MATH 322 , SPRING 2013. PRACTICE PROBLEMS FOR MIDTERM 1

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Problem Find the steady-state solution of the heat equation with boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = h\ (T_0 - u(0,t)),\ \frac{\partial u}{\partial x}(L,t) = \Phi,$$

where T_0 and Φ are constants

Problem Suppose that $u(x, y)$ is a solution of Laplace's equation. If θ is a fixed real number, define the function $v(x, y) = u(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$. Show that $v(x, y)$ is a solution of Laplace's equation.

Problem Apply the result from the previous exercise to the separated solutions of Laplace's equation of the form $u(x, y) = (A_1e^{kx} + A_2e^{-kx})(A_3 \cos(ky) + A_4 \sin(ky))$, to obtain additional solutions of Laplace's equation. Are these new solutions separated?

Problem Which of the following pairs of functions are orthogonal on the interval $0 \le x \le 1$?

$$\varphi_1 = \sin 2\pi x,\ \varphi_2 = x,\ \varphi_3 = \cos 2\pi x,\ \varphi_4 = 1.$$

Problem Let $(\varphi_1, \varphi_2, \varphi_3)$ be an orthonormal set of real-valued function on the interval $-1 \le x \le 1$. That is, $\langle\varphi_i, \varphi_j\rangle = 0$ for $i \ne j$ and $\langle\varphi_i, \varphi_j\rangle = 1$ for $i = j$, $i, j = 1, 2, 3$. Here

$$\langle\varphi_i, \varphi_j\rangle = \frac{1}{2}\int_{-1}^{1} \varphi_i(x)\varphi_j(x)dx.$$

Let f and g be any functions of the form $f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x)$, $g(x) = b_1\varphi_1(x) + b_2\varphi_2(x) + b_3\varphi_3(x)$.

(a) Show that $||f||^2 + \langle f, f\rangle = a_1^2 + a_2^2 + a_3^2$.
(b) Show that $\langle f, \varphi_1\rangle = a_1, \langle f, \varphi_2\rangle = a_2, \langle f, \varphi_3\rangle = a_3$.
(c) Show that $\langle f, g\rangle = a_1b_1 + a_2b_2 + a_3b_3$.

Problem Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4$, $u(x, 0) = f(x)$, $\frac{\partial u}{\partial x}(0, t) = 5$, $\frac{\partial u}{\partial x}(L, t) = 6$. Calculate the total thermal energy in the one-dimensional rod as a function of time.

Problem Isobars are lines of constant temperature. Show that isobars are perpendicular to any part of the boundary that is insulated.

Problem Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$

Determine the eigenvalues λ, and the corresponding eigenfunctions if ϕ satisfies the following boundary condition

$$\frac{d\phi}{dx}(0) = 0,\ \frac{d\phi}{dx}(L) = 0.$$

Problem Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary condition

$$\frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

$$u(x,0) = \begin{cases} 0, & 0 < x \le \frac{L}{2} \\ 1, & \frac{L}{2} < x < L \end{cases}$$

Problem Solve Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the following boundary condition

$$\frac{\partial u}{\partial x}(0,y) = g(y),\ \frac{\partial u}{\partial x}(L,y) = 0,\ u(x,0) = 0,\ u(x,H) = 0.$$

Problem Solve Laplace's equation outside a circular disk ($r \ge a$) subject to the boundary condition

(a) $u(a,\theta) = \ln(2) + 4\cos(3\theta)$
(b) $u(a,\theta) = f(\theta)$

Problem Solve Laplace's equation inside a semi-infinite strip ($0 < x < \infty$, $0 < y < H$) subject to the boundary condition

$$u(x,0) = 0, u(x,H) = 0, u(0,y) = g(y)$$

Problem Solve Laplace's equation inside a semi-infinite strip ($0 < x < \infty$, $0 < y < H$) subject to the boundary condition

$$\frac{\partial u}{\partial x}(x,0) = 0, \frac{\partial u}{\partial x}(x,H) = 0, u(0,y) = g(y)$$

Problem For what values of θ will $u_r = 0$ off the cylinder? For these values of θ, where (for what values of r) will $u_\theta = 0$ also?

Problem Show that $\psi = \alpha \frac{\sin\theta}{r}$ satisfies Laplace's equation. Show that the streamlines are circles. Graph the streamlines.