MATH 322 - SEC 001, SPRING 2013. HOMEWORK 8

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Due : Friday, April 5

Please show all your work and/or justify your answers for full credit.

Problem 1: (*Textbook problem 4.4.3*) Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

(b) Determine the solution (by separation of variables)that satisfies the boundary conditions

$$u(0,t) = 0,\ u(L,t) = 0,$$

and the initial conditions

$$u(x,0) = f(x),\ \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi\rho_0 T_0/L^2$).

Problem 2: (*Textbook problem 4.4.8*) If a vibrating string satisfying (4.4.1)-(4.4.3) is initially unperturbed, $f(x) = 0$, with the initial velocity given, show that

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} G(\bar{x}) d\bar{x},$$

where $G(\bar{x})$ is the odd periodic extension of $g(x)$. *Hints:*

1 For all x, $G(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L}$

2 $\sin a \sin b = \frac{1}{2} [\cos(a-b) - \cos(a+b)]$

Problem 3: (*Textbook problem 4.4.9*) From (4.4.1), derive conservation of energy for a vibrating string,

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \Big|_0^L,$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2} \left(\frac{\partial u}{\partial t}\right)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2} \left(\frac{\partial u}{\partial x}\right)^2 dx$.

Problem 4: Let

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

be a solution of the vibrating string problem. Suppose that the string is constrained so that $u(L/3, t) = 0$ for all t. What conditions does this impose on the coefficients B_n?

Problem 5: The voltage $v(x,t)$ in a transmition cable is known to satisfy the partial differential equation

$$v_{tt} + 2av_t + a^2 v = c^2 v_{xx},$$

where a and c are positive constants. Let $u(x,t) = e^{at} v(x,t)$ and show that u satisfies the wave equation $u_{tt} = c^2 u_{xx}$.