Math 322 : Midterm 1

Instructor: Gerardo Hernández

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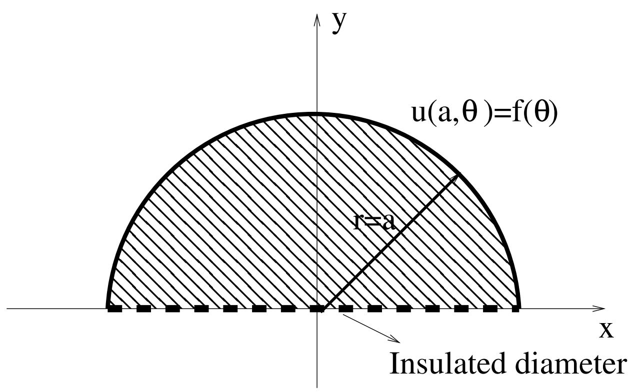
YOUR NAME:

PLEASE WRITE YOUR NAME ON EVERY PAGE.

Prob 1 /25	
Prob 2 /20	
Prob 3 /20	
Prob 4 /20	
Prob 5 /20	
TOTAL /100	



Problem 1. Solve Laplace's equation inside a semicircle of radius a ($0 < r < a, 0 < \theta < \pi$) subject to the boundary conditions (see figure above).

$$\begin{cases} \text{the diameter is insulated, and} \\ \\ u(a,\theta) = f(\theta). \end{cases}$$

Hint: $\frac{\partial u}{\partial y} = \sin(\theta)\frac{\partial u}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial u}{\partial \theta}$.
You can assume that the solution remains finite at the origin.

Problem 2. Find the steady-state solution of the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with the boundary conditions

$$\begin{aligned} -\tfrac{\partial u}{\partial x}(0,t) &= h(T_0 - u(0,t)), \\ \tfrac{\partial u}{\partial x}(L,t) &= h(T_1 - u(L,t)), \end{aligned}$$

where T_0, T_1, h are constants with $h > 0$.

Problem 3. This problem shows that not every linear PDE has non-trivial separated solutions. Suppose that $u(x, y) = a(x)b(y)$ is a solution of the equation

$$\frac{\partial u}{\partial x} + (x + y)\,\frac{\partial u}{\partial y} = 0.$$

Show that $a(x)$ and $b(y)$ are both constant.

Problem 4. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary condition

$$u(0,t) = 0, \; u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

$$u(x,0) = \begin{cases} -1, & 0 < x \leq \frac{L}{2} \\ 1, & \frac{L}{2} < x < L \end{cases}$$

Problem 5. Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = \beta$, $\frac{\partial u}{\partial x}(L,t) = 1$.

(a) Calculate the total thermal energy in the one-dimensional rod as a function of time

(b) From part (a), determine a value of β for which an equilibrium exists. For this value of β, determine $\lim_{t\to\infty} u(x,t)$.