MATH 322 - SEC 001, SPRING 2013. HOMEWORK 3

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Due : Friday, February 22

Please show all your work and/or justify your answers for full credit.

Problem 1: (*Textbook problem 2.3.2*) Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and the corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0$, $\lambda = 0$, $\lambda < 0$). You may assume that the eigenvalues are real.

(f) $\phi(a) = 0$, and $\phi(b) = 0$ (You may assume that $\lambda > 0$)

(g) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) + \phi(L) = 0$

Problem 2: (*Textbook problem 2.3.3*) Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions

$$u(0,t) = 0 \text{ and } u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

(b) $u(x,0) = 3\sin(\frac{\pi x}{L}) - \sin(\frac{3\pi x}{L})$

(c) $u(x,0) = 2\cos\frac{3\pi x}{L}$

(d)

$$u(x,0) = \begin{cases} 1, & 0 < x \leq \frac{L}{2} \\ 2, & \frac{L}{2} < x < L \end{cases}$$

Problem 3: (*Textbook problem 2.3.6*) Evaluate

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx \text{ for } n \geq 0, m \geq 0.$$

Use the trigonometric identity

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$

(Be careful if $a - b = 0$, or $a + b = 0$).

Problem 4: (*Textbook problem 2.3.8*) Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u.$$

This corresponds to a one-dimensional rod either with heat loss through the lateral sides with outside temperature 0° or with insulated lateral sides with a heat sink proportional to the temperature. Suppose that the boundary conditions are

$$u(0,t) = 0, \text{ and } u(L,t) = 0.$$

(a) What are the possible equilibrium temperature distributions if $\alpha > 0$?

(b) Solve the time-dependent problem $[u(x, 0) = f(x)]$ if $\alpha > 0$. Analyze the temperature for large time ($t \to \infty$) and compare to part (a).

Problem 5: (*Textbook problem 2.4.4*) Explicitly show that there are no negative eigenvalues for

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \text{ subject to } \frac{d\phi}{dx}(0) = 0, \; \frac{d\phi}{dx}(L) = 0.$$

Problem 6: (*Textbook problem 2.5.12*)

(a) Using the divergence theorem, determine an alternative expression for

$$\int\int\int u\nabla^2 u dx dy dz$$

(b) Using part (a), prove that the solution of Laplace's equation $\nabla^2 u = 0$ (with u given on the boundary) is unique

Problem 7: (*Textbook problem 2.5.14*) Show that the "backward" heat equation

$$\frac{\partial u}{\partial t} = -k\frac{\partial^2 u}{\partial x^2},$$

subject to $u(0, t) = u(L, t) = 0$ and $u(x, 0) = f(x)$, is *not* well posed. [*Hint:* Show that if the data are changed an arbitrary small amount, for example,

$$f(x) \to f(x) + \frac{1}{n}\sin\frac{n\pi x}{L}$$

for large n, then the solution $u(x, t)$ changes by a large amount.]