Math 322 : Midterm 1

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YOUR NAME: Solutions.

PLEASE WRITE YOUR NAME ON EVERY PAGE.

Prob 1 /20	
Prob 2 /20	
Prob 3 /20	
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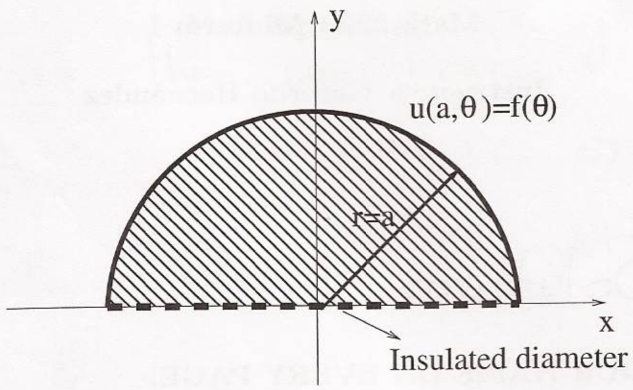
TOTAL /100	



Problem 1. Solve Laplace's equation inside a semicircle of radius a $(0 < r < a, 0 < \theta < \pi)$ subject to the boundary conditions: (see figure above)

$$\begin{cases} \text{the diameter is insulated, and} \\ u(a,\theta) = f(\theta). \end{cases}$$

Hint: $\frac{\partial u}{\partial y} = \sin(\theta)\frac{\partial u}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial u}{\partial \theta}$.
You can assume that the solution is bounded at the origin.

Since the solutron is bounded at the origin, we know that the solutron in polar coordinates takes the form:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)\right)$$

Since the diameter is insulated, then $\nabla u \cdot \hat{n} = 0$ wher $\hat{n} = (0,-1)$ is the outward normal vector at the diameter.

$\Rightarrow \frac{\partial u}{\partial y} = 0$

$\Rightarrow$ In polar coordinates $\sin(\theta)\frac{\partial u}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial u}{\partial \theta} = 0$ at $\theta = 0, \pi$.

$\sin(0) = \sin(\pi) = 0$, $\cos(0) = 1$, $\cos(\pi) = -1$

$\Rightarrow \frac{\partial u}{\partial \theta} = 0$ at $\theta = 0, \pi$

$$\frac{\partial u}{\partial \theta}\Big|_{\theta=0,\pi} = \sum_{n=1}^{\infty} -A_n r^n n \sin(n\theta) + B_n r^n n \cos(n\theta) \Big|_{\theta=0,\pi}$$

$$= \begin{cases} \sum_{n=1}^{\infty} n B_n r^n, & \theta = 0 \\ \sum_{n=1}^{\infty} n B_n (-1)^n r^n & \theta = \pi \end{cases}$$

$\Rightarrow B_n = 0$ for all $n = 1, 2, \ldots$

$$\Rightarrow u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta)$$

Using the boundary conditions we get:

$$f(\theta) = u(a,\theta) = A_0 + \sum_{n=1}^{\infty} A_n a^n \cos(n\theta)$$

Using the formulas seen in class, we can obtain the coefficients A_n as:

$$A_0 = \frac{1}{\pi}\int_0^{\pi} f(\theta)\,d\theta \quad ; \quad a^n A_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

Therefore, the solution is given as:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta)$$

where

$$A_0 = \frac{1}{\pi}\int_0^{\pi} f(\theta)\,d\theta$$

$$A_n = \frac{2}{\pi} a^{-n} \int_0^{\pi} f(\theta)\cos(n\theta)\,d\theta .$$

Problem 2. Find the steady-state solution of the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with the following boundary conditions

$$-\frac{\partial u}{\partial x}(0,t) = h(T_0 - u(0,t)),$$

$$\frac{\partial u}{\partial x}(L,t) = h(T_1 - u(L,t)),$$

where T_0, T_1, h are constants with $h > 0$.

The steady-state solution satisfies:

$$\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u(x) = c_1 x + c_2 \qquad \frac{\partial u}{\partial x} = c_1$$

Using the boundary conditions we get

$$-c_1 = h(T_0 - c_2) \Rightarrow c_1 = h(c_2 - T_0) = hc_2 - hT_0$$

$$c_1 = h(T_1 - c_1 L - c_2) = h(T_1 - c_2) - hLc_1$$

$$\Rightarrow (1+hL)c_1 = h(T_1 - c_2) = hT_1 - (c_1 + hT_0) = h(T_1 - T_0) - c_1$$

$$\Rightarrow (2+hL)c_1 = h(T_1 - T_0) \Rightarrow c_1 = \frac{h(T_1 - T_0)}{2+hL}$$

and $$c_2 = \frac{c_1}{h} + T_0 = \frac{T_1 - T_0}{2+hL} + T_0 = \frac{T_1 - T_0 + (2+hL)T_0}{2+hL}$$

$$= \frac{T_1 + (1+hL)T_0}{2+hL}$$

$\therefore$ The steady-state solution is

$$u(x) = \frac{h(T_1 - T_0)}{2+hL}x + \frac{T_1 + (1+hL)T_0}{2+hL}$$

Problem 3. This problem provides an example of a homogeneous linear PDE with no separated solutions other than $u(x,y) =$ constant. Suppose that $u(x,y) = a(x)b(y)$ is a solution of the equation

$$\frac{\partial u}{\partial x} + (x+y) \cdot \frac{\partial u}{\partial y} = 0.$$

Show that $a(x)$ and $b(y)$ are both constant.

$u(x,y) = a(x)\,b(y)$

Subtituting this in the PDE we get

$a'(x)\,b(y) + (x+y)\,a(x)\,b'(y) = 0$

We can assume $a(x) \neq 0$, $b(y) \neq 0$ without loss of generality, otherwise we divide in regions where a and b are not zero, where we will show it is constant.

$\Rightarrow$ Dividing by $a(x)b(y)$ we get:

$$\frac{a'(x)}{a(x)} + (x+y)\frac{b'(y)}{b(y)} = 0$$

Taking a derivative w.r.t. x, we get:

$$\left(\frac{a'(x)}{a(x)}\right)' + \frac{b'(y)}{b(y)} = 0 \Rightarrow \left(\frac{a'(x)}{a(x)}\right)' = -\lambda, \quad \frac{b'(y)}{b(y)} = \lambda \text{ are constants}$$

$$\Rightarrow \frac{a'(x)}{a(x)} + (x+y)\lambda = 0$$

Taking the derivative w.r.t. y to the equation above, we get:

$$0 + \lambda = 0 \Rightarrow \frac{b'(y)}{b(y)} = 0 \Rightarrow \frac{a'(x)}{a(x)} = 0$$

$\Rightarrow a'(x) = 0$, $b'(y) = 0$

$\Rightarrow a(x)$ an $b(y)$ are both constant fonctrons.

Problem 4. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0,t) = 0, \; u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

$$u(x,0) = \begin{cases} -1, & 0 < x \le \frac{L}{2} \\ 1, & \frac{L}{2} < x < L \end{cases}$$

Note: Compute the Fourier coefficients explicitly for full credit.

We know that in this case the solution has the general form: $u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

where:

$$B_n = \frac{2}{L}\int_0^L u(x,0)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^{L/2} -\sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{L/2}^{L}\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^{L/2} - \frac{2}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{L/2}^{L}$$

$$= \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right] - \frac{2}{n\pi}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{4}{n\pi}\cos\frac{n\pi}{2} - \frac{2}{n\pi} - \frac{2}{n\pi}(-1)^n$$

$$\cos\frac{n\pi}{2} = \begin{cases} 1 & \text{if } n = 4k \text{ is a multiple of } 4 \\ 0 & \text{if } n = 4k+1 \\ -1 & \text{if } n = 4k+2 \\ 0 & \text{if } n = 4k+3 \end{cases}$$

Then for $n = 4k$ $B_{4k} = \frac{4}{n\pi} - \frac{4}{n\pi} = 0$

$B_{4k+1} = 0 - \frac{2}{n\pi} + \frac{2}{n\pi} = 0$, $B_{4k+2} = -\frac{4}{n\pi} - \frac{2}{n\pi} - \frac{2}{n\pi} = -\frac{8}{n\pi}$

$B_{4k+3} = 0 - \frac{2}{n\pi} + \frac{2}{n\pi} = 0$

Therefore $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$, $B_n = \begin{cases} -\frac{8}{n\pi} & \text{if } n = 4k+2 \\ 0 & \text{otherwise} \end{cases}$

or

$$u(x,t) = \sum_{k=1}^{\infty} -\frac{8}{\pi(4k+2)}\sin\frac{(4k+2)\pi x}{L} e^{-k\left(\frac{(4k+2)\pi}{L}\right)^2 t}$$

Problem 5. Consider a one-dimensional rod with constant thermal properties: $c = 1, \rho = 1, K_0 = 1$, and heat source $Q = 1$. Suppose that the temperature satisfies the heat equation with boundary conditions $\frac{\partial u}{\partial x}(0,t) = \beta$, $\frac{\partial u}{\partial x}(L,t) = 1$, and that the temperature is initially $u(x,0) = f(x)$.

(a) Calculate the total thermal energy in the one-dimensional rod as a function of time

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1$$

$$\Rightarrow \frac{d}{dt}\int_0^L c\rho u\,dx = \int_0^L \frac{\partial u}{\partial t}dx = \int_0^L\left(\frac{\partial^2 u}{\partial x^2} + 1\right)dx$$

$$= \frac{\partial u}{\partial x}\Big|_0^L + L = 1 - \beta + L$$

$$\Rightarrow \int_0^L c\rho u\,dx = (1+L-\beta)t + c, \quad c = \int_0^L c\rho u(x,0)\,dx = \int_0^L f(x)dx$$

Therefore the total thermal energy as a function of time is given by

$$\int_0^L c\rho\, u(x,t)\,dx = \int_0^L f(x)dx + (1+L-\beta)t.$$

(b) From part (a), determine a value of β for which an equilibrium exists. For this value of β, and assuming that $u(x,t)$ converges to the equilibrium distribution as $t \to \infty$, determine the limit $\lim_{t\to\infty} u(x,t)$.

A necessary condition for an equilibrium distribution to exist is that the thermal energy is constant in time, then we need

$$(1+L-\beta) = 0 \Rightarrow \beta = 1+L.$$

Let's now consider a time dependent solution $u(x,t)$ that converges to the

Let's compute the equilibrium distribution:

$$\frac{\partial^2 u}{\partial x^2}+1=0 \Rightarrow \frac{\partial u}{\partial x}=-x+c_1 \Rightarrow u=-\frac{x^2}{2}+c_1x+C_2$$

Using the boundary conditions we get:

$$\frac{\partial u}{\partial x}(0)=c_1=\beta$$

$$\frac{\partial u}{\partial x}(L)=c_1-L=1 \Rightarrow c_1=\beta=1+L.$$

In order to compute C_2, we take a time dependent solution $u(x,t)$ that converges to the equilibrium distribution.

Since the thermal energy is constant in time, we get

$$\int_0^L u(x,0)dx=\int_0^L u(x,t)dx$$

$\Rightarrow$ as $t\to\infty$ $\displaystyle\int_0^L f(x)dx=\int_0^L\left(-\frac{x^2}{2}+c_1x+C_2\right)dx$

$$=-\frac{x^3}{6}\Big|_0^L+(1+L)\frac{L^2}{2}+C_2L \Rightarrow C_2=\frac{1}{L}\int_0^L f(x)+\frac{\cancel{L^2}}{\cancel{6}}-(1+\cancel{L})\frac{L}{2}$$

$$\Rightarrow \lim_{t\to\infty}u(x,t)=-\frac{x^2}{2}+(1+L)x+\frac{L^2}{6}-(1+L)\frac{L}{2}+\frac{1}{L}\int_0^L f(x)dx$$

$$=-\frac{x^2}{2}+(1+L)x-\frac{L^2}{3}-\frac{L}{2}+\frac{1}{L}\int_0^L f(x)dx$$