$$\frac{1}{2c}\int_{x-ct}^{x+ct} G(\bar{x})\,d\bar{x} = \frac{1}{2c}\int_{x-ct}^{x+ct}\sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\frac{n\pi \bar{x}}{L}\,d\bar{x}$$

$$= \frac{1}{\cancel{2c}}\sum_{n=1}^{\infty} B_n \frac{\cancel{n\pi c}}{\cancel{L}}\,\frac{-\cancel{L}}{\cancel{n\pi}}\cos\frac{n\pi\bar{x}}{L}\Bigg|_{x-ct}^{x+ct}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} -B_n\cos\frac{n\pi\bar{x}}{L}\Bigg|_{x-ct}^{x+ct} = \frac{1}{2}\sum_{n=1}^{\infty}\left(B_n\cos\left(\frac{n\pi}{L}(x-ct)\right) - B_n\cos\left(\frac{n\pi}{L}(x+ct)\right)\right)$$

$$= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi ct}{L}\sin\frac{n\pi x}{L} = u(x,t)$$

Therefore $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\bar{x})\,d\bar{x}$.

Problem 3: From (4.4.1) derive conservation of energy for a vibrating string,

$$\frac{dE}{dt} = c^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Bigg|_0^L,$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$.

Answer:

$$E(t) = \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$\Rightarrow \frac{dE}{dt} = \int_0^L \frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2\right) dx$$

$$\frac{dE(t)}{dt} = \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x \partial t}\, dx$$

$$= \int_0^L \frac{\partial u}{\partial t}\, c^2 \frac{\partial^2 u}{\partial x^2} + c^2 \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\, dx \leftarrow$$ Because u satisfies the wave eqn.

$$= c^2 \int_0^L \cancel{\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\, dx$$

If you don't know what to do, integrate by parts:

$$c^2 \int_0^L \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\, dx = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^L \cancel{- c^2 \int_0^L \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t}\, dx}$$

$$\Rightarrow \frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^L$$

Problem 4: Let $u(x,t) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$ be a solution of the vibrating string problem. Suppose the string is constrained so that $u(\frac{L}{3},t) = 0$ for all t. What conditions does this impose on the coefficients B_n?

$$0 = u\left(\tfrac{L}{3}, t\right) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi}{3}\right)$$

$\Rightarrow$ Since $\cos\left(\frac{n\pi ct}{L}\right)$ form a linearly independent set of functions, then

$$B_n \sin\left(\frac{n\pi}{3}\right) = 0$$

$\sin\left(\frac{n\pi}{3}\right) = 0$ only when $\frac{n}{3}$ is an integer

$\Rightarrow B_n = 0$ if n is not a multiple of 3

That is, $B_n = 0$ if $n = 3k+1$ or $3k+2$, k integer.

Problem 5: The voltage $v(x,t)$ in a transmission cable is known to satisfy the PDE:

$$\cancel{\ } v_{tt} + 2a v_t + a^2 v = c^2 v_{xx},$$

where a and c are positive constants. Let $u(x,t) = e^{a\cancel{t}}\, v(x,t)$ and show ~~that~~ that u satisfies the wave eqn. $u_{tt} = c^2 u_{xx}$.

$$u_t = a e^{at} v + e^{at} v_t$$

$$u_{tt} = a^2 e^{at} v + a e^{at} v_t + a e^{at} v_t + e^{at} v_{tt}$$

$$= a^2 e^{at} v + 2a e^{at} v_t + e^{at} v_{tt}$$

$$u_{xx} = e^{at} v_{xx}$$

$$\Rightarrow u_{tt} - c^2 u_{xx} = e^{at}\left(a^2 v + 2a v_t + v_{tt} - c^2 v_{xx}\right)$$

$$= e^{at} \cdot 0 = 0$$

$$\Rightarrow u_{tt} = c^2 u_{xx}.$$