Problem 1. Consider

$$\rho \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha u + \beta \frac{\partial u}{\partial t}$$

(a) Give a brief physical interpretation. What signs must α and β have to be physical

Answer:

$Q = \alpha u$ represents a body force acting on the string, which in this case is proportional to the displacement u.

In order to see the effect this term has on the whole equation, let's ignore the other terms for now: $\rho \frac{\partial^2 u}{\partial t^2} = \alpha u$

If $\alpha < 0 \Rightarrow u$ consists of sines and cosines.

If $\alpha > 0 \Rightarrow u$ consist of a linear combination of two exponentials, one increasing and one decreasing.

So if $\alpha > 0$, we should have some boundary conditions that rules out the exponentially growing part for it to be physical.

$\beta \frac{\partial u}{\partial t}$ represents friction. In order to see this, let's ignore the other term

$\Rightarrow \rho \frac{\partial^2 u}{\partial t^2} = \beta \frac{\partial u}{\partial t} \Rightarrow \rho \frac{\partial u}{\partial t} = \beta u + \text{constant}$

$\Rightarrow u$ is an exponential function

If $\beta > 0 \Rightarrow u$ increases exponentially

If $\beta \leq 0 \Rightarrow u$ decays exponentially

$\Rightarrow \beta \leq 0$ gives the physical interpretation

(b) Allow ρ, α, β to be functions of x.
show that separation of variables works
if $\beta = c\rho$, c a constant.

Answer:

$u = \phi(x) h(t)$

$\Rightarrow \rho(x)\phi(x)h''(t) = T_0 \phi''(x) h(t) + \alpha(x)\phi(x) h(t) + \beta(x)\phi(x) h'(t)$

$= h(t)\left(T_0 \phi''(x) + \alpha(x)\phi(x)\right) + \beta(x)\phi(x) h'(t)$

$\Rightarrow \phi(x)\left(\underbrace{\rho(x) h''(t) - \beta(x) h'(t)}_{\text{we need } \beta(x) = c\rho(x)}\right) = h(t)\left(T_0 \phi''(x) + \alpha(x)\phi(x)\right)$ to separate variables here

Then if $\beta(x) = c\rho(x)$, we get

$\phi(x)\rho(x)\left(h''(t) - c h'(t)\right) = h(t)\left(T_0 \phi''(x) + \alpha(x)\phi(x)\right)$

$$\Rightarrow \frac{h''(t) - c h'(t)}{h(t)} = \frac{T_0 \phi''(x) + \alpha(x)\phi(x)}{\phi(x)\rho(x)} = -\lambda, \ \lambda \text{ const.}$$

(c) If $\beta = c\rho$, show that the Sturm-Liouville differential equation. Solve the time equation.

Answer: $T_0 \dfrac{d^2\phi}{dx^2} + \alpha(x)\phi(x) + \lambda \rho(x)\phi(x) = 0$

which is of Sturm-Liouville type.

To solve the time equation:

$$\frac{d^2h}{dt^2} - c\frac{dh}{dt} + \lambda h(t) = 0$$

Look for solutions of the form:

$$h(t) = e^{rt}$$

$$\Rightarrow r^2 e^{rt} - cr e^{rt} + \lambda e^{rt} = 0$$

$$\Rightarrow r^2 - cr + \lambda = 0 \quad \Rightarrow \quad r = \frac{+c \pm \sqrt{c^2 - 4\lambda}}{2} = \frac{c}{2} \pm \sqrt{\frac{c^2}{4} - \lambda}$$

If $\frac{c^2}{4} - \lambda > 0 \Rightarrow r_1 = \frac{c}{2} + \sqrt{\frac{c^2}{4} - \lambda} \in \mathbb{R}$, $r_2 = \frac{c}{2} + \sqrt{\frac{c^2}{4} - \lambda} \in \mathbb{R}$

$$\Rightarrow h(t) = a e^{r_1 t} + b e^{r_2 t}.$$

$$\boxed{\text{if } \frac{c^2}{4} = \lambda \Rightarrow h(t) = e^{\frac{c}{2}t}\cdot(a + bt)}$$

If $\frac{c^2}{4} - \lambda < 0 \Rightarrow r_1 = \frac{c}{2} + \sqrt{\lambda - \frac{c^2}{4}}\, i$, $r_2 = \frac{c}{2} - \sqrt{\lambda - \frac{c^2}{4}}\, i$

$$\Rightarrow h(t) = a e^{\frac{c}{2}t} \cos\sqrt{\lambda - \frac{c^2}{4}}\, t + b e^{\frac{c}{2}t} \sin\sqrt{\lambda - \frac{c^2}{4}}\, t.$$

Problem 2: Consider the non-Sturm-Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0.$$

Answer:

$$\frac{d^2\phi}{dx^2} H(x) + H(x)\alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x)H(x) + \gamma(x)H(x)]\phi = 0$$

Notice that

$$\frac{d}{dx}\left[H(x)\frac{d\phi}{dx}\right] = H(x)\frac{d^2\phi}{dx^2} + \frac{d\phi}{dx}\frac{dH}{dx}$$

$$\Rightarrow \frac{d}{dx}\left[H(x)\frac{d\phi}{dx}\right] - \frac{d\phi}{dx}\frac{dH}{dx} + H(x)\alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x)H(x)+\gamma(x)H(x)]\phi = 0$$

$$\Rightarrow \frac{d}{dx}\left[H(x)\frac{d\phi}{dx}\right] + \left[-\frac{dH}{dx} + H(x)\alpha(x)\right]\frac{d\phi}{dx} + [\lambda\beta(x)H(x)+\gamma(x)H(x)]\phi = 0$$

The $\frac{d\phi}{dx}$ shouldn't be there, so choose $H(x)$ such that:

$$\frac{dH}{dx} = \alpha(x)H(x) \Rightarrow \frac{d}{dx}(\log H) = \frac{1}{H}\frac{dH}{dx} = \alpha(x)$$

$\Rightarrow$ choose $H(x) = e^{\int_a^x \alpha(y)dy}$, a constant.

$$\Rightarrow \frac{d}{dx}\left[H(x)\frac{d\phi}{dx}\right] + \gamma(x)H(x)\phi + \lambda\beta(x)H(x)\phi = 0$$

Which is of Sturm-Liouville type.

$\Rightarrow$ Here $p(x) = H(x) = e^{\int_a^x \alpha(y)dy}$

$q(x) = \gamma(x)H(x)$

$\sigma(x) = \beta(x)H(x)$.

Problem 3: For the Sturm-Liouville eigenvalue Problem $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$, with $\frac{d\phi}{dx}(0) = 0$, $\phi(L) = 0$,

verify the properties below.

Answer:

Let's first find the eigenvalues and eigenfunctions.

We can show that $\lambda > 0$ in this case.

$\Rightarrow \phi(x) = C_1 \cos\sqrt{\lambda}\,x + C_2 \sin\sqrt{\lambda}\,x$.

$\frac{d\phi}{dx} = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x$

$\frac{d\phi}{dx}(0) = C_2\sqrt{\lambda} = 0 \Rightarrow C_2 = 0$ Can assume $C_1 = 1$

$\Rightarrow \phi(x) = \cos\sqrt{\lambda}\,x$

$\phi(L) = \cos\sqrt{\lambda}\,L = 0 \Rightarrow \sqrt{\lambda}\,L = -\frac{\pi}{2} + n\pi$ (so that $n=1$ is the first e-value)

$\Rightarrow \lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$

$\phi_n(x) = \cos\frac{(2n-1)\pi}{2L}x$

Verify the following properties:

(a) There is an infinite number of eigenvalue with a smallest but no largest.

Answer: $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$ increases ~~...~~ a n increases.

$\lambda_1 = \left(\frac{\pi}{2L}\right)^2$ is the first e-value

Clearly $\lambda_n \to \infty$ as $n \to \infty$.

(b) The nth eigenfunction has n-1 zeros.

Answer: $\phi_n(x) = \cos\frac{(2n-1)\pi}{2L} x$

The range in the expression inside the cosine is $[0, \frac{-\pi}{2} + n\pi]$.

For $n=1$, $\cos\frac{\pi}{2L}x$ doesn't have any zeros for $x \in (0, L)$

Each time we increase n by 1, we cross the angle $-\frac{\pi}{2}$ or $\frac{\pi}{2}$ once, which adds one zero to the previous eigenfunction.

(c) The functions are complete and orthogonal.

Answer: We can skip the completeness part because we haven't seen any of these proofs in class.

For $n \neq m$

$$\int_0^L \cos\left(\frac{(2n-1)\pi}{2L}x\right)\cos\left(\frac{(2m-1)\pi}{2L}x\right)dx$$

$$= \int_0^L \frac{1}{2}\cos\left(\frac{(2(m+n-1))\pi}{2L}x\right)dx + \int_0^L \frac{1}{2}\cos\left(\frac{2(m-n)\pi}{2L}x\right)dx$$

Since m, n are positive integers and $m \neq n \Rightarrow m+n \geq 2$

$\Rightarrow m+n-1 \neq 0$

$$\Rightarrow = \frac{1}{2}\frac{L}{(m+n-1)\pi}\sin\left(\frac{(m+n-1)\pi}{L}x\right)\Big|_0^L + \frac{1}{2}\frac{L}{(m-n)\pi}\sin\left(\frac{(m-n)\pi}{L}x\right)\Big|_0^L$$

$= 0$ if $n \neq m$.

(d) What does the Rayleigh quotient say concerning negative and zero eigenvalues?

Answer:

Here $p=1$, $q=0$, $\sigma=1$

Since $\frac{d\phi}{dx}(0)=0$, and $\phi(L)=0$

$$\Rightarrow -p\phi\frac{d\phi}{dx}\Big|_0^L=0$$

$\Rightarrow$ Rayleigh quotient is:

$$\lambda=\frac{-p\phi\frac{d\phi}{dx}\Big|_0^L+\int_0^L\left[p\left[\frac{d\phi}{dx}\right]^2-q\phi^2\right]dx}{\int_0^L\phi^2\sigma\,dx}$$

$$=\frac{\int_0^L\left[\frac{d\phi}{dx}\right]^2dx}{\int_0^L\phi^2\,dx}\geq 0$$

Note: If $\lambda=0\Rightarrow\frac{d\phi}{dx}=0$
$\Rightarrow\phi=$ constant
Since $\phi(L)=0\Rightarrow\phi\equiv 0$
$\Rightarrow\lambda=0$ is not an eigenvalue

So all eigenvalues are strictly positive.

Problem 4: Which of the statements 1-5 of the theorems of this section are valid for the following eigenvalue problem?

$$\frac{d^2\phi}{dx^2}+\lambda\phi=0$$

$$\phi(-L)=\phi(L)$$

$$\frac{d\phi}{dx}(-L)=\frac{d\phi}{dx}(L).$$

Answer: This are periodic boundary conditions, which are not of Sturm-Liouville type.

The eigenvalues and eigenfunction are:

$\lambda_0 = 0, \quad \phi_0 = 1$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \phi_n(x) = \cos\frac{n\pi x}{L}, \quad \varphi_n = \sin\frac{n\pi x}{L}$

Then λ_n for $n \geq 1$ is a double eigenvalue

λ_0 is the first one.

Statements:

1. It is clear that all eigenvalues are real ✓

2. $\lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$

So we have an infinite number of e-values.

$\lambda_0 = 0$ is the first e-value.

$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \longrightarrow \infty$ as $n \to \infty$.

3. For $n \geq 1$, we have two eigenfunctions

$\phi_n(x) = \cos\frac{n\pi x}{L}, \quad \varphi_n = \sin\frac{n\pi x}{L}$.

$\Rightarrow$ We have no uniqueness here.

$\phi_0 = 1$ is the first e-function and has no zeros.

For $n=1$, $\sin\frac{\pi x}{L}$ has no zeros, and it is supposed to have one, so the theorem is not valid here.

4. This is valid. We know this is a complete set of eigenfunctions, which gives the Fourier series.

5. We also know that different eigenfunction are orthogonal, even if they have the same eigenvalues.

Problem 5: Show that $\lambda \geq 0$ for the eigenvalue problem:

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0, \quad \text{with } \frac{d\phi}{dx}(0) = 0, \; \frac{d\phi}{dx}(1) = 0.$$

Is $\lambda = 0$ an eigenvalue?

Answer:

Here $p = 1$, $q = -x^2$, $\sigma = 1$

Using Rayleigh quotient, we get:

$$\lambda = \frac{-\phi \frac{d\phi}{dx}\Big|_0^1 + \int_0^1 \left(\frac{d\phi}{dx}\right)^2 + x^2\phi^2 \, dx}{\int_0^1 \phi^2 \, dx}$$

$$= \frac{\int_0^1 \left(\frac{d\phi}{dx}\right)^2 + x^2\phi^2 \, dx}{\int_0^1 \phi^2 \, dx} \geq 0$$

$\lambda = 0$ is not an eigenvalue because $\int_0^1 x^2\phi^2 \, dx > 0$ since $\phi \neq 0$