HW1.1
02/08/13

Problem 1

Let's consider the equatron:

$$\frac{d}{dt}\int_a^b e(x,t)\,dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)\,dx$$

and take the derivative with respect to b:

$$\frac{\partial}{\partial b}\left[\frac{d}{dt}\int_a^b e(x,t)\,dx\right] = \frac{\partial}{\partial b}\left[\int_a^b c\rho\,\frac{\partial u(x,t)}{\partial t}\,dx\right]$$

$$= c(b)\rho(b)\,\frac{\partial u(b,t)}{\partial t}$$ by the fundamental theorem of calculus.

$$\frac{\partial}{\partial b}\left[\phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)\,dx\right]$$

$$= 0 - \frac{\partial \phi(b,t)}{\partial b} + Q(b,t).$$

Therefore: $$c(b)\rho(b)\,\frac{\partial u(b,t)}{\partial b} = -\frac{\partial \phi(b,t)}{\partial b} + Q(b,t)$$

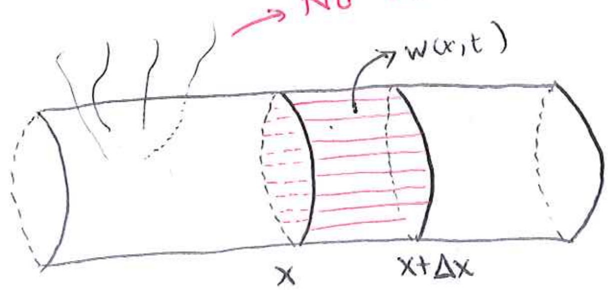
Finally replace b by x:

$$c(x)\rho(x)\,\frac{\partial u(x,t)}{\partial x} = -\frac{\partial \phi(x,t)}{\partial x} + Q(x,t)$$

Problem 2:

Part (a)

No insulated lateral surface.



$w(x,t)$ = energy flowing out of lateral sides per unit surface are per unit time.

if $w(x,t) > 0 \Rightarrow$ energy is flowing out of the rod.

Take ~~an~~ small slice of the rod, from x to $x+\Delta x$, as in the figure above.

Conservation of energy now reads: → No heat source

rate of change of thermal energy = energy flowing through boundaries + energy flowing through lateral surface.

In the small slice, the energy density is approximately constant. Then

if $\omega > 0$ ⇒ loose energy.

$$\frac{d}{dt}\big[\underbrace{e(x,t)}_{\text{energy per unit volume}}\ \underbrace{A\Delta x}_{\text{volume}}\big] = \underbrace{\phi(x,t)}_{\text{energy per unit surface are per unit time}}\underbrace{A}_{\text{area}} - \phi(x+\Delta x,t)A - \underbrace{\omega(x,t)}_{\text{energy per unit surface area per unit time}}\ \underbrace{P\Delta x}_{\text{Lateral surface area}}$$

P = Lateral perimeter

$$\Rightarrow \frac{d}{dt}\left[e(x,t)\right] = \frac{\phi(x,t)-\phi(x+\Delta x,t)}{\Delta x} - \frac{P}{A}\,\omega(x,t)$$

As $\Delta x \to 0$, we get

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} - \frac{P}{A}\,\omega(x,t).$$

Part (b) ω is proportional to the temperature difference between the rod and ~~and~~ a known outside temp. $\gamma(x,t)$

$$\Rightarrow \omega(x,t) = \left[u(x,t)-\gamma(x,t)\right]h(x),$$

$h(x)$ = constant of proportionality

$\phi = K_0(x)\dfrac{\partial u}{\partial x}$ Therefore $$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) \cancel{\tfrac{P}{A}} - \frac{P}{A}\left[u(x,t)-\gamma(x,t)\right]$$

Part (c)

The extra term now acts as a heat source, so

$$Q(x,t) = -\frac{P}{A}\left[u(x,t) - \gamma(x,t)\right] h(x).$$

Part (d)

Circular cross section $\Rightarrow$ $P = 2\pi r$, $A = \pi r^2$; r = radius of cross section.

c, ρ, K_0 constants, $\gamma(x,t) = 0$, $h(x) = h_0$ constant.

K_0 = constant.

$$\Rightarrow c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} - \frac{2\pi r}{\pi r^2} u(x,t) h_0.$$

$$\therefore c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} - \frac{2}{r} u(x,t) h_0.$$

Part (e)

Uniform temperature: $u(x,t) = u(t)$

$$\Rightarrow \frac{\partial u}{\partial x} = 0. \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow c\rho \frac{du}{dt} = -\frac{2h_0}{r} u(t) \Rightarrow \frac{du}{dt} = -\frac{2h_0}{c\rho r} u$$

Therefore $u(t) = u_0 e^{-\frac{2h_0}{c\rho r} t}$

Problem 3:

If no energy is lost at $x = x^0$, that means that the flux is continuous at $x = x_0$

$$\phi(x_0^-, t) = \phi(x_0^+, t).$$

Under what conditions is $\frac{\partial u}{\partial x}$ continuous at $x = x_0$?

See next page.

$$\phi = K_0(x)\frac{\partial u(x,t)}{\partial x}$$

and we assume $K_0(x) \neq 0$ ~~and~~

~~$\Rightarrow \phi(x,t)$ is continuous at $x=x_0$~~

~~only~~ We know $\phi(x,t) = K_0(x)\frac{\partial u(x,t)}{\partial x}$ is continuous at $x=x_0$

Then $\frac{\partial u(x,t)}{\partial x}$ is continuous at $x=x_0$ if and only if $K(x)$ is continuous at $x=x_0$.

Problem 4
$$\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & k>0. \\ u(0,t)=0 \\ u(2\pi,t)=0 \end{cases}$$

Steady state solutions: $\frac{\partial^2 u_{steady}}{\partial x^2} = 0 \Rightarrow u_{steady} = C_1 x + C_2$.

Using the boundary conditions, we get:

$0 = u_{steady}(0,t) = C_2 \Rightarrow C_2 = 0$

$0 = u_{steady}(2\pi,t) = C_1 2\pi \Rightarrow C_1 = 0$

Therefore $u_{steady} \equiv 0$.

Now we look for non-steady solutions of the form

$u(x,t) = \phi(t)\sin x$

$\Rightarrow \frac{\partial u}{\partial t} = \phi'(t)\sin x, \quad \frac{\partial u}{\partial x} = \phi(t)(+\cos x), \quad \frac{\partial^2 u}{\partial x^2} = \phi(t)(-\sin x)$

$\Rightarrow \phi'(t)\sin x = -k\phi(t)\sin x \Rightarrow \phi'(t) = -k\phi(t)$

$\Rightarrow \phi(t) = Ce^{-kt} \Rightarrow u(x,t) = Ce^{-kt}\sin x$

Since $k>0 \Rightarrow \lim_{t\to\infty} u(x,t) = 0 = u_{steady}(x)$.