Problem 1: Compute the Fourier series of the indicated functions:

(a) $f(x) = x^2, \quad -L \le x \le L$

Answer:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

where:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} x^2\,dx = \frac{1}{2L}\frac{x^3}{3}\Big|_{-L}^{L} = \frac{1}{2L}\frac{2L^3}{3}$$

$$= \frac{L^2}{3}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x^2\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\left[\cancelto{0}{\frac{L}{n\pi}x^2\sin\frac{n\pi x}{L}\Big|_{-L}^{L}} - \int_{-L}^{L}\frac{L}{n\pi}2x\sin\frac{n\pi x}{L}\,dx\right]$$

$$\int_{-L}^{L} x\sin\frac{n\pi x}{L}\,dx = x\cdot\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{-L}^{L} - \int_{-L}^{L}\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{-L^2}{n\pi}\cos(n\pi) - (-L)\frac{-L}{n\pi}\cos(n\pi) + \frac{L}{n\pi}\cancelto{0}{\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{-L}^{L}}$$

$$= -\frac{2L^2}{n\pi}(-1)^n$$

$$\Rightarrow a_n = \frac{1}{L}\left[\frac{-2L}{n\pi}\;\frac{-2L^2}{n\pi}(-1)^n\right] = \frac{4L^2}{n^2\pi^2}(-1)^n.$$

$b_n = 0$ since x^2 is even.

$$\Rightarrow x^2 \sim \frac{L^2}{6} + \sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2}(-1)^n\cos\frac{n\pi x}{L} \qquad -L \le x \le L.$$

(b) $f(x) = x^3, \quad -L \le x < L.$

$f(x)$ is odd, so $a_0 = 0$, $a_n = 0$.

$$b_n = \frac{1}{L}\int_{-L}^{L} x^3\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L x^3\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[-x^3\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L -3x^2\frac{L}{n\pi}\cos\frac{n\pi x}{L}\,dx\right]$$

$$= \frac{2}{L}\left[-\frac{L^4}{n\pi}\cos(n\pi) + \frac{3L}{n\pi}\int_0^L x^2\cos\frac{n\pi x}{L}\,dx\right]$$

From previous exercise: $\int_0^L x^2 \cos\frac{n\pi x}{L}\,dx = \frac{2L^3}{n^2\pi^2}(-1)^n$

$$\Rightarrow b_n = \frac{2}{L}\left[-\frac{L^4}{n\pi}(-1)^n + \frac{3L}{n\pi}\,\frac{2L^3}{n^2\pi^2}(-1)^n\right]$$

$$= \frac{2}{L}(-1)^n\frac{L^4}{n\pi}\left[-1+\frac{6}{n^2\pi^2}\right] = \frac{2L^3}{n\pi}(-1)^n\left[\frac{6}{n^2\pi^2}-1\right]$$

(c) $f(x) = |x|^3$, $-L \le x < L$.

Answer:

Since $|x|^3$ is even, then $b_n = 0 \quad \forall n \ge 1$.

$$a_0 = \frac{1}{L}\int_0^L |x|^3 dx = \frac{1}{L}\int_0^L x^3 dx = \frac{1}{L}\,\frac{x^4}{4}\Big|_0^L = \frac{L^3}{4}$$

$$a_n = \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L}|x|^3 dx = \frac{2}{L}\int_0^L x^3\cos\frac{n\pi x}{L}\,dx =$$

$$= \frac{2}{L}\left[\cancelto{0}{x^3\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L} - \int_0^L 3x^2\frac{L}{n\pi}\sin\frac{n\pi x}{L}\,dx\right]$$

$$= -\frac{6}{n\pi}\int_0^L x^2\sin\frac{n\pi x}{L}\,dx$$

$$\int_0^L x^2\sin\frac{n\pi x}{L}\,dx = x^2\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L 2x\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\,dx$$

$$= -\frac{L^3}{n\pi}(-1)^n + \frac{2L}{n\pi}\int_0^L x\cos\frac{n\pi x}{L}\,dx$$

$$\int_0^L x\cos\frac{n\pi x}{L}\,dx = \cancelto{0}{x\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L} - \int_0^L\frac{L}{n\pi}\sin\frac{n\pi x}{L}\,dx$$

$$= -\frac{L}{n\pi}\,\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L = \frac{L^2}{n^2\pi^2}\left((-1)^n-1\right)$$

$$\Rightarrow \int_0^L x^2\sin\frac{n\pi x}{L}\,dx = \frac{L^3}{n\pi}(-1)^{n+1} + \frac{2L}{n\pi}\,\frac{L^2}{n^2\pi^2}\left((-1)^n-1\right)$$

$$\Rightarrow a_n = -\frac{6}{n\pi}\,\frac{L^3}{n\pi}\left[(-1)^{n+1} + \frac{2}{n^2\pi^2}\left((-1)^n-1\right)\right]$$

$$\Rightarrow |x|^3 \approx \frac{L^3}{4} + \sum_{n=1}^{\infty} -\frac{6L^3}{n^2\pi^2}\left[(-1)^{n+1} + \frac{2}{n^2\pi^2}\left((-1)^n-1\right)\right]\cos\frac{n\pi x}{L}$$

HW 5.2

(d) $f(x) = \sin^2(2x)\ , \ -L \le x < L$

Answer:

This is an even function $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{L}\int_0^L \sin^2(2x)\,dx$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha \Rightarrow \sin^2\alpha = \frac{1-\cos(2\alpha)}{2}$$

$$\sin^2(2x) = \frac{1-\cos(4x)}{2}$$

$$\Rightarrow a_0 = \frac{1}{L}\int_0^L \frac{1-\cos(4x)}{2}\,dx = \frac{1}{2L}\left[x\Big|_0^L - \frac{1}{4}\sin(4x)\Big|_0^L\right]$$

$$= \frac{1}{2L}\left[L - \frac{1}{4}\sin(4L)\right]$$

$$a_n = \frac{2}{L}\int_0^L \sin^2(2x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\,\frac{1}{2}\int_0^L\left(1-\cos(4x)\right)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\cancelto{0}{\int_0^L \cos\frac{n\pi x}{L}\,dx} - \frac{1}{L}\int_0^L \cos(4x)\cos\frac{n\pi x}{L}\,dx$$

Using the trigonometric identity $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$

we get

$$\int_0^L \cos(4x)\cos\frac{n\pi x}{L}\,dx = \int_0^L \frac{1}{2}\left[\cos\left(\left(4+\frac{n\pi}{L}\right)x\right) + \cos\left(\left(4-\frac{n\pi}{L}\right)x\right)\right]dx$$

if $n \ne \frac{4L}{\pi}$

$$= \frac{1}{2}\,\frac{1}{4+\frac{n\pi}{L}}\sin\left(\left(4+\frac{n\pi}{L}\right)x\right)\Big|_0^L + \frac{1}{2}\,\frac{1}{4-\frac{n\pi}{L}}\sin\left(\left(4-\frac{n\pi}{L}\right)x\right)\Big|_0^L$$

$$= \frac{1}{8+\frac{2n\pi}{L}}\sin(4L+n\pi) + \frac{1}{8-\frac{2n\pi}{L}}\sin(4L-n\pi)$$

$$\sin(\alpha+n\pi) = \sin\alpha\cos n\pi + \cos\alpha\cancelto{0}{\sin n\pi} = (-1)^n\sin\alpha$$

$$= \frac{1}{8+\frac{2n\pi}{L}}(-1)^n\sin(4L) + \frac{1}{8-\frac{2n\pi}{L}}(-1)^n\sin(4L)$$

$$= (-1)^n\sin(4L)\,\frac{1}{2}\cdot\frac{4}{16-\frac{n^2\pi^2}{L^2}} = \frac{2(-1)^n\sin(4L)}{16-\frac{n^2\pi^2}{L^2}}$$

If $n = \frac{4L}{\pi}$

$$\Rightarrow \int_0^L \cos(4x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\frac{1}{4+\frac{n\pi}{L}}\sin\left(\left(4+\frac{n\pi}{L}\right)x\right)\Big|_0^L + \frac{L}{2}$$

$$= \frac{1}{2}\frac{1}{4+\frac{n\pi}{L}}(-1)^n\sin(4L) + \frac{L}{2} = \frac{1}{2}\frac{1}{4+\frac{n\pi}{L}}(-1)^n\cancel{\sin(n\pi)}^{\,0} + \frac{L}{2}$$

$$= \frac{L}{2}$$

$$\Rightarrow \quad a_n = \begin{cases} \dfrac{2(-1)^{n+1}\sin(4L)}{\left(16 - \frac{n^2\pi^2}{L^2}\right)L} & \text{if } n \neq \frac{4L}{\pi} \\ -\frac{1}{2} & \text{if } n = \frac{4L}{\pi} \end{cases}$$

Problem 2: Prove the following facts about even an odd functions.

(a) The product of two even functions is even

Answer:

Let f_1, f_2 be even functions, and $f = f_1 f_2$

Then $f(-x) = f_1(-x)f_2(-x) = f_1(x)f_2(x)$ because f_1, f_2 are even

$= f(x) \Rightarrow f$ is even

(b) The product of two odd functions is even.

Let $f = f_1 f_2$, f_1, f_2 odd function

$\Rightarrow f(-x) = f_1(-x)f_2(-x) = (-f_1(x))(-f_2(x))$ because f_1, f_2 are both odd

$= f_1(x)f_2(x) = f(x) \Rightarrow f$ is even

(c) The product of an even function and and odd function is odd.

Let $f = f_1 f_2$ f_1 even, f_2 odd

$\Rightarrow f(-x) = f_1(-x)f_2(-x) = f_1(x)(-f_2(x))$ because f_1 is even f_2 is odd

$= -f(x) \Rightarrow f$ is odd

(d) which of the statements (a), (b), (c) remain true if the word "product" is replaced by "sum".

Answer:

(a) is true: if $f = f_1 + f_2$, f_1 even, f_2 even

$\Rightarrow f(-x) = f_1(-x) + f_2(-x) = f_1(x) + f_2(x) = f(x)$.

(b) is ~~true~~ false: $f = f_1 + f_2$ f_1, f_2 odd

$\Rightarrow f(-x) = f_1(-x) + f_2(-x) = -f_1(x) - f_2(+x) = -f(x)$

$\Rightarrow f$ is odd, not even. (unless $f \equiv 0$).

(c) False: Counterexample;

$f_1 = x^2$ even, $f_2 = 0$ odd

$\Rightarrow f_1 + f_2 = x^2 + 0 = x^2$, which is even.

Problem 3; Let f be an arbitrary function. Show that there is and odd function f_1 and an even function f_2 such that $f = f_1 + f_2$

Answer:

Given f, define $f_1(x) = \frac{f(x) - f(-x)}{2}$, $f_2(x) = \frac{f(x) + f(-x)}{2}$

$\Rightarrow f_1(x) + f_2(x) = \frac{2f(x)}{2} = f(x)$

In addition:

$f_1(-x) = \frac{f(-x) - f(-(-x))}{2} = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -f_1(x)$

$f_2(x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = f_2(x)$

$\Rightarrow f = f_1 + f_2$, f_1 odd, f_2 even.

Problem 4: Which of the following functions are even, odd, or neither?

(a) $f(x) = \cos(3x)$

$f(-x) = \cos(-3x) = \cos(3x) = f(x) \Rightarrow f$ is even.

(b) $f(x) = x^3 - 3x$

$f(-x) = (-x)^3 - 3(-x) = -x^3 + 3x = -f(x) \Rightarrow f$ is odd

(c) $f(x) = \sin x - 3x^5$

$f(-x) = \sin(-x) - 3(-x)^{-5} = -\sin x + 3x^5 = -f(x) \Rightarrow f$ is odd

(d) $f(x) = \underset{\text{even}}{x^2} - \underset{\text{even}}{\cos(x)} \Rightarrow f(x)$ is even by ~~prev~~ problem 2 (d).

(e) $f(x) = x^3 - x^2$

$f(1) = 0$, $f(-1) = -1 - 1 = -2 \Rightarrow f(-1) \neq \pm f(1)$

$\Rightarrow f$ is neither even nor odd.

(f) $f(x) = \underset{\text{even}}{|x|}\,\underset{\text{odd}}{\sin x} \Rightarrow f(x)$ is odd by problem 2 (c).

Problem 5: Let $f(x)$, $-l \le x < l$, be an odd function that satisfies the symmetric condition

$$f(l-x) = f(x).$$

Show that the sine and cosine coefficients in the Fourier series $A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}$ satisfy

$A_n = 0 \ \forall n$, $B_n = 0 \ \forall n$ even.

Anwer: Since f is odd $\Rightarrow A_n = 0 \ \forall n$.

$\Rightarrow f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$

Notice that

$$f(L-x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(L-x)}{L}\right)$$

and

$$\sin\left(\frac{n\pi(L-x)}{L}\right) = \sin\left(n\pi - \frac{n\pi x}{L}\right)$$

$$= \cancelto{0}{\sin(n\pi)} \cos \frac{n\pi x}{L} + \cos(n\pi)\sin\left(-\frac{n\pi x}{L}\right)$$

$$= (-1)^{n+1} \sin \frac{n\pi x}{L}$$

$$\Rightarrow f(L-x) \sim \sum_{n=1}^{\infty} B_n (-1)^{n+1} \sin \frac{n\pi x}{L}$$

Since $f(x) = f(L-x)$, the coefficients need to be the same in both Fourier series

$\Rightarrow B_n = (-1)^{n+1} B_n$

For n even $B_n = -B_n \Rightarrow B_n = 0$

Therefore $A_n = 0$ for all n, and $B_n = 0$ for all even n.

Problem 6: For each of the functions below, state wether or not it is periodic and find the smallest period.

(a) $f(x) = \sin(\pi x)$

Answer: Horizontal stretch of $\sin(x)$ by a factor of π

$\Rightarrow f$ is periodic with period $\frac{2\pi}{\pi} = 2$.

(b) $f(x) = \sin(2x) + \sin(3x)$

$\sin(2x)$ has period π

$\sin(3x)$ has period $\frac{2\pi}{3}$

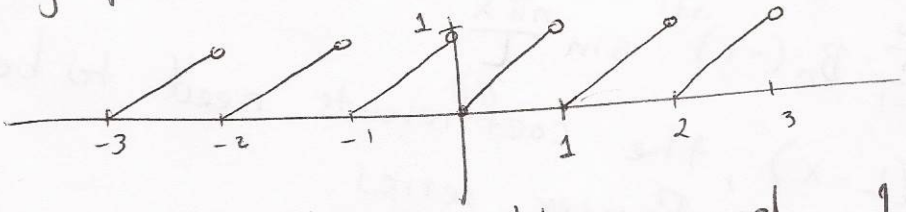
$\Rightarrow f(x) = \sin(2x) + \sin(3x)$ is perrodic with perrod the least common multiple of π and $\frac{2\pi}{3}$, which is 2π.

(c) $f(x) = \sin(x) + \sin(\pi x)$

If $f(x)$ was perrodic, the perrod would be the least common multiple of 2π and 2, which is $+\infty$ because π is irratronal and 2 is an integer.

(d) $f(x) = x - [x]$, where $[x]$ = integer part of x

The graph is:



$\Rightarrow f(x)$ is periodic with perrod 1.

(e) $f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

This is the Taylor serves of $\cos x - 1$

$\Rightarrow f(x)$ is perrodic.

(f) $f(x) = \sin(x^2)$

Not perrodic because sin is and x^2 is not.