HW1.3

Problem 5: $c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q$, $c, \rho, K_0 =$ constants.

Part (a) $Q = 0$, $u(0) = 0$, $u(L) = T$

$$\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = c_1 x + c_2$$

$$0 = u(0) = c_2 \Rightarrow c_2 = 0$$

$$T = u(L) = c_1 L \Rightarrow c_1 = \frac{T}{L}$$

$$\Rightarrow u = \frac{T}{L} x + 0 = \frac{T}{L} x.$$

Part (d) $Q = 0$, $u(0) = T$, $\frac{\partial u}{\partial x}(L) = \alpha$

$$\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u(x) = c_1 x + c_2$$

$$T = u(0) = c_2 \Rightarrow c_2 = T$$

$$\alpha = \frac{\partial u}{\partial x}(L) = c_1 \Rightarrow c_1 = \alpha$$

therefore $u = \alpha x + T$.

Part (f) $0 = K_0 \frac{\partial^2 u}{\partial x^2} + Q \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{Q}{K_0} = -x^2$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{x^3}{3} + c_1 \Rightarrow u = -\frac{x^4}{12} + c_1 x + c_2$$

$$T = u(0) = c_2 \Rightarrow c_2 = T$$

$$0 = \frac{\partial u}{\partial x}(L) = -\frac{L^3}{3} + c_1 \Rightarrow c_1 = \frac{L^3}{3}$$

$$\Rightarrow u = -\frac{x^4}{12} + \frac{L^3}{3} x + T.$$

Part (h) $\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = c_1 x + c_2$

$$\alpha = \frac{\partial u}{\partial x}(L) = c_1 \Rightarrow c_1 = \alpha$$

$$0 = \frac{\partial u}{\partial x}(0) - [u(0) - T] = c_1 - [c_2 - T] = \alpha - [c_2 - T] \Rightarrow c_2 = T + \alpha$$

$$\therefore u = \alpha x + T + \alpha.$$

Problem 6:

Part (a) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 1$, $\frac{\partial u}{\partial x}(L,t) = \beta$.

$\frac{\partial^2 u}{\partial x^2} = -1 \Rightarrow \frac{\partial u}{\partial x} = -x + C_1 \Rightarrow u = -\frac{x^2}{2} + C_1 x + C_2$

$1 = \frac{\partial u}{\partial x}(0,t) = -0 + C_1 \Rightarrow C_1 = 1.$

$\beta = \frac{\partial u}{\partial x}(L,t) = -L + C_1 = -L + 1$

If $\beta = 1 - L$, there is an equilibrium solution, which is $\left(u = -\frac{x^2}{2} + x + C_2\right)$.

If $\beta \neq 1 - L$, there isn't an equilibrium solution.

The difficulty is caused by the heat flow being specified at both ends and a source specified inside. An equilibrium will exist only if these three are in balance. This balance can be mathematically verified from conservation of energy:

$$\frac{d}{dt}\int_0^L c\rho u\,dx = \int_0^L c\rho \frac{\partial u}{\partial t}\,dx = \int_0^L -\frac{\partial \phi}{\partial x} + Q\,dx \quad , \quad Q = 1$$

$$= \phi(0) - \phi(L) + L = -\frac{\partial u}{\partial x}(0) + \frac{\partial u}{\partial x}(L) + L = -1 + \beta + L$$

If $\beta = 1 - L \Rightarrow$ the total energy is constant

$$\Rightarrow \int_0^L f(x) = \int_0^L u(x)\,dx = \int_0^L -\frac{x^2}{2} + x + C_2 = -\frac{L^3}{6} + \frac{L^2}{2} + C_2 L$$

$$\Rightarrow C_2 = -\frac{1}{L}\left(-\frac{L^3}{6} + \frac{L^2}{2}\right) \cancel{\frac{L^2}{6} - \frac{L}{2}} + \frac{1}{L}\int_0^L f(x)\,dx$$

$$\therefore u = -\frac{x^2}{2} + x + \frac{L^2}{6} - \frac{L}{2} + \frac{1}{L}\int_0^L f(x)\,dx$$

HW1.4

Part (b) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 1$, $\frac{\partial u}{\partial x}(L,t) = \beta$

$$\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = c_1 x + c_2$$

$$1 = \frac{\partial u}{\partial x}(0) = c_1 \Rightarrow c_1 = 1$$

$$\beta = \frac{\partial u}{\partial x}(L,t) = c_1 \Rightarrow \beta = c_1 = 1$$

If $\beta = 1$, there is an equilibrium solution

$u = x + c_2$. and if $\beta \neq 1$, there are no equilibrium solutions.

To determine, we check that the total energy is constant

$$\frac{d}{dt}\int_0^L u\,dx = \int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \frac{\partial^2 u}{\partial x^2}\,dx$$

$$= \frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) = 1 - \beta = 0$$

$\Rightarrow \int_0^L u\,dx$ is constant in time

$$\Rightarrow \int_0^L f(x)\,dx = \int_0^L x + c_2\,dx = \frac{L^2}{2} + c_2 L$$

$$\Rightarrow c_2 = -\frac{L}{2} + \frac{1}{L}\int_0^L f(x)\,dx$$

$$\therefore u = x - \frac{L}{2} + \frac{1}{L}\int_0^L f(x)\,dx.$$

Part (c) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x - \beta$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$

$$\frac{\partial^2 u}{\partial x^2} = \beta - x \Rightarrow \frac{\partial u}{\partial x} = \beta x - \frac{x^2}{2} + C_1 \Rightarrow u = \beta \frac{x^2}{2} - \frac{x^3}{6} + C_1 x + C_2$$

$$0 = \frac{\partial u}{\partial x}(0) = C_1 \Rightarrow C_1 = 0$$

$$0 = \frac{\partial u}{\partial x}(L) = \beta L - \frac{L^2}{2} + C_1 = \beta L - \frac{L^2}{2} \Rightarrow \beta = \frac{L}{2}$$

Only if $\beta = \frac{L}{2}$, we can find equilibrium solutions.

To determine C_2, we need to check that the total energy is constant in time

$$\frac{d}{dt}\int_0^L u\,dx = \int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \frac{\partial^2 u}{\partial x^2} + x - \beta\,dx$$

$$= \frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) + \frac{x^2}{2}\Big|_0^L - \beta x\Big|_0^L$$

$$= 0 - 0 + \frac{L^2}{2} - \beta L = \frac{L^2}{2} - \frac{L}{2}L = 0$$ ✓

$$\Rightarrow \int_0^L f(x)\,dx = \int_0^L u(x)\,dx = \int_0^L \beta\frac{x^2}{2} - \frac{x^3}{6} + C_2\,dx$$

$$= \frac{L}{4}\frac{x^3}{3}\Big|_0^L - \frac{x^4}{24}\Big|_0^L + C_2 L = \frac{L}{4}\frac{L^3}{3} - \frac{L^4}{24} + C_2 L$$

$$= \frac{L^4}{24} + C_2 L \Rightarrow C_2 = -\frac{L^3}{24} + \frac{1}{L}\int_0^L f(x)\,dx$$

$$\Rightarrow u = \frac{L}{4}x^2 - \frac{x^3}{6} - \frac{L^3}{24} + \frac{1}{L}\int_0^L f(x)\,dx$$