67}$$

In order to estimate the last integral of (65), we first note that

$$\begin{aligned}
&\partial_z \chi_\varepsilon(q_{t1} - q_{vs}(z)) - \partial_z \chi_\varepsilon(q_{t2} - q_{vs}(z)) \\
&\quad = \chi_\varepsilon'(q_{t1} - q_{vs}(z)) \partial_z (q_{t1} - q_{vs}(z)) - \chi_\varepsilon'(q_{t2} - q_{vs}(z)) \partial_z (q_{t2} - q_{vs}(z)) \\
&\qquad \pm \chi_\varepsilon'(q_{t1} - q_{vs}(z)) \partial_z (q_{t2} - q_{vs}(z)) \\
&\quad = \chi_\varepsilon'(q_{t1} - q_{vs}(z))\, \partial_z \varepsilon^{q_t} + [\chi_\varepsilon'(q_{t1} - q_{vs}(z)) - \chi_\varepsilon'(q_{t2} - q_{vs}(z))] \partial_z (q_{t2} - q_{vs}(z)).
\end{aligned}$$

By Hölder's inequality and Lipschitz continuity assumption (6), we have

$$\begin{aligned}
&\int_{\mathcal{M}} V_T (\partial_z \chi_\varepsilon(q_{t1} - q_{vs}(z)) - \partial_z \chi_\varepsilon(q_{t2} - q_{vs}(z))) \varepsilon^{q_t} d\mathcal{M} \\
&= \int_{\mathcal{M}} V_T \chi_\varepsilon'(q_{t1} - q_{vs}(z)) (\partial_z \varepsilon^{q_t}) \varepsilon^{q_t} d\mathcal{M} \\
&\qquad + \int_{\mathcal{M}} V_T [\chi_\varepsilon'(q_{t1} - q_{vs}(z)) - \chi_\varepsilon'(q_{t2} - q_{vs}(z))] (\partial_z q_{t2}) \varepsilon^{q_t} d\mathcal{M} \\
&\qquad - \int_{\mathcal{M}} V_T [\chi_\varepsilon'(q_{t1} - q_{vs}(z)) - \chi_\varepsilon'(q_{t2} - q_{vs}(z))] (\partial_z q_{vs}(z)) \varepsilon^{q_t} d\mathcal{M} \\
&\leq V_T\, c_\varepsilon \|\partial_z \varepsilon^{q_t}\|_{L^2(\mathcal{M})} \|\varepsilon^{q_t}\|_{L^2(\mathcal{M})} \\
&\qquad + V_T L_2 \int_{\mathcal{M}} |(q_{t1} - q_{vs}(z)) - (q_{t2} - q_{vs}(z))| |\partial_z q_{t2}| |\varepsilon^{q_t}| d\mathcal{M} \\
&\qquad + V_T L_2 \int_{\mathcal{M}} |(q_{t1} - q_{vs}(z)) - (q_{t2} - q_{vs}(z))| |\partial_z q_{vs}(z)| |\varepsilon^{q_t}| d\mathcal{M} \\
&\leq V_T\, c_\varepsilon \|\varepsilon^{q_t}\|_{H^1(\mathcal{M})} \|\varepsilon^{q_t}\|_{L^2(\mathcal{M})} + V_T L_2 \|\partial_z q_{t2}\|_{L^4(\mathcal{M})} \|\varepsilon^{q_t}\|_{L^4(\mathcal{M})} \|\varepsilon^{q_t}\|_{L^2(\mathcal{M})} \\
&\qquad + V_T L_2 \|\partial_z q_{vs}(z)\|_{L^4(\mathcal{M})} \|\varepsilon^{q_t}\|_{L^4(\mathcal{M})} \|\varepsilon^{q_t}\|_{L^2(\mathcal{M})}.
\end{aligned}$$

Then, applying the continuous injection $\boldsymbol{i}_4$ of $H^1(\cdot)$ into $L^4(\cdot)$ (see, e.g., [26, Theorem 1.3.4]), we deduce that

$$\begin{aligned}
&\int_{\mathcal{M}} V_T(\partial_z\chi_\varepsilon(q_{t1}-q_{vs}(z))-\partial_z\chi_\varepsilon(q_{t2}-q_{vs}(z)))\varepsilon^{q_t}d\mathcal{M}\\
&\le V_T\, c_\varepsilon\|\varepsilon^{q_t}\|_{H^1(\mathcal{M})}\|\varepsilon^{q_t}\|_{L^2(\mathcal{M})}\\
&\quad + V_T L_2\|\boldsymbol{i}_4\|^2\left(\|\partial_z q_{t2}\|_{H^1(\mathcal{M})}+\|\partial_z q_{vs}(z)\|_{H^1(\mathcal{M})}\right)\|\varepsilon^{q_t}\|_{H^1(\mathcal{M})}\|\varepsilon^{q_t}\|_{L^2(\mathcal{M})}\\
&\le V_T\left[c_\varepsilon+L_2\|\boldsymbol{i}_4\|^2\left(\|\partial_z q_{t2}\|_{H^1(\mathcal{M})}+\|\partial_z q_{vs}(z)\|_{H^1(\mathcal{M})}\right)\right]\|\varepsilon^{q_t}\|^2_{L^2(\mathcal{M})}\\
&\quad + V_T\left[c_\varepsilon+L_2\|\boldsymbol{i}_4\|^2\left(\|\partial_z q_{t2}\|_{H^1(\mathcal{M})}+\|\partial_z q_{vs}(z)\|_{H^1(\mathcal{M})}\right)\right]\|\varepsilon^{q_t}\|_{L^2(\mathcal{M})}\|\nabla\varepsilon^{q_t}\|_{L^2(\mathcal{M})}.
\end{aligned}$$

Thanks to estimate of the Lemma 6 and Young's inequality, we get

$$\begin{aligned}
&\int_{\mathcal{M}} V_T(\partial_z\chi_\varepsilon(q_{t1}-q_{vs}(z))-\partial_z\chi_\varepsilon(q_{t2}-q_{vs}(z)))\varepsilon^{q_t}d\mathcal{M}\\
&\qquad\le c_{q_t}(t)\left(1+\tfrac{c_{q_t}(t)}{2\widetilde{\delta}_4}\right)\|\varepsilon^{q_t}\|^2_{L^2(\mathcal{M})}+\tfrac{\widetilde{\delta}_4}{2}\|\nabla\varepsilon^{q_t}\|^2_{L^2(\mathcal{M})},
\end{aligned}\tag{68}$$

where

$$c_{q_t}(t):=V_T\left[c_\varepsilon+L_2\|\boldsymbol{i}_4\|^2\left(\left(\widetilde{C}_{q_t}+\frac{2}{\mu_2}(c_1+c_2c_1(t))\right)^{1/2}+\|\partial_z q_{vs}(z)\|_{H^1(\mathcal{M})}\right)\right].$$

Substituting (66), (67) and (68) into (65), it yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\varepsilon^{q_t}\|^2_{L^2(\mathcal{M})}+(\min\{\kappa_1,\kappa_2\})\|\nabla\varepsilon^{q_t}\|^2_{L^2(\mathcal{M})}\\
&\le\left[\widetilde{C}_1\left(1+\tfrac{\widetilde{C}_1}{4\widetilde{\delta}_1}\right)+\tfrac{1}{\widetilde{\delta}_3}\left(1+\tfrac{1}{2\widetilde{\delta}_2}\right)^2+c_{q_t}(t)\left(1+\tfrac{\kappa_{q_t}(t)}{2\widetilde{\delta}_4}\right)\right]\|\varepsilon^{q_t}\|^2_{L^2(\mathcal{M})}\\
&\quad+\widetilde{C}_1\left(1+\tfrac{\widetilde{C}_1}{4\widetilde{\delta}_1}\right)\|\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})}+\left[\widetilde{\delta}_1+\tfrac{\widetilde{\delta}_2^2}{4\widetilde{\delta}_3}+\tfrac{\widetilde{\delta}_4}{2}\right]\|\nabla\varepsilon^{q_t}\|^2_{L^2(\mathcal{M})}\\
&\quad+\left[\widetilde{\delta}_1+\tfrac{\widetilde{C}_2^2}{2}\widetilde{\delta}_3\right]\|\nabla\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})}.
\end{aligned}\tag{69}$$

For the uniqueness of θ_e, we proceed in a similar way as for q_t in (65), getting

$$\begin{aligned}
&\frac{1}{2}\ \tfrac{d}{dt}\|\varepsilon^{\theta_e}\|^2_{L^2(\mathcal{M})}+(\min\{\mu_1,\mu_2\})\|\nabla\varepsilon^{\theta_e}\|^2_{L^2(\mathcal{M})}\\
&\le\left[\widetilde{C}_1\left(1+\tfrac{\widetilde{C}_1}{4\widetilde{\delta}_1}\right)+\tfrac{1}{\widetilde{\delta}_3}\left(1+\tfrac{1}{2\widetilde{\delta}_2}\right)^2\right]\|\varepsilon^{\theta_e}\|^2_{L^2(\mathcal{M})}+\widetilde{C}_1\left(1+\tfrac{\widetilde{C}_1}{4\widetilde{\delta}_1}\right)\|\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})}\\
&\quad+\left[\widetilde{\delta}_1+\tfrac{\widetilde{\delta}_2^2}{4\widetilde{\delta}_3}\right]\|\nabla\varepsilon^{\theta_e}\|^2_{L^2(\mathcal{M})}+\left[\widetilde{\delta}_1+\tfrac{\widetilde{C}_2^2}{2}\widetilde{\delta}_3\right]\|\nabla\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})}.
\end{aligned}\tag{70}$$

Finally, for the uniqueness of $\mathbf{u}_h$, we multiply equation (63a) for the difference $\mathbf{u}_{h1}-\mathbf{u}_{h2}$ by the error $\varepsilon^{\mathbf{u}_h}$. Integrating over $\mathcal{M}$, we get

$$\begin{aligned}
&\int_{\mathcal{M}}(\partial_t\varepsilon^{\mathbf{u}_h})\varepsilon^{\mathbf{u}_h}d\mathcal{M}\\
&\quad+\int_{\mathcal{M}}((\mathbf{u}_{h1}\cdot\nabla_h)\mathbf{u}_{h1}+w_1\partial_z\mathbf{u}_{h1}-((\mathbf{u}_{h2}\cdot\nabla_h)\mathbf{u}_{h2}+w_2\partial_z\mathbf{u}_{h2}))\varepsilon^{\mathbf{u}_h}d\mathcal{M}\\
&\qquad=-\int_{\mathcal{M}}(\nabla_h\varepsilon^{\mathrm{p}})\varepsilon^{\mathbf{u}_h}+\int_{\mathcal{M}}(\nu_1\Delta_h\varepsilon^{\mathbf{u}_h}+\nu_2\partial_z^2\varepsilon^{\mathbf{u}_h})\varepsilon^{\mathbf{u}_h}d\mathcal{M}.
\end{aligned}\tag{71}$$

Here, the term that involves the Coriolis parameter vanishes since

$$f_0(\hat{z} \times \mathbf{u}_1 - \hat{z} \times \mathbf{u}_2)_h \cdot \varepsilon^{\mathbf{u}_h} = f_0(v_2 - v_1, u_1 - u_2) \cdot (u_1 - u_2, v_1 - v_2) = 0,$$

and $\varepsilon^{\mathrm{p}} = \mathrm{p}_1 - \mathrm{p}_2$. As a consequence of Lemma 15 (see Appendix A), we deduce that

$$\int\limits_{\mathcal{M}} (\nu_1 \Delta_h \varepsilon^{\mathbf{u}_h} + \nu_2 \partial_z^2 \varepsilon^{\mathbf{u}_h}) \varepsilon^{\mathbf{u}_h}\, d\mathcal{M} \leq -\nu_1 \|\nabla_h \varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})} - \nu_2 \|\partial_z \varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})}. \tag{72}$$

Then, we can apply Lemma 19 (see Appendix A), obtaining

$$\begin{aligned} &\int\limits_{\mathcal{M}} \left[((\mathbf{u}_{h1} \cdot \nabla_h)\mathbf{u}_{h1} + w_1 \partial_z \mathbf{u}_{h1}) - ((\mathbf{u}_{h2} \cdot \nabla_h)\mathbf{u}_{h2} + w_2 \partial_z \mathbf{u}_{h2})\right] \varepsilon^{\mathbf{u}_h} d\mathcal{M} \\ &\quad \leq \left[2\widetilde{C}_1 \left(1 + \frac{\widetilde{C}_1}{4\widetilde{\delta}_1}\right) + \frac{1}{\widetilde{\delta}_3} \left(1 + \frac{1}{2\widetilde{\delta}_2}\right)^2 \right] \|\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})} \\ &\qquad + \left[2\widetilde{\delta}_1 + \tfrac{\widetilde{\delta}_2^2}{4\widetilde{\delta}_3} + \tfrac{\widetilde{C}_2^2}{2} \widetilde{\delta}_3 \right] \|\nabla \varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})}. \end{aligned} \tag{73}$$

Integration by parts gives us the boundary conditions $\partial_{\boldsymbol{n}} \varepsilon^{\mathbf{u}_h} = 0$ and (8e). We obtain

$$\begin{aligned} -\int\limits_{\mathcal{M}} \nabla_h \varepsilon^{\mathrm{p}} \cdot \varepsilon^{\mathbf{u}_h} d\mathcal{M} &= \int\limits_{\mathcal{M}} \varepsilon^{\mathrm{p}} (\nabla_h \cdot \varepsilon^{\mathbf{u}_h}) d\mathcal{M} - \int\limits_{\Gamma_s} \varepsilon^{\mathrm{p}}\, \partial_{\boldsymbol{n}} \varepsilon^{\mathbf{u}_h} d\Gamma_s = \int\limits_{\mathcal{M}} \varepsilon^{\mathrm{p}} (\nabla_h \cdot \varepsilon^{\mathbf{u}}_h) d\mathcal{M} \\ &= -\int\limits_{\mathcal{M}} \varepsilon^{\mathrm{p}}\, \partial_z \varepsilon^w d\mathcal{M} = \int\limits_{\mathcal{M}} \varepsilon^w \partial_z \varepsilon^{\mathrm{p}} d\mathcal{M} - \int\limits_{\mathcal{M}'} \varepsilon^{\mathrm{p}} \varepsilon^w \Big|_{z_0}^{z_1} d\mathcal{M}'. \end{aligned}$$

Since $\varepsilon^w := w_1 - w_2 = 0$ at $z = z_0$ and $z = z_1$, together with equation (8b), we get

$$-\int\limits_{\mathcal{M}} \nabla_h \varepsilon^{\mathrm{p}} \cdot \varepsilon^{\mathbf{u}_h} d\mathcal{M} = \int\limits_{\mathcal{M}} \varepsilon^w \partial_z \varepsilon^{\mathrm{p}} d\mathcal{M} = \int\limits_{\mathcal{M}} \varepsilon^w (b(q_{t1}, \theta_{e1}; z) - b(q_{t2}, \theta_{e2}; z)) d\mathcal{M}.$$

Also,

$$\begin{aligned} &\int\limits_{\mathcal{M}} \varepsilon^w (b(q_{t1}, \theta_{e1}; z) - b(q_{t2}, \theta_{e2}; z)) d\mathcal{M} = \int\limits_{\mathcal{M}} \frac{g}{\theta_0} \varepsilon^{\theta_e} \varepsilon^w d\mathcal{M} - \int\limits_{\mathcal{M}} g \left(\frac{L}{c_p \theta_0} - \epsilon_0\right) \varepsilon^{q_t} \varepsilon^w d\mathcal{M} \\ &\quad + \int\limits_{\mathcal{M}} g \left(\frac{L}{c_p \theta_0} - \epsilon_0 - 1\right) (\chi_\varepsilon(q_{t1} - q_{vs}(z)) - \chi_\varepsilon(q_{t2} - q_{vs}(z))) \varepsilon^w d\mathcal{M} \\ &\leq g C_{\theta_0, \epsilon_0} \max\{1, L_1\} \left(\|\varepsilon^{\theta_e}\|_{L^2(\mathcal{M})} + \|\varepsilon^{q_t}\|_{L^2(\mathcal{M})}\right) \|\varepsilon^w\|_{L^2(\mathcal{M})} \\ &\leq \frac{1}{2\widetilde{\delta}_5} g^2 C^2_{\theta_0, \epsilon_0} \left(\max\{1, L_1\}\right)^2 \left(\|\varepsilon^{\theta_e}\|_{L^2(\mathcal{M})} + 2\|\varepsilon^{q_t}\|_{L^2(\mathcal{M})}\right)^2 + \frac{\widetilde{\delta}_5}{2} \|\varepsilon^w\|^2_{L^2(\mathcal{M})}. \end{aligned}$$

In the case where rain is known to be bounded and so the transition function χ_ε can be assumed of compact support, the Lipschitz continuity condition is not longer necessary. Furthermore, the involved coefficients greatly simplify as the term $\|\varepsilon^{q_t}\|$ does not show up anymore.

Since $(a+2b)^2 \le 3(a^2+2b^2)$ for $a,b>0$, we obtain

$$-\int_{\mathcal{M}} \nabla_h \varepsilon^{\mathrm{p}} \cdot \varepsilon^{\mathbf{u}_h} d\mathcal{M} \le \frac{3g^2 C^2_{\theta_0,\epsilon_0}(\max\{1,L_1\})^2}{2\widetilde{\delta}_5}\left(\|\varepsilon^{\theta_e}\|^2_{L^2(\mathcal{M})} + \|\varepsilon^{q_t}\|^2_{L^2(\mathcal{M})}\right) + \frac{(z_1-z_0)^2)\widetilde{\delta}_5}{2}\|\nabla_h \varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})}. \tag{74}$$

Substituting (72), (73) and (74) into (71), it yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})} &+(\min\{\nu_1,\nu_2\})\|\nabla\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})}\\ &\le \frac{3g^2 C^2_{\theta_0,\epsilon_0}(\max\{1,L_1\})^2}{2\widetilde{\delta}_5}\left(\|\varepsilon^{\theta_e}\|^2_{L^2(\mathcal{M})} + \|\varepsilon^{q_t}\|^2_{L^2(\mathcal{M})}\right)\\ &\quad + \left[2\widetilde{C}_1\left(1+\frac{\widetilde{C}_1}{4\widetilde{\delta}_1}\right) + \frac{1}{\widetilde{\delta}_3}\left(1+\frac{1}{2\widetilde{\delta}_2}\right)^2\right]\|\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})}\\ &\quad + \left[2\widetilde{\delta}_1 + \frac{\widetilde{\delta}_2^2}{4\widetilde{\delta}_3} + \frac{\widetilde{C}_2^2}{2}\widetilde{\delta}_3 + \frac{\widetilde{\delta}_5}{2}(z_1-z_0)^2\right]\|\nabla\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})}.\end{aligned} \tag{75}$$

Finally, the total energy estimate is obtained. Combining the estimates (69), (70) and (75) together with the definition of the total error given in (64), we get

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon^{\mathrm{tot}}\|^2_{L^2(\mathcal{M})} + \min_{j\in\{1,2\}}\{\nu_j,\mu_j,\kappa_j\}\|\nabla\varepsilon^{\mathrm{tot}}\|^2_{L^2(\mathcal{M})} \le C_{1,\delta}(t)\|\varepsilon^{\mathrm{tot}}\|^2_{L^2(\mathcal{M})} + C_{2,\delta}\|\nabla\varepsilon^{\mathrm{tot}}\|^2_{L^2(\mathcal{M})}, \tag{76}$$

where,

$$\begin{aligned} C_{1,\delta}(t) &:= 3\widetilde{C}_1\left(1+\frac{\widetilde{C}_1}{4\widetilde{\delta}_1}\right) + \frac{1}{\widetilde{\delta}_3}\left(1+\frac{1}{2\widetilde{\delta}_2}\right)^2 + c_{q_t}(t)\left(1+\frac{c_{q_t}(t)}{2\widetilde{\delta}_4}\right) + \frac{3g^2 C^2_{\theta_0,\epsilon_0}(\max\{1,L_1\})^2}{2\widetilde{\delta}_5},\\ C_{2,\delta} &:= 4\widetilde{\delta}_1 + \frac{\widetilde{\delta}_2^2}{4\widetilde{\delta}_3} + \frac{3\widetilde{C}_2^2}{2}\widetilde{\delta}_3 + \frac{\widetilde{\delta}_4}{2} + \frac{\widetilde{\delta}_5}{2}(z_1-z_0)^2. \end{aligned}$$

For simplicity, we choose the parameters $\widetilde{\delta}_j,\ j\in\{1,2,3,4,5\}$, as follows

$$\begin{aligned} \widetilde{\delta}_1 &:= \frac{1}{40}\min_{j\in\{1,2\}}\{\nu_j,\mu_j,\kappa_j\}, \quad \widetilde{\delta}_2 := \frac{2}{5\sqrt{6}\widetilde{C}_2}\min_{j\in\{1,2\}}\{\nu_j,\mu_j,\kappa_j\},\\ \widetilde{\delta}_3 &:= \frac{1}{15\widetilde{C}_2^2}\min_{j\in\{1,2\}}\{\nu_j,\mu_j,\kappa_j\},\\ \widetilde{\delta}_4 &:= \frac{1}{5}\min_{j\in\{1,2\}}\{\nu_j,\mu_j,\kappa_j\}, \qquad \widetilde{\delta}_5 := \frac{1}{5(z_1-z_0)^2}\min_{j\in\{1,2\}}\{\nu_j,\mu_j,\kappa_j\}. \end{aligned}$$

This choice of the parameters $\widetilde{\delta}_j$ allows the coefficients for the term $\|\nabla\varepsilon^{\mathrm{tot}}\|^2_{L^2(\mathcal{M})}$ to be combined and appear on the left. This way, the inequality (76) reduces to

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon^{\mathrm{tot}}\|^2_{L^2(\mathcal{M})} + \frac{1}{2}\min_{j\in\{1,2\}}\{\nu_j,\mu_j,\kappa_j\}\|\nabla\varepsilon^{\mathrm{tot}}\|^2_{L^2(\mathcal{M})} \le C_{\mathrm{tot}}(t)\|\varepsilon^{\mathrm{tot}}\|^2_{L^2(\mathcal{M})}, \tag{77}$$

with

$$C_{\mathrm{tot}}(t) := 3\widetilde{C}_1\left(1+\frac{10\widetilde{C}_1}{\min_{j\in\{1,2\}}\{\nu_j,\mu_j,\kappa_j\}}\right) + \frac{15\widetilde{C}_2^2}{\min_{j\in\{1,2\}}\{\nu_j,\mu_j,\kappa_j\}}\left(1+\frac{5\sqrt{6}\widetilde{C}_2}{4\min_{j\in\{1,2\}}\{\nu_j,\mu_j,\kappa_j\}}\right)^2$$

$$+ c_{q_t}(t)\left(1 + \frac{5c_{q_t}(t)}{2\min\limits_{j\in\{1,2\}}\{\nu_j,\mu_j,\kappa_j\}}\right) + \frac{15(z_1-z_0)^2 g^2 C^2_{\theta_0,\epsilon_0}\left(\max\{1,L_1\}\right)^2}{\min\limits_{j\in\{1,2\}}\{\nu_j,\mu_j,\kappa_j\}}.$$

Integrating (77) over $[0,t]$, we get

$$\|\varepsilon^{\mathrm{tot}}(t)\|^2_{L^2(\mathcal{M})} \le \|\varepsilon^{\mathrm{tot}}(0)\|^2_{L^2(\mathcal{M})} + \int_0^t 2C_{\mathrm{tot}}(s)\|\varepsilon^{\mathrm{tot}}(s)\|^2_{L^2(\mathcal{M})}ds.$$

Finally, by virtue the Gronwall inequality (see [2, Lemma 1.1, eq. (1.77)]), we get

$$\|\varepsilon^{\mathrm{tot}}(t)\|^2_{L^2(\mathcal{M})} \le \|\varepsilon^{\mathrm{tot}}(0)\|^2_{L^2(\mathcal{M})}\exp\left(\int_0^t 2C_{\mathrm{tot}}(s)ds\right) = 0,$$

implying $\varepsilon^{\mathrm{tot}} \equiv 0$, and thus $(\mathbf{u}_{h1}, q_{t1}, \theta_{e1}) \equiv (\mathbf{u}_{h2}, q_{t2}, \theta_{e2})$.

We now proceed to prove the global regularity of the solution.

Proof of Theorem 1. The uniqueness is a direct consequence of the subsection 3.2.3. On the order hand, in Lemma 1 we have demonstrated that there is a unique local strong solution $(\mathbf{u}, q_t, \theta_e)$, satisfying

$$u, v, q_t, \theta_e, \in C([0,\mathcal{T}_0]; H^2(\mathcal{M})), w \in C([0,\mathcal{T}_0]; H^1(\mathcal{M}))$$

We extend the unique strong solution $(\mathbf{u}, q_t, \theta_e)$ to the maximal time of existence $\mathcal{T}_{\max}$. To obtain a global strong solution we need to prove $\mathcal{T}_{\max} = +\infty$. Suppose, by contradiction, that $\mathcal{T}_{\max} < +\infty$, then

$$\lim_{\mathcal{T}\to\mathcal{T}^-_{\max}} \|(\mathbf{u}_h, q_t, \theta_e)\|_{L^\infty(0,\mathcal{T}_0,H^2(\mathcal{M}))} = +\infty.$$

From the above it can be deduced that

$$\limsup_{\mathcal{T}\to\mathcal{T}^-_{\max}} \|(\mathbf{u}_h, q_t, \theta_e)\|_{\boldsymbol{H}^2(\mathcal{M})} = +\infty. \tag{78}$$

By Lemmas 3-5, we have that for some $\mathcal{T} \in (0,\mathcal{T}_0) \subset (0,\mathcal{T}_{\max})$

$$\sup_{0<t\le\mathcal{T}} \|(\mathbf{u}_h, q_t, \theta_e)\|_{\boldsymbol{H}^2(\mathcal{M})} \le \mathcal{K}, \tag{79}$$

where $\mathcal{K}$ is a positive constant independent of $\mathcal{T}$. i.e. depending only on initial and boundary data. The corresponding limits for the vertical velocity are analogous. Applying limits to (79) when $\mathcal{T} \to \mathcal{T}^-_{\max}$, we arrive at a contradiction of (78). Thus, $\mathcal{T}_{\max} = +\infty$. □

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Appendix A

In this Appendix, several lemmas needed for the existence and uniqueness proofs are proved. Lemmas that do not include a proof have been demonstrated in [5,14].

Linear Parabolic equations: Given a positive time $\mathcal{T}$, set $Q_{\mathcal{T}} := \Omega\times(0,\mathcal{T})$. Consider the parabolic problem

$$\begin{cases} \partial_t + \mathcal{L}u = F & \text{in } Q_{\mathcal{T}}, \\ Bu = G & \text{on } \Gamma\times(0,\mathcal{T}), \\ u(\cdot,0) = u_0, \end{cases}$$

where $\mathcal{L}$ is a elliptic operator. The boundary operator B and the boundary function G are given by

$$Bu := \begin{cases} \partial_n u + \alpha u = 0 & \text{on } \Gamma_s, \\ \partial_n u + \beta u = 0 & \text{on } \Gamma_0\cup\Gamma_1. \end{cases} \quad \text{and} \quad G := \begin{cases} \phi & \text{on } \Gamma_s, \\ \psi & \text{on } \Gamma_0\cup\Gamma_1. \end{cases}$$

Lemma 9. *[5] Consider a positive time $\mathcal{T}\in(0,\infty)$ and the initial data $u_0\in L^2(\Omega)$. We assume that*

$$0\le\alpha,\beta\in W^{1,\infty}(\Omega),\quad \phi\in L^2(0,T,H^{1/2}(\Gamma_s)),$$
$$\psi\in L^2(0,T,H^{1/2}(\Gamma_0\cup\Gamma_1)),\quad \partial_t B\in L^2(\partial\Omega\times(0,T))$$

hold, and that $F\in L^2(Q_{\mathcal{T}})$. Then, there is a unique weak solution to (8), satisfying

$$\begin{aligned}&\|u\|_{L^\infty(0,T,L^2(\Omega))}+\|u\|_{L^2(0,T,H^1(\Omega))}\\ &\quad\le C\left(\|F\|_{L^2(Q_{\mathcal{T}})}+\|u_0\|_{L^2(\Omega)}+\|\phi\|_{L^2(0,T,H^{1/2}(\Gamma_s))}+\|\psi\|_{L^2(0,T,H^{1/2}(\Gamma_0\cup\Gamma_1))}+\|\partial_t G\|_{L^2(0,T,L^2(\partial\Omega))}\right).\end{aligned}$$

Moreover, if we assume that $u_0\in H^1(\Omega)$, then the unique weak solution is a strong one, and satisfies

$$\begin{aligned}&\|u\|_{L^\infty(0,T;H^1(\Omega))}+\|u\|_{L^2(0,T,H^2(\Omega))}+\|\partial_t u\|_{L^2(Q_{\mathcal{T}}}\\ &\quad\le C\left(\|F\|_{L^2(Q_{\mathcal{T}})}+\|u_0\|_{H^1(\Omega)}+\|\phi\|_{L^2(0,T,H^{1/2}(\Gamma_s))}+\|\psi\|_{L^2(0,T,H^{1/2}(\Gamma_0\cup\Gamma_1))}+\|\partial_t G\|_{L^2(0,T,L^2(\partial\Omega))}\right).\end{aligned}$$

Proof. See [5, Corollary A.1] □

Lemma 10. *[27] Let $\mathcal{T}\in(0,+\infty)$ and let us consider X, Y and B three Banach spaces with $X\subset B\subset Y$ and compact embedding $X\to B$. Then,*

- *If $\mathcal{L}$ is a bounded subset of $L^p(0,\mathcal{T},X)$, with $1\le p<+\infty$ and $\frac{\partial f}{dt}$ is bounded in $L^1(0,\mathcal{T},Y)$, for all $f\in\mathcal{L}$. Then, $\mathcal{L}$ is relatively compact in $L^p(0,\mathcal{T};B)$.*
- *If $\mathcal{L}$ is bounded in $L^\infty(0,\mathcal{T},X)$ and $\frac{\partial f}{dt}$ is bounded in $L^r(0,\mathcal{T},Y)$, for all $f\in\mathcal{L}$ and $r>1$. Then, $\mathcal{L}$ is relatively compact in $C([0,\mathcal{T}];B)$*

Proof. See [27, Corollary 4] □

Lemma 11. *Let $\mathcal{M}'$ be a bounded domain of $\mathbb{R}^n, n\in\{2,3\}$, with Lipchitz continuous boundary $\partial\mathcal{M}'$. Then, given $f\in L^2(\mathcal{M}')$, there exists a unique $\bar{\mathrm{p}}\in H_0^1(\mathcal{M}') := \{v\in H^1(\mathcal{M}') : v=0 \text{ on } \partial\mathcal{M}'\}$ satisfying the Poisson problem*

$$\Delta\bar{\mathrm{p}} = f \text{ in } \mathcal{M}' \qquad \bar{\mathrm{p}} = 0 \text{ on } \partial\mathcal{M}'.$$

Moreover, there exists a constant $C_\mathrm{p} > 0$ *such that*

$$\|\nabla \bar{\mathrm{p}}\|_{L^2(\mathcal{M}')} \le C_\mathrm{p} \|f\|_{L^2(\mathcal{M}')}.$$

Proof. Using one of the Green identities (cf [8, Corollary 1.2 or Theorem 1.8]) we can derive the following variational formulation of the Poisson problem

$$\int_{\mathcal{M}'} \nabla \bar{\mathrm{p}} \cdot \nabla v = -\int_{\mathcal{M}'} f v, \qquad \forall\, v \in H_0^1(\mathcal{M}'). \tag{A.1}$$

A direct application of Lax–Milgram's Lemma (cf [8, Theorem 1.1] or [9, Theorem 4.1]) on the bilinear form $B(\bar{\mathrm{p}}, v) := \int_{\mathcal{M}'} \nabla \bar{\mathrm{p}} \cdot \nabla v$ and the functional $F(v) = -\int_{\mathcal{M}'} f v$, leads us to the existence and uniqueness of the solution. Furthermore, taking $v = \bar{\mathrm{p}}$ in (A.1) together with the Friedrichs–Poincaré inequality (cf [9, Lemma 4.1]), we arrive at

$$\|\nabla \bar{\mathrm{p}}\|_{L^2(\mathcal{M}')} \le \|f\|_{L^2(\mathcal{M}')} \|\bar{\mathrm{p}}\|_{L^2(\mathcal{M}')} \le C_\mathrm{p} \|f\|_{L^2(\mathcal{M}')} \|\nabla \bar{\mathrm{p}}\|_{L^2(\mathcal{M}')},$$

where C_p is the constant from the Friedrichs–Poincaré inequality. Finally by Young's inequality, we get

$$\|\nabla \bar{\mathrm{p}}\|_{L^2(\mathcal{M}')} \le \frac{1}{2\delta} C_\mathrm{p}^2 \|f\|^2_{L^2(\mathcal{M}')} + \frac{\delta}{2} \|\nabla \bar{\mathrm{p}}\|^2_{L^2(\mathcal{M}')}.$$

The proof is concluded by choosing δ small enough. □

Lemma 12. *Let* $\bar{\mathrm{p}}$ *be the pressure at the top surface, satisfying the Poisson equation* (13)*. Then, given* $\boldsymbol{S}^0 = (u_0, v_0, q_{t0}, \theta_{e0}) \in \boldsymbol{H}^2(\mathcal{M})$*, there exists a constant* $\bar{C}_{\bar{\mathrm{p}}} > 0$ *independent of the solution vector such that*

$$\|\nabla \bar{\mathrm{p}}\|_{L^2(\mathcal{M}')} \le \bar{C}_{\bar{\mathrm{p}}} \|\boldsymbol{S}^n\|_{\boldsymbol{H}^2(\mathcal{M})}.$$

Proof. We will start the proof by rewriting (13) as

$$\Delta \bar{\mathrm{p}} = (z_1 - z_0) \int_{z_0}^{z_1} F dz, \tag{A.2}$$

where

$$F = \nabla_h \cdot ((\mathbf{u}^n \cdot \nabla) \mathbf{u}_h^n) + \int_{z_0}^{z} \Delta_h b(q_t^n, \theta_e^n; \sigma) d\sigma - f_0 (\partial_x v^n - \partial_y u^n). \tag{A.3}$$

To avoid proliferation of unimportant constants, we will use the terminology $a \lesssim b$ whenever $a \le Cb$ and C is a positive constant independent of the solution. In what follows, we will prove that $F \in L^2(\mathcal{M})$. To do this, note that due to the fact that $(u_0, v_0, q_{t0}, \theta_{e0}) \in \boldsymbol{H}^2(\mathcal{M})$, and that the buoyancy force term involves the Laplacian of q_t, and θ_e, then we have

$$\begin{aligned} \left\| \int_{z_0}^{z} \Delta_h b(q_t^n, \theta_e^n; \sigma) d\sigma \right\|_{L^2(\mathcal{M})} &\lesssim \sqrt{6} |g| (z_1 - z_0)^2 C_{\theta_0, \epsilon_0} \|\boldsymbol{S}^n\|_{\boldsymbol{H}^2(\mathcal{M})}. \\ \|f_0 (\partial_x v^n - \partial_y u^n)\|_{L^2(\mathcal{M})} &\le 2|f_0| \|\boldsymbol{S}^n\|_{\boldsymbol{H}^2(\mathcal{M})}. \end{aligned} \tag{A.4}$$

In order to estimate the first term of F, defined in (A.3), we use the incompressibility condition (11e) to get

$$\nabla_h \cdot ((\mathbf{u}^n \cdot \nabla)\mathbf{u}_h^n) = 2(\partial_x u^n)^2 + 2(\partial_y v^n)^2 + 2u^n(\partial_{xx}u^n + \partial_{xy}v^n) + 2v^n(\partial_{yy}v^n + \partial_{xy}u^n)$$
$$+2\partial_x v^n \partial_y u^n + 2\partial_x u^n \partial_y v^n + \partial_x\partial_z(uw) + \partial_y\partial_z(vw).$$

The first few terms on the right hand side of the above identity can be estimated thanks to the continuous injection $\boldsymbol{i}_4$ of $\boldsymbol{H}^1(\Omega)$ into $\boldsymbol{L}^4(\Omega)$ with Sobolev constant C_{Sob} (see [26, Theorem 1.3.4]). For instance, $\|(\partial_x u^n)^2\|_{L^2(\mathcal{M})} = \|\partial_x u^n\|^2_{L^4(\mathcal{M})} \leq C^2_{\text{sob}}\|\partial_x u^n\|^2_{H^1(\mathcal{M})} \leq C^2_{\text{sob}}\|\boldsymbol{S}^n\|^2_{\boldsymbol{H}^2(\mathcal{M})}$ (the same estimate holds for $(\partial_y v^n)^2$). For other terms, we use the logarithmic type embedding inequality for anisotropic Sobolev spaces (see Lemma 18). That is, $\|u^n\partial_{xy}v^n\|_{L^2(\mathcal{M})} \leq \|u^n\|_{L^\infty(\mathcal{M})}\|\partial_{xy}v^n\|_{L^2(\mathcal{M})} \lesssim c_{\mathbf{u},\infty}(t)\|\boldsymbol{S}^n\|_{\boldsymbol{H}^2(\mathcal{M})}$. The terms $u^n\partial_{xx}u^n, v^n(\partial_{yy}v^n + \partial_{xy}u^n), 2\partial_x v^n\partial_y u^n$ and $2\partial_x u^n\partial_y v^n$ are estimated in a manner analogous to the above. Then, due to the fact that w^n vanishes in z_0 and z_1 (see Lemma 2), the last terms involved in the decomposition of $\nabla_h \cdot ((\mathbf{u}^n \cdot \nabla)\mathbf{u}_h^n)$ vanish when integrating with respect to z, (so they no longer appear in (A.2)), that is, $\int_{z_0}^{z_1}(\partial_x\partial_z(uw) + \partial_y\partial_z(vw))dz = \partial_x\left((uw)\big|_{z_0}^{z_1}\right) + \partial_y\left((vw)\big|_{z_0}^{z_1}\right) = 0$. Thus,

$$\|\nabla_h \cdot ((\mathbf{u}^n \cdot \nabla)\mathbf{u}_h^n)\|_{L^2(\mathcal{M})} \lesssim \|\boldsymbol{S}^n\|_{\boldsymbol{H}^2(\mathcal{M}} \tag{A.5}$$

According to the estimates of (A.4) and (A.5), and the definition of F in (A.3), we have shown that $F \in L^2(\mathcal{M})$. Thanks to the above fact, it is not difficult to see that the right side of (A.2) belongs to $L^2(\mathcal{M}')$. Indeed,

$$\left\|(z_1 - z_0)\int_{z_0}^{z_1} F dz\right\|^2_{L^2(\mathcal{M}')} = \int_{\mathcal{M}'}(z_1 - z_0)^2\left(\int_{z_0}^{z_1} F dz\right)^2 d\mathcal{M}'$$
$$\leq \int_{\mathcal{M}'}(z_1 - z_0)^4 \int_{z_0}^{z_1} F^2 dz d\mathcal{M}$$
$$= (z_1 - z_0)^4 \int_{\mathcal{M}} F^2 d\mathcal{M} \lesssim \|\boldsymbol{S}^n\|^2_{\boldsymbol{H}^2(\mathcal{M})}.$$

Finally applying Lemma 12 to the identity (A.2), we conclude $\|\nabla_h \bar{\text{p}}\|_{L^2(\mathcal{M}')} \leq \bar{C}_{\bar{\text{p}}}\|\boldsymbol{S}^n\|^2_{\boldsymbol{H}^2(\mathcal{M})}$, where $\bar{C}_{\bar{\text{p}}}$ is a positive constant that depends only on $C_p, C_{\text{sob}}, |f_0|, C_{\theta_0,\epsilon_0}, g, c_{\mathbf{u},\infty}$ and of some power of $(z_1 - z_0)$. □

Lemma 13. *Suppose that w satisfies the incompressibility condition (8e). Then*

$$\|w\|_{L^{r_1}(\mathcal{M})} \leq |z_1 - z_0|\|\nabla_h \cdot \mathbf{u}_h\|_{L^{r_1}(\mathcal{M})} = |z_1 - z_0|\|\partial_z w\|_{L^{r_1}(\mathcal{M})}.$$

Proof. Let Ω be any standard domain in $\mathbb{R}^n$ with $n \in \mathbb{N}$. By Hölder's inequality, it is known that

$$\int_\Omega |f| \leq \left(\int_\Omega |f|^{r_1}\right)^{1/r_1}\left(\int_\Omega 1^{r_2}\right)^{1/r_2} = |\Omega|^{1/r_2}\|f\|_{L^{r_1}(\Omega)} \qquad \forall\, f \in L^{r_1}(\Omega),$$

where $r_1, r_2 \in (0, +\infty)$ satisfying $\frac{1}{r_1} + \frac{1}{r_2} = 1$. So,

$$\left(\int_\Omega |f|\right)^{r_1} \leq |\Omega|^{r_1/r_2}\|f\|^{r_1}_{L^{r_1}(\Omega)}, \qquad \forall\, f \in L^{r_1}(\Omega). \tag{A.6}$$

On the other hand, from the incompressibility condition (8e), we get for $z \in (z_0, z_1)$

$$w(x,y,z,t) = \int_z^{z_1} \nabla_h \cdot \mathbf{u}_h(x,y,s,t)ds,$$

from which it follows that

$$\begin{aligned}\|w\|^{r_1}_{L^{r_1}(\mathcal{M})} &= \int_{\mathcal{M}} |w|^{r_1} d\mathcal{M} \leq \int_{\mathcal{M}'} \int_{z_0}^{z_1} \left(\int_z^{z_1} |\nabla_h \cdot \mathbf{u}_h| ds \right)^{r_1} dz d\mathcal{M}' \\ &\leq \int_{\mathcal{M}'} \int_{z_0}^{z_1} \left(\int_{z_0}^{z_1} |\nabla_h \cdot \mathbf{u}_h| ds \right)^{r_1} dz d\mathcal{M}'.\end{aligned}$$

Applying (A.6) for $\Omega = [z_0, z_1]$, we get

$$\begin{aligned}\|w\|^{r_1}_{L^{r_1}(\mathcal{M})} &\leq \int_{\mathcal{M}'} \int_{z_0}^{z_1} |z_1 - z_0|^{r_1/r_2} \|\nabla_h \cdot \mathbf{u}_h\|^{r_1}_{L^{r_1}([z_0,z_1])} dz d\mathcal{M}' \\ &\leq |z_1 - z_0|^{r_1/r_2} \int_{\mathcal{M}'} \left(\|\nabla_h \cdot \mathbf{u}_h\|^{r_1}_{L^{r_1}([z_0,z_1])} \int_{z_0}^{z_1} dz \right) d\mathcal{M}' \\ &= |z_1 - z_0|^{\frac{r_1+r_2}{r_2}} \int_{\mathcal{M}'} \int_{z_0}^{z_1} |\nabla_h \cdot \mathbf{u}_h|^{r_1} dz d\mathcal{M} \\ &= |z_1 - z_0|^{\frac{r_1+r_2}{r_2}} \|\nabla_h \cdot \mathbf{u}_h\|^{r_1}_{L^{r_1}(\mathcal{M})}.\end{aligned}$$

The proof is concluded by combining the relationship between r_1 and r_2 and the previous estimate. □

Lemma 14. *Suppose that (8e) and the boundary conditions (10) are satisfied. Then, for any measurable function $f \in H^1(\mathcal{M})$, the following identify holds*

$$\int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h f + w\partial_z f) f d\mathcal{M} = 0.$$

Proof. It follows from integration by parts and the identity $(\partial_{x_i} f)f = \frac{1}{2}\partial_{x_i}(f^2)$, that

$$\begin{aligned}&\int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h f + w\partial_z f) f d\mathcal{M} \\ &= \int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h f) f d\mathcal{M} + \int_{\mathcal{M}} (w\partial_z f) f d\mathcal{M} = \frac{1}{2} \int_{\mathcal{M}} \mathbf{u}_h \cdot \nabla_h (f^2) d\mathcal{M} + \frac{1}{2} \int_{\mathcal{M}} w\partial_z (f^2) d\mathcal{M} \\ &= \frac{1}{2} \left(- \int_{\mathcal{M}} (\nabla_h \cdot \mathbf{u}_h) f^2 d\mathcal{M} + \int_{\partial\mathcal{M}} (\mathbf{u}_h \cdot \boldsymbol{n}) f^2 d(\partial\mathcal{M}) - \int_{\mathcal{M}} (\partial_z w) f^2 d\mathcal{M} + \int_{\mathcal{M}'} (wf^2)\Big|_{z_0}^{z_1} d\mathcal{M}' \right).\end{aligned}$$

By the boundary conditions (10), the second and fourth integrals vanish, since $\mathbf{u} \cdot \boldsymbol{n} = \partial_{\boldsymbol{n}} \mathbf{u} = 0$ over $\partial\mathcal{M}$ and $w = 0$ at $z = z_0$ and $z = z_1$. Then

$$\int_{\mathcal{M}} (\mathbf{u}_h \cdot \nabla_h f + w\partial_z f) f d\mathcal{M} = -\frac{1}{2}\left(\int_{\mathcal{M}} (\nabla_h \cdot \mathbf{u}_h + \partial_z w) f^2 d\mathcal{M}\right) = 0,$$

where in the last step we have used the incompressibility condition given in (8e). □

Lemma 15. *Suppose that (8e) is satisfied. Let f be one of the solution variables u, v, q_t or θ_e satisfying the boundary conditions (10). Then, the following estimate holds*

$$\int_{\mathcal{M}} (\lambda_1 \Delta_h f + \lambda_2 \partial_z^2 f) f d\mathcal{M} \leq -\lambda_1 \|\nabla_h f\|^2_{L^2(\mathcal{M})} - \lambda_2 \|\partial_z f\|^2_{L^2(\mathcal{M})},$$

with λ_1 and λ_2 are positive constants.

Proof. It follows from integration by parts that

$$\begin{aligned}
\int_{\mathcal{M}} (\lambda_1 \Delta_h f + \lambda_2 \partial_z^2 f) f d\mathcal{M} &= \lambda_1 \int_{\mathcal{M}} (\Delta_h f) f \, d\mathcal{M} + \lambda_2 \int_{\mathcal{M}} (\partial_z^2 f) f \, d\mathcal{M} \\
&= -\lambda_1 \int_{\mathcal{M}} (\nabla_h f \cdot \nabla_h f) \, d\mathcal{M} + \lambda_1 \int_{\Gamma_s} (\partial_n f) f \, d\Gamma_s - \lambda_2 \int_{\mathcal{M}} (\partial_z f)(\partial_z f) \, d\mathcal{M} \\
&\quad + \lambda_2 \int_{\mathcal{M}'} (\partial_z f) f \Big|_{z_0}^{z_1} d\mathcal{M}'.
\end{aligned}$$

Then, using the boundary conditions (10) for $f \in \{\mathbf{u}, q_t, \theta_e\}$, we deduce that

$$\begin{aligned}
\int_{\mathcal{M}} (\lambda_1 \Delta_h f + \lambda_2 \partial_z^2 f) f \, d\mathcal{M} = -\lambda_1 \|\nabla_h f\|^2_{L^2(\mathcal{M})} - \lambda_2 \|\partial_z f\|^2_{L^2(\mathcal{M})} - \alpha_{\diamond s} \lambda_1 \int_{\Gamma_s} f^2 d\Gamma_s \\
- \alpha_{\star 1} \lambda_2 \int_{\mathcal{M}'} f^2 \Big|_{z=z_1} d\mathcal{M}' - \alpha_{\star 0} \lambda_2 \int_{\mathcal{M}'} f^2 \Big|_{z=z_0} d\mathcal{M}'.
\end{aligned}$$

The proof is concluded, thanks to the fact that the last three integrals are non-negative since the scalars $\lambda_j, \alpha_{\star 0}, \alpha_{\star 1}, \alpha_{s\diamond} \geq 0$ for all $j \in \{1, 2\}, \star \in \{q_t, \theta_e\}, \diamond \in \{q_t, \theta_e\}$. □

Lemma 16. *[14] For any measurable function f satisfying $f, \partial_z f \in L^1(\mathcal{M})$ the following estimates hold*

$$\sup_{z_0 \leq z \leq z_1} \|f\|_{L^1(\mathcal{M}')} \leq \frac{1}{z_1 - z_0} \|f\|_{L^1(\mathcal{M})} + \|\partial_z f\|_{L^1(\mathcal{M})}.$$

Proof. See [14, Lemma 1-Appendix] □

Lemma 17. *[5] Let $f_1, f_2 \in L^2(\mathcal{M})$ be such that $\nabla_h f_1, \nabla_h f_2 \in L^2(\mathcal{M})$. Then, the following inequalities hold*

$$\begin{aligned}
&\int_{\mathcal{M}'} \left(\int_{z_0}^{z_1} |f_1| dz\right) \left(\int_{z_0}^{z_1} |f_2 f_3| dz\right) d\mathcal{M}' \\
&\leq \widetilde{c}_1 \|f_1\|_{L^2(\mathcal{M})} \|f_2\|^{1/2}_{L^2(\mathcal{M})} \left(\|f_2\|^{1/2}_{L^2(\mathcal{M})} + \|\nabla_h f_2\|^{1/2}_{L^2(\mathcal{M})}\right) \|f_3\|^{1/2}_{L^2(\mathcal{M})} \left(\|f_3\|^{1/2}_{L^2(\mathcal{M})} + \|\nabla_h f_3\|^{1/2}_{L^2(\mathcal{M})}\right),
\end{aligned}$$

and

$$\int_{\mathcal{M}'} \left(\int_{z_0}^{z_1} |f_1| dz \right) \left(\int_{z_0}^{z_1} |f_2 f_3| dz \right) d\mathcal{M}'$$
$$\leq \widetilde{c}_2 \|f_1\|_{L^2(\mathcal{M})}^{1/2} \left(\|f_1\|_{L^2(\mathcal{M})}^{1/2} + \|\nabla_h f_1\|_{L^2(\mathcal{M})}^{1/2} \right) \|f_2\|_{L^2(\mathcal{M})}^{1/2} \left(\|f_2\|_{L^2(\mathcal{M})}^{1/2} + \|\nabla_h f_2\|_{L^2(\mathcal{M})}^{1/2} \right) \|f_3\|_{L^2(\mathcal{M})}.$$

Proof. See [5, Lemma 2.1]. □

Lemma 18. *[4] Let* $\mathbf{r} = (r_1, r_2, r_3)$ *with* $r_i \in (1, +\infty)$ *such that* $\sum_{i=1}^{3} \frac{1}{r_i} < 1$. *Then, for any function* f *on* $\Omega \subseteq \mathbb{R}^3$, *we have for any* $\lambda > 0$ *that*

$$\|f\|_{L^\infty(\Omega)} \leq C_\lambda \max \left\{ 1, \sup_{s \geq 2} \frac{\|f\|_{L^s(\Omega)}}{s^\lambda} \right\} \log^\lambda \left(\sum_{i=1}^{3} (\|f\|_{L^{r_i}(\Omega)} + \|\partial_i f\|_{L^{r_i}(\Omega)}) + e \right)$$

Proof. See [4, Lemma 2.4]. □

Lemma 19. *Suppose that (8e) is satisfied. Consider measurable functions* $f_j \in H^1(\mathcal{M})$ *with* $j \in \{1, 2\}$ *satisfying the boundary conditions (10) and*

$$\|\nabla_h \partial_z f_j\|_{L^2(\mathcal{M})} \leq \text{constant}.$$

Then, there are positive constants $\widetilde{C}_j,\ j \in \{1, 2\}$ *and* $\widetilde{\delta}_j\ j \in \{1, 2, 3\}$, *such that the following estimate holds*

$$\int_{\mathcal{M}} \left[(\mathbf{u}_{h1} \cdot \nabla_h f_1 + w_1 \partial_z f_1) - (\mathbf{u}_{h2} \cdot \nabla_h f_2 + w_2 \partial_z f_2) \right] \varepsilon^f \, d\mathcal{M}$$
$$\leq \left[\widetilde{C}_1 \left(1 + \frac{\widetilde{C}_1}{4\widetilde{\delta}_1} \right) + \frac{1}{\widetilde{\delta}_3} \left(1 + \frac{1}{2\widetilde{\delta}_2} \right)^2 \right] \|\varepsilon^f\|_{L^2(\mathcal{M})}^2 + \widetilde{C}_1 \left(1 + \frac{\widetilde{C}_1}{4\widetilde{\delta}_1} \right) \|\varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})}^2$$
$$+ \left[\widetilde{\delta}_1 + \frac{\widetilde{\delta}_2^2}{4\widetilde{\delta}_3} \right] \|\nabla_h \varepsilon^f\|_{L^2(\mathcal{M})}^2 + \left[\widetilde{\delta}_1 + \frac{\widetilde{C}_2^2}{2} \widetilde{\delta}_3 \right] \|\nabla_h \varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})}^2,$$

where $\varepsilon^f = f_1 - f_2$.

Proof. We first note that

$$\begin{aligned}
&(\mathbf{u}_{h1} \cdot \nabla_h f_1 + w_1 \partial_z f_1) - (\mathbf{u}_{h2} \cdot \nabla_h f_2 + w_2 \partial_z f_2) \\
&\quad = (\mathbf{u}_{h1} \cdot \nabla_h f_1 + w_1 \partial_z f_1) - (\mathbf{u}_{h2} \cdot \nabla_h f_2 + w_2 \partial_z f_2) \pm (\mathbf{u}_{h1} \cdot \nabla_h f_2 + w_1 \partial_z f_2) \\
&\quad = \mathbf{u}_{h1} \cdot \nabla_h \varepsilon^f + w_1 \partial_z \varepsilon^f + \varepsilon^{\mathbf{u}_h} \cdot \nabla_h f_2 + \varepsilon^w \partial_z f_2.
\end{aligned}$$

Multiplying by ε^f and integrating over $\mathcal{M}$, we get

$$\begin{aligned}
&\int_{\mathcal{M}} \left[(\mathbf{u}_{h1} \cdot \nabla_h f_1 + w_1 \partial_z f_1) - (\mathbf{u}_{h2} \cdot \nabla_h f_2 + w_2 \partial_z f_2) \right] \varepsilon^f \, d\mathcal{M} \\
&\quad = \int_{\mathcal{M}} [\mathbf{u}_{h1} \cdot \nabla_h \varepsilon^f + w_1 \partial_z \varepsilon^f + \varepsilon^{\mathbf{u}_h} \cdot \nabla_h f_2 + \varepsilon^w \partial_z f_2] \varepsilon^f \, d\mathcal{M} \\
&\quad = \int_{\mathcal{M}} [\mathbf{u}_{h1} \cdot \nabla_h \varepsilon^f + w_1 \partial_z \varepsilon^f] \varepsilon^f \, d\mathcal{M} + \int_{\mathcal{M}} [\varepsilon^{\mathbf{u}_h} \cdot \nabla_h f_2 + \varepsilon^w \partial_z f_2] \varepsilon^f \, d\mathcal{M}.
\end{aligned}$$

The first integral on the right hand side vanishes thanks to Lemma 14. Thus,

$$\int_{\mathcal{M}} [(\mathbf{u}_{h1} \cdot \nabla_h f_1 + w_1 \partial_z f_1) - (\mathbf{u}_{h2} \cdot \nabla_h f_2 + w_2 \partial_z f_2)]\, \varepsilon^f d\mathcal{M} = \int_{\mathcal{M}} \varepsilon^{\mathbf{u}_h} \cdot \nabla_h f_2 d\mathcal{M} + \int_{\mathcal{M}} \varepsilon^w \partial_z f_2] \varepsilon^f d\mathcal{M}. \tag{A.7}$$

In order to bound the second integral, we decompose it into two parts

$$\begin{aligned}
&\int_{\mathcal{M}} (\varepsilon^{\mathbf{u}_h} \cdot \nabla_h f_2) \varepsilon^f d\mathcal{M} \\
&\leq \int_{\mathcal{M}'} \left[\frac{1}{z_1 - z_0} \int_{z_0}^{z_1} |\nabla_h f_2| dz + \int_{z_0}^{z_1} |\nabla_h \partial_z f_2| dz \right] \int_{z_0}^{z_1} |\varepsilon^{\mathbf{u}_h}| |\varepsilon^f| dz d\mathcal{M}' \\
&\leq \widetilde{c}_1 \left(\frac{1}{z_1 - z_0} + 1 \right) (\|\nabla_h f_2\|_{L^2(\mathcal{M})} + \|\partial_z \nabla_h f_2\|_{L^2(\mathcal{M})}) \times \|\varepsilon^f\|^{1/2}_{L^2(\mathcal{M})} \left(\|\varepsilon^f\|^{1/2}_{L^2(\mathcal{M})} + \|\nabla_h \varepsilon^f\|^{1/2}_{L^2(\mathcal{M})} \right) \times \\
&\quad \times \|\varepsilon^{\mathbf{u}_h}\|^{1/2}_{L^2(\mathcal{M})} \left(\|\varepsilon^{\mathbf{u}_h}\|^{1/2}_{L^2(\mathcal{M})} + \|\nabla_h \varepsilon^{\mathbf{u}_h}\|^{1/2}_{L^2(\mathcal{M})} \right).
\end{aligned}$$

Owing to the fact that $f_2 \in H^1(\mathcal{M})$ and the term $\|\partial_z \nabla_h f_2\|_{L^2(\mathcal{M})}$ is bounded, then there exists a constant $\widetilde{C}_1 > 0$ such that

$$\widetilde{c}_1 \left(\frac{1}{z_1 - z_0} + 1 \right) (\|\nabla_h f_2\|_{L^2(\mathcal{M})} + \|\partial_z \nabla_h f_2\|_{L^2(\mathcal{M})}) \leq \widetilde{C}_1.$$

Therefore, combining the above together with the fact that $(a^{1/2} + b^{1/2})(c^{1/2} + d^{1/2}) \leq a + b + c + d$, we arrive at

$$\begin{aligned}
&\int_{\mathcal{M}} (\varepsilon^{\mathbf{u}_h} \cdot \nabla_h f_2) \varepsilon^f d\mathcal{M} \\
&\leq \widetilde{C}_1 \|\varepsilon^f\|^{1/2}_{L^2(\mathcal{M})} \|\varepsilon^{\mathbf{u}_h}\|^{1/2}_{L^2(\mathcal{M})} \left(\|\varepsilon^f\|_{L^2(\mathcal{M})} + \|\nabla_h \varepsilon^f\|_{L^2(\mathcal{M})} + \|\varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})} + \|\nabla_h \varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})} \right) \\
&\leq \frac{\widetilde{C}_1}{2} \left(\|\varepsilon^f\|_{L^2(\mathcal{M})} + \|\varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})} \right) \left(\|\varepsilon^f\|_{L^2(\mathcal{M})} + \|\nabla_h \varepsilon^f\|_{L^2(\mathcal{M})} + \|\varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})} + \|\nabla_h \varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})} \right) \\
&\leq \frac{\widetilde{C}_1}{2} \left(\|\varepsilon^f\|_{L^2(\mathcal{M})} + \|\varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})} \right)^2 + \frac{\widetilde{C}_1}{2} \left(\|\varepsilon^f\|_{L^2(\mathcal{M})} + \|\varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})} \right) \left(\|\nabla_h \varepsilon^f\|_{L^2(\mathcal{M})} + \|\nabla_h \varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})} \right).
\end{aligned}$$

Now, by Young's inequality, one arrives at

$$\begin{aligned}
\int_{\mathcal{M}} (\varepsilon^{\mathbf{u}_h} \cdot \nabla_h f_2) \varepsilon^f d\mathcal{M} &\leq \widetilde{C}_1 \left(\|\varepsilon^f\|^2_{L^2(\mathcal{M})} + \|\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})} \right) + \frac{\widetilde{C}_1^2}{4\widetilde{\delta}_1} \left(\|\varepsilon^f\|^2_{L^2(\mathcal{M})} + \|\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})} \right) \\
&\quad + \widetilde{\delta}_1 \left(\|\nabla_h \varepsilon^f\|^2_{L^2(\mathcal{M})} + \|\nabla_h \varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})} \right).
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\int_{\mathcal{M}} (\varepsilon^{\mathbf{u}_h} \cdot \nabla_h f_2) \varepsilon^f d\mathcal{M} &\leq \widetilde{C}_1 \left(1 + \tfrac{\widetilde{C}_1}{4\widetilde{\delta}_1} \right) \left(\|\varepsilon^f\|^2_{L^2(\mathcal{M})} + \|\varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})} \right) \\
&\quad + \widetilde{\delta}_1 \left(\|\nabla_h \varepsilon^f\|^2_{L^2(\mathcal{M})} + \|\nabla_h \varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})} \right).
\end{aligned} \tag{A.8}$$

Analogously, we can deduce that

$$\begin{aligned}\int_{\mathcal{M}} (\varepsilon^w \partial_z f_2)\varepsilon^f \, d\mathcal{M} &\leq \int_{\mathcal{M}'} \int_{z_0}^{z_1} |\nabla_h \varepsilon^{\mathbf{u}_h}| dz \int_{z_0}^{z_1} |\partial_z f_2||\varepsilon^f| dz \\ &\leq \widetilde{C}_2 \|\nabla_h \varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})} \|\varepsilon^f\|^{1/2}_{L^2(\mathcal{M})} \left(\|\varepsilon^f\|^{1/2}_{L^2(\mathcal{M})} + \|\nabla_h \varepsilon^f\|^{1/2}_{L^2(\mathcal{M})} \right),\end{aligned}$$

where $\widetilde{C}_2$ is a positive constant satisfying

$$\widetilde{c}_1 \|\partial_z f_2\|^{1/2}_{L^2(\mathcal{M})} \left(\|\partial_z f_2\|^{1/2}_{L^2(\mathcal{M})} + \|\nabla_h \partial_z f_2\|^{1/2}_{L^2(\mathcal{M})} \right) \leq \widetilde{C}_2.$$

Applying Young's inequality and algebraic arrangements, it turns out that

$$\begin{aligned}\int_{\mathcal{M}} (\varepsilon^w \partial_z f_2)\varepsilon^f \, d\mathcal{M} &\leq \widetilde{C}_2 \|\nabla_h \varepsilon^{\mathbf{u}_h}\|_{L^2(\mathcal{M})} \left(\left(1 + \frac{1}{2\widetilde{\delta}_2}\right) \|\varepsilon^f\|_{L^2(\mathcal{M})} + \frac{\widetilde{\delta}_2}{2} \|\nabla_h \varepsilon^f\|_{L^2(\mathcal{M})} \right) \\ &\leq \frac{1}{2\widetilde{\delta}_3} \left(\left(1 + \frac{1}{2\widetilde{\delta}_2}\right) \|\varepsilon^f\|_{L^2(\mathcal{M})} + \frac{\widetilde{\delta}_2}{2} \|\nabla_h \varepsilon^f\|_{L^2(\mathcal{M})} \right)^2 + \frac{\widetilde{C}_2^2}{2} \widetilde{\delta}_3 \|\nabla_h \varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})}.\end{aligned}$$

This fact together with the inequality $(a+b)^2 \leq 2(a^2+b^2)$ for all $a, b > 0$, it implies that

$$\int_{\mathcal{M}} (\varepsilon^w \partial_z f_2)\varepsilon^f \, d\mathcal{M} \leq \frac{1}{\widetilde{\delta}_3} \left(1 + \frac{1}{2\widetilde{\delta}_2}\right)^2 \|\varepsilon^f\|^2_{L^2(\mathcal{M})} + \frac{\widetilde{\delta}_2^2}{4\widetilde{\delta}_3} \|\nabla_h \varepsilon^f\|^2_{L^2(\mathcal{M})} + \frac{\widetilde{C}_2^2}{2} \widetilde{\delta}_3 \|\nabla_h \varepsilon^{\mathbf{u}_h}\|^2_{L^2(\mathcal{M})}. \tag{A.9}$$

The proof is concluded by replacing the inequalities (A.8) and (A.9) into (A.7). □

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