MATH 322 , SPRING 2013
PRACTICE PROBLEMS FOR CHAPTERS 5 AND 6

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Problem (*Textbook problem 5.3.4*)
Consider heat flow with convection

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - V_0\frac{\partial u}{\partial x}$$

(a) Show that the spatial ordinary differential equation obtained by separation of variables is not in Sturm-Liouville form.
(b) Solve the initial value problem

$$u(0,t) = 0, u(L,t) = 0, u(x,0) = f(x).$$

(c) Solve the initial value problem

$$\frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0, u(x,0) = f(x).$$

Problem (*Textbook problem 5.3.9*)
COnsider the eigenvalue problem

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0$$

with

$$\phi(1) = 0,\ \phi(b) = 0.$$

(a) Show that multiplying by $1/x$ puts this in the Sturm-Liouville form
(b) Show that $\lambda \geq 0$
(c) Since (5.3.10) is an equidimensional equation, determine all positive eigenvalues. Is $\lambda = 0$ an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest, but no largest.
(d) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.
(e) Show that the nth eigenfunction has $n-1$ zeros.

Problem (*Textbook problem 5.4.2*) Consider

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)$$

where c, ρ, K_0 are functions of x, subject to

$$\frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0, u(x,0) = f(x).$$

Assume that the appropriate eigenfunctions are known. Solve the initial value problem, briefly discuss $\lim_{t\to\infty} u(x,t)$.

Problem (*Textbook problem 5.4.6*) Consider the vibrations of a nonuniform string of mass density $\rho_0(x)$. Suppose that the left end at $x = 0$ is fixed and the right end obeys the elastic boundary condition: $\partial u/\partial x = -(k/T_0)u$ at $x = L$. Suppose that the string is initially at rest with a known initial position $f(x)$. Solve the initial value problem. (*Hints:* Assume that the appropriate eigenvalues and corresponding eigenfunctions are known. What differential equations with what boundary conditions do they satisfy? The eigenfunctions are orthogonal with what weighting function?)

Problem (*Textbook problem 5.5.1*) A Sturm-Liouville eigenvalue problem is called self-adjoint if

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b = 0$$

since then $\int_a^b [uL(v) - vL(u)]dz = 0$ for any two functions u and v satisfying the boundary conditions. Show that the following yield self-adjoint problems:

(a) $\phi(0) = 0$ and $\phi(L) = 0$
(b) $\frac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$
(c) $\frac{d\phi}{dx}(0) - h\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$
(d) $\phi(a) = \phi(b)$ and $p(a)\frac{d\phi}{dx}(a) = p(b)\frac{d\phi}{dx}(b)$
(e) $\phi(a) = \phi(b)$ and $\frac{d\phi}{dx}(a) = \frac{d\phi}{dx}(b)$ [self adjoint only if $p(a) = p(b)$]
(f) $\phi(L) = 0$ and [in the situation which $p(0) = 0$] $\phi(0)$ bounded and $\lim_{x\to 0} p(x)\frac{d\phi}{dx} = 0$
(g) Under what conditions is the following self-adjoint (if p is constant)?

$$\phi(L) + \alpha\phi(0) + \beta\frac{d\phi}{dx}(0) = 0$$
$$\frac{d\phi}{dx}(L) + \gamma\phi(0) + \delta\frac{d\phi}{dx}(0) = 0$$

Problem (*Textbook problem 5.5.4*) Give an example of an eigenvalue with more than one eigenfunction corresponding to an eigenvalue.

Problem (*Textbook problem 5.5.9*) For the eigenvalue problem

$$\frac{d^4\phi}{dx^4} + \lambda e^x\phi = 0$$

subject to the boundary conditions

$$\phi(0) = 0,\ \phi(1) = 0$$
$$\frac{d\phi}{dz}(0) = 0,\ \frac{d^2\phi}{dx^2}(1) = 0,$$

show that the eigenvalues are less than or equal to zero ($\lambda \leq 0$). (Don't worry; in a physical contact that is exactly what is expected.) Is $\lambda = 0$ an eigenvalue?

Problem (*Textbook problem 5.6.1*) Use the Rayleigh quotient to obtain a (reasonably accrue) upper bound for the lowest eigenvalue of

(a)
$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0,\ \frac{d\phi}{dx}(0) = 0,\ \phi(1) = 0$$

(b)
$$\frac{d^2\phi}{dx^2} + (\lambda - x)\phi = 0,\ \frac{d\phi}{dx}(0) = 0,\ \frac{d\phi}{dx}(1) + 2\phi(1) = 0$$

(c)
$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,\ \phi(0) = 0,\ \frac{d\phi}{dx}(1) + \phi(1) = 0$$

Problem (*Textbook problem 5.7.2*) Consider heat flow in a one-dimensional rod without sources with nonconstant thermal properties. Assume that the temperature is zero at $x = 0$ and $x = L$. Suppose that $c\rho_{\min} \leq c\rho \leq c\rho_{\max}$, and $K_{\min} \leq K_0(x) \leq K_{\max}$. Obtain an upper and (nonzero) lower bound on the lowest exponential rate of decay of the product solution.

Problem (*Textbook problem 5.8.9*) Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,\ \phi(0) = \frac{d\phi}{dx}(0),\ \phi(1) = \beta\frac{d\phi}{dx}(1)$$

For what values (if any) of β is $\lambda = 0$ an eigenvalue?

Problem (*Textbook problem 5.9.2*) Consider

$$\frac{d^2\phi}{dx^2} + \lambda(1+x)\phi = 0$$

subject to $\phi(0) = 0$ and $\phi(1) = 0$. Roughly sketch the eigenfunctions for λ large. Take into account amplitude and period variations.

Problem (*Textbook problem 6.2.6*) Derive an approximation for $\partial^2 u/\partial x \partial y$ whose truncation error is $O(\Delta x^2)$. (*Hint:* Twice apply the centered difference approximations for first-order partial derivatives.)

Problem (*Textbook problem 6.2.7*) How well does $\frac{1}{2}[f(x)+f(x+\Delta x)]$ approximate $f(x+\Delta x/2)$ (i.e. what s the truncation error)?

Problem (*Textbook problem 6.3.5*) Show that at each successive mesh point the sign go the solution alternates for the most unstable mode (of our numerical scheme for the heat equation, $s > 1/2$).

Problem (*Textbook problem 6.3.8*) Under what conditions will an initial positive solution $[u(x,0) > 0]$ remain positive $[u(x,t) > 0]$ for our numerical scheme (6.3.9) for the heat equation?

Problem (*Textbook problem 6.3.9*) Consider

$$\frac{d^2u}{dx^2} = f(x), \; u(0) = 0, \; u(L) = 0.$$

(a) Using the centered difference approximation for the second-derivative and dividing the length L into three equal mesh lengths derive a system of linear equations for an approximation to $u(x)$. USe the notation $x_i = i\Delta x$, $f_i = f(x_i)$ and $u_i = u(x_i)$. (*Note:* $x_0 = 0$, $x_1 = \frac{1}{3}L, x_2 = \frac{2}{3}L, x_3 = L$.)

(b) Write the system as a matrix system $\mathbf{Au} = \mathbf{f}$. What is $\mathbf{A}$?

(c) Solve for u_1 and u_2.

(d) Show that a "Green's function" matrix $\mathbf{G}$ can be defined:

$$u_i = \sim_j G_{ij} f_j \; (\mathbf{u} = \mathbf{Gf}).$$

What is $\mathbf{G}$? Show that it is symmetric, $G_{ij} = G_{ji}$.

Problem (*Textbook problem 6.3.16*) Using forward differences in time and centered differences in space, analyze carefully the stability of the difference scheme if the boundary condition for the heat equation is

$$\frac{\partial u}{\partial x}(0) = 0, \; \frac{\partial u}{\partial x}(L) = 0.$$

(*Hint:* See sec 6.3.9) Compare your result to the one for the boundary conditions $u(0) = 0$ and $u(L) = 0$.