Hw 11.1

2013

Homework 11:

Problem 1: Consider (with $h>0$)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ \begin{cases} \dfrac{\partial u}{\partial x}(0,t) - h u(0,t) = 0 \\ \dfrac{\partial u}{\partial x}(L,t) = 0 \end{cases} \\ \begin{cases} u(x,0) = f(x) \\ \dfrac{\partial u}{\partial x}(x,0) = g(x) \end{cases} \end{cases}$$

(a) Show that there are an infinite number of different frequencies of oscillation.

Answer:

$$u(x,t) = \phi(x) G(t) \Rightarrow \dfrac{d^2 G}{dt^2} = -\lambda c^2 G$$

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + \lambda \phi = 0 \\ \dfrac{d\phi}{dx}(0) - h\phi(0) = 0 \\ \dfrac{d\phi}{dx}(L) = 0 \end{cases}$$

$$\phi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x .$$

$$\frac{d\phi}{dx} = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$\Rightarrow \sqrt{\lambda}c_2 - h c_1 = 0$$

$$\Rightarrow c_1 = \frac{\sqrt{\lambda}}{h} c_2 .$$

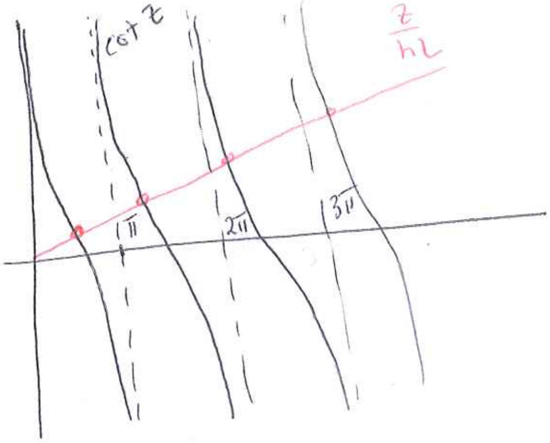
On the other hand, $-c_1\sqrt{\lambda}\sin\sqrt{\lambda}L + c_2\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$

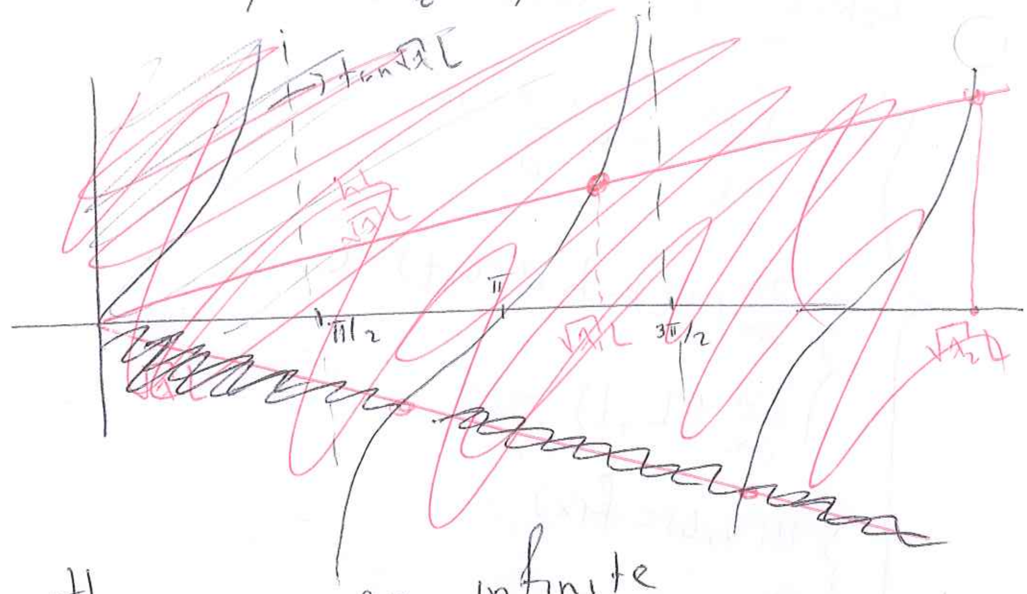
$$\Rightarrow \tan\sqrt{\lambda}L = \frac{c_2}{c_1} = \frac{h}{\sqrt{\lambda}} = \frac{hL}{\sqrt{\lambda}L}$$

$\circledast$

The eigenvalues are located at the intersection of the graph of $\tan z$, an $\frac{hL}{z}$, for $z = \sqrt{\lambda}\,L$.

or $\cot z = \frac{z}{hL}$





It is clear that there is an infinite number of oscillations.

(b) Estimate the large frequencies of oscillation

Answer:

We can see from the graph that the points of intersection are closer and closer to $\lambda_n \approx \left(\frac{(n-1)\pi}{L}\right)^2$ for large values of λ_n.

$(n-1)\pi$

$\Rightarrow$

(c) Solve the initial value problem.

Answer: Assume you know the eigenvalues.

$\Rightarrow \phi_n(x) = c_1 \cos\sqrt{\lambda_n}\,x + \frac{\sqrt{\lambda_n}}{h} c_1 \sin\sqrt{\lambda_n}\,x = c_1\left(\cos\sqrt{\lambda_n}\,x + \frac{\sqrt{\lambda_n}}{h}\sin\sqrt{\lambda_n}\,x\right)$

The time dependent part is of the form:

$G(t) = d_1 \cos\sqrt{\lambda_n}\,ct + d_2 \sin\sqrt{\lambda_n}\,ct$.

$\Rightarrow$ The principle of superposition gives:

$$u(x,t) = \sum_{n=1}^{\infty}\left(\cos\sqrt{\lambda_n}\,x + \frac{\sqrt{\lambda_n}}{h}\sin\sqrt{\lambda_n}\,x\right)\left(a_n\cos\sqrt{\lambda_n}\,ct + b_n\sin\sqrt{\lambda_n}\,ct\right)$$

$$= \sum_{n=1}^{\infty}\phi_n(x)\left(a_n\cos\sqrt{\lambda_n}\,t + b_n\sin\sqrt{\lambda_n}\,t\right).$$

To find the coefficients, we need the initial conditions

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n\,\phi_n(x)$$

$$\Rightarrow a_n = \frac{\int_0^L f(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n(x)^2\,dx}$$

$$g(x) = \frac{\partial u}{\partial x}(x,0) = \sum_{n=1}^{\infty} b_n\sqrt{\lambda_n}\;\phi_n(x)$$

$$\Rightarrow b_n = \frac{1}{\sqrt{\lambda_n}}\int_0^L g(x)\,\phi_n(x)\,dx \Big/ \int_0^L \phi_n(x)^2\,dx.$$

Problem 2: Consider the boundary value problem:

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + \lambda\phi = 0 \\ \phi(0) - \dfrac{d\phi}{dx}(0) = 0 \\ \phi(1) + \dfrac{d\phi}{dx}(1) = 0 \end{cases}$$

(a) Using the Rayleigh quotient, show that $\lambda \geq 0$. Why is $\lambda > 0$?

$$-\phi(1)\frac{d\phi}{dx}(1)+\phi(0)\frac{d\phi}{dx}(0) = +\phi(1)^2+\phi(0)^2$$

Answer:

$$\lambda = \frac{-\phi\frac{d\phi}{dx}\Big|_0^1 + \int_0^1 \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^1 \phi^2\, dx}$$

$$= \frac{\phi(1)^2+\phi(0)^2 + \int_0^1 \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^1 \phi^2\, dx} \geq 0$$

because $\phi(1)^2+\phi(0)^2 \geq 0$

$\int_0^1 \left(\frac{d\phi}{dx}\right)^2 dx \geq 0$.

Why is $\lambda > 0$?

Suppose $\lambda = 0$ is an eigenvalue

$\Rightarrow \frac{d^2\phi}{dx^2} = 0 \Rightarrow \phi(x) = c_1x + c_2$

$\Rightarrow c_2 - c_1 = 0 \Rightarrow c_1 = c_2$

$c_1 + c_2 + c_1 = 0 \Rightarrow c_1 = c_2 = 0$

$\Rightarrow \phi = 0$, a contradiction to the fact that we can find non-trivial solutions!

$\Rightarrow \lambda > 0$.

(b) Prove that eigenfunctions corresponding to different eigenvalues are orthogonal.

Answer: We can do it directly, or we can follow the analysis we did in class.

Let ϕ_1, ϕ_2 be two eigenfunction with eigenvalues $\lambda_1 \neq \lambda_2$

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$\Rightarrow \int_0^1 \phi_1'' \phi_2 \, dx = \phi_1' \phi_2 \Big|_0^1 - \int_0^1 \phi_1' \phi_2' dx$

$= \phi(1)\phi_2(1) - \phi_1(0)\phi_2(0) - \left(\phi_1 \phi_2' \Big|_0^1 - \int_0^1 \phi_1 \phi_2'' dx \right)$

$= -\cancel{\phi_1(1)\phi_2(1)} - \phi_1(0)\phi_2(0) + \cancel{\phi_1(1)\phi_2(1)} + \cancel{\phi_1(0)}\phi_2(0)$

$+ \int_0^1 \phi_1 \phi_2'' dx$

$\Rightarrow \int_0^1 \phi_1'' \phi_2 dx = \int_0^1 \phi_1 \phi_2'' dx$

$\Rightarrow \int_0^1 \lambda_1 \phi_1 \phi_2 dx = \int_0^1 \lambda_2 \phi_1 \phi_2 dx$

$\Rightarrow (\lambda_1 - \lambda_2) \int_0^1 \phi_1 \phi_2 dx = 0$

$\Rightarrow \int_0^1 \phi_1 \phi_2 dx = 0$ because $\lambda_1 \neq \lambda_2$.

(c) Show that

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Answer:

$\phi(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

$\frac{d\phi}{dx} = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x$

$\Rightarrow 0 = c_1 - c_2\sqrt{\lambda} \Rightarrow c_1 = \sqrt{\lambda}\, c_2$

and

$0 = c_1 \cos\sqrt{\lambda} + c_2 \sin\sqrt{\lambda} - c_1\sqrt{\lambda}\sin\sqrt{\lambda} + c_2\sqrt{\lambda}\cos\sqrt{\lambda}$

$= (c_1 + c_2\sqrt{\lambda})\cos\sqrt{\lambda} + (c_2 - c_1\sqrt{\lambda})\sin\sqrt{\lambda}$

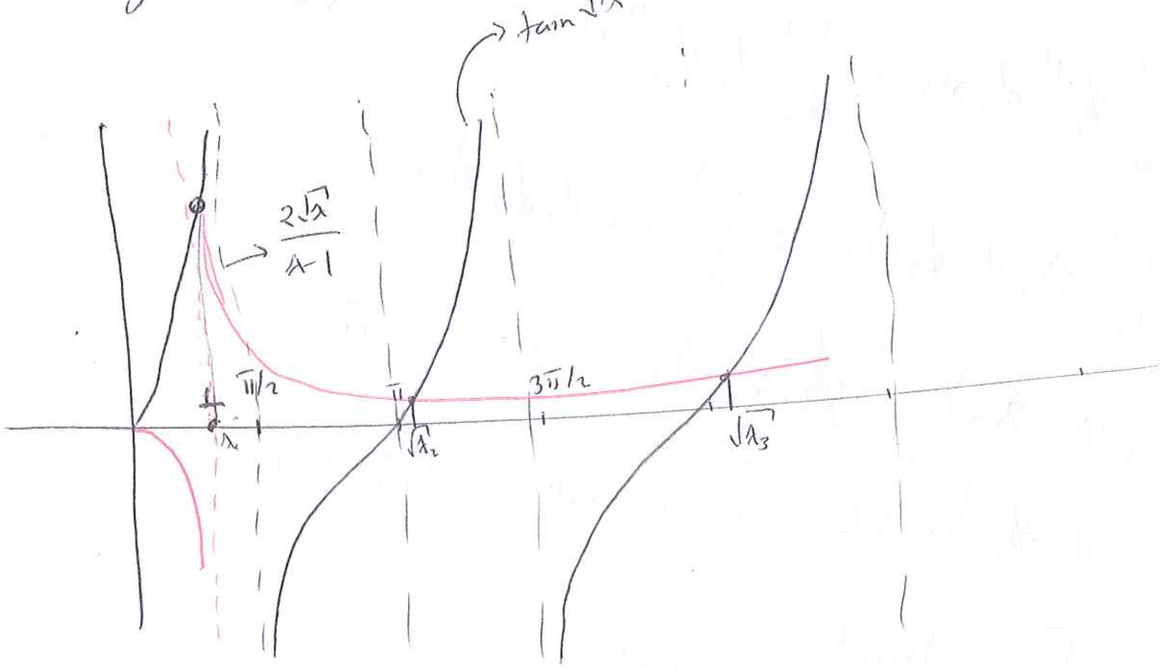
$=$

$\Rightarrow 0 = 2c_2\sqrt{\lambda}\cos\sqrt{\lambda} + c_2(1-\lambda)\sin\sqrt{\lambda}$

Since the sol. is not trivial $\Rightarrow c_2 \neq 0$

$$\Rightarrow \tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Determine the eigenvalues graphically. Estimate the large eigenvalues (using the graph.)



$\Rightarrow \sqrt{\lambda_n} \approx (n-1)\pi \quad \Rightarrow \lambda_n \approx ((n-1)\pi)^2.$

(d) Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ with

$$\begin{cases} u(0,t) - \frac{\partial u}{\partial x}(0,t) = 0 \\ u(1,t) + \frac{\partial u}{\partial x}(1,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

You may call the relevant eigenfunctions $\phi_n(x)$ and assume they are known.

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The time dependent part of the solution satisfies:

$$\frac{dG}{dt} = -\lambda k G \Rightarrow G(t) = e^{-\lambda_n k t}$$

and ϕ_n satisfies the Sturm-Liouville problem from above, which is assumed to be known

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n k t}$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

$$a_n = \frac{\int_0^1 f(x)\phi_n(x)dx}{\int_0^1 \phi_n(x)^2 dx}.$$

Problem 3: Estimate (to leading order) the large eigenvalues and corresponding eigenfunction for

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + [\lambda\sigma(x) + q(x)]\phi = 0$$

if the boundary conditions are as follows.

Answer:

The general form for the leading order approximation is:

$$\phi(x) \approx (\sigma p)^{-1/4}\left[c_1 \cos\left(\sqrt{\lambda}\int_0^x \left(\frac{\sigma}{p}\right)^{1/2} dz\right) + c_2 \sin\left(\sqrt{\lambda}\int_0^x \left(\frac{\sigma}{p}\right)^{1/2} dz\right)\right]$$

Now we need the boundary conditions.

(a) $\frac{d\phi}{dx}(0)=0, \quad \frac{d\phi}{dx}(L)=0$

Answer:

$$\frac{d\phi}{dx} = \frac{d}{dx}\left((\sigma\rho)^{-1/4}\right)\left[c_1\cos\left(\sqrt{\lambda}\int_0^x \left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)+c_2\sin\left(\sqrt{\lambda}\int_0^x\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)\right]$$

$$+(\sigma\rho)^{-1/4}\left[-c_1\sqrt{\lambda}\left(\frac{\sigma}{\rho}\right)^{1/2}\sin\left(\sqrt{\lambda}\int_0^x\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)+c_2\sqrt{\lambda}\left(\frac{\sigma}{\rho}\right)^{1/2}\cos\left(\sqrt{\lambda}\int_0^x\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)\right]$$

$$\Rightarrow -\frac{1}{4}(\sigma\rho)^{-5/4}(0)\,(\sigma\rho)'(0)\,c_1+c_2\sqrt{\lambda}\left(\frac{\sigma}{\rho}\right)^{1/4}(0)=0$$

$$\Rightarrow c_2=\lambda^{-1/2}\frac{1}{4}(\sigma\rho)^{-3/2}(0)\,(\sigma\rho)'(0)\,c_1$$

and

$$-\frac{1}{4}(\sigma\rho)^{-5/4}(L)\,(\sigma\rho)'(0)\left[c_1\cos\left(\sqrt{\lambda}\int_0^L\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)+c_2\sin\left(\sqrt{\lambda}\int_0^L\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)\right]$$

$$+(\sigma\rho)^{-1/4}\left[-c_1\sqrt{\lambda}\left(\frac{\sigma}{\rho}\right)^{1/2}(L)\sin\left(\sqrt{\lambda}\int_0^L\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)+c_2\sqrt{\lambda}\left(\frac{\sigma}{\rho}\right)^{1/2}(L)\cos\left(\sqrt{\lambda}\int_0^L\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)\right]$$

$$=0$$

$$=c_1\left[\left((\sigma\rho)^{-1/4}\right)'(L)\cos\left(\sqrt{\lambda}\int_0^L\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)-(\sigma\rho)^{-1/4}(L)\sqrt{\lambda}\left(\frac{\sigma}{\rho}\right)^{1/2}(L)\sin\left(\sqrt{\lambda}\int_0^L\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)\right]$$

$$+\lambda^{-1/2}\frac{\left((\sigma\rho)^{-1/4}\right)'(0)}{\left(\frac{\sigma}{\rho}\right)^{1/4}(0)}\,c_1\left[\left((\sigma\rho)^{-1/4}\right)'(L)\,\cancel{(\sigma\rho)}\sin\left(\sqrt{\lambda}\int_0^L\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)+(\sigma\rho)^{-1/4}(L)\sqrt{\lambda}\left(\frac{\sigma}{\rho}\right)^{1/2}(L)\cos\left(\sqrt{\lambda}\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)\right]$$

Then λ must satisfy (approximately) the following equation:

$$\left((\sigma\rho)^{-1/4}\right)'(L)\cos\left(\sqrt{\lambda}\int_0^L\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)-(\sigma\rho)^{-1/4}(L)\sqrt{\lambda}\left(\frac{\sigma}{\rho}\right)^{1/2}(L)\sin\left(\sqrt{\lambda}\int_0^L\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)$$

$$+\lambda^{-1/2}\left((\sigma\rho)^{-1/4}\right)'(0)\left(\frac{\sigma}{\rho}\right)^{-1/4}(0)\left[\left((\sigma\rho)^{-1/4}\right)'(L)\sin\left(\sqrt{\lambda}\int_0^L\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)+(\sigma\rho)^{-1/4}(L)\sqrt{\lambda}\left(\frac{\sigma}{\rho}\right)^{1/2}(L)\cos\left(\sqrt{\lambda}\left(\frac{\sigma}{\rho}\right)^{1/2}dx\right)\right]$$

$$=0.$$

(b) $\phi(0)=0$ and $\frac{d\phi}{dx}(L)=0$

Answer:

Since $\phi(0)=0 \Rightarrow c_1=0$

$$\Rightarrow \phi(x) \approx (\sigma p)^{-1/4} c_2 \sin\left(\sqrt{\lambda}\int_0^x \left(\frac{\sigma}{p}\right)^{1/2} d\tau\right).$$

and $\left((\sigma p)^{-1/4}\right)'(L)\, c_2 \sin\left(\sqrt{\lambda}\int_0^L \left(\frac{\sigma}{p}\right)^{1/2} dx\right)$

$$+ (\sigma p)^{-1/4}(L)\, c_2 \sqrt{\lambda}\cdot\left(\frac{\sigma}{p}\right)^{1/2}(L) \cos\left(\sqrt{\lambda}\int_0^L \left(\frac{\sigma}{p}\right)^{1/2} dx\right) = 0$$

$$\Rightarrow \cot\left(\sqrt{\lambda}\int_0^L \left(\frac{\sigma}{p}\right)^{1/2} dx\right) = \frac{-\left((\sigma p)^{-1/4}\right)'(L)}{(\sigma p)^{-1/4}(L)\sqrt{\lambda}\left(\frac{\sigma}{p}\right)^{1/2}(L)}$$

$$= \frac{-\left((\sigma p)^{-1/4}\right)'(L)}{\sqrt{\lambda}\,\sigma^{1/4}(L)\, p^{-3/4}(L)}.$$

(c) $\phi(0)=0$, $\frac{d\phi}{dx}(L)+h\phi(L)=0$

Answer:

$\Rightarrow c_1=0$, and

$$\left((\sigma p)^{-1/4}\right)'(L)\, c_2 \sin\left(\sqrt{\lambda}\int_0^L \left(\frac{\sigma}{p}\right)^{1/2} dx\right)$$

$$+(\sigma p)^{-1/4}(L)\, c_2 \sqrt{\lambda}\left(\frac{\sigma}{p}\right)^{1/2}(L)\cos\left(\sqrt{\lambda}\int_0^L \left(\frac{\sigma}{p}\right)^{1/2} dx\right)$$

$$+h\,(\sigma p)^{-1/4}\, c_2 \sin\left(\sqrt{\lambda}\int_0^L \left(\frac{\sigma}{p}\right)^{1/2} dx\right).$$

$$\Rightarrow \left(\left((\sigma p)^{-1/4}\right)'(L) + h\,(\sigma p)^{-1/4}(L) \right) \sin\left(\sqrt{\lambda}\int_0^L \left(\frac{\sigma}{p}\right)^{1/2} dx\right)$$

$$= -\,(\sigma p)^{-1/4}(L)\,\sqrt{\lambda}\,\left(\frac{\sigma}{p}\right)^{1/2}(L)\,\cos\left(\sqrt{\lambda}\int_0^L \left(\frac{\sigma}{p}\right)^{1/2} dx\right)$$

Therefore

$$\tan\left(\sqrt{\lambda}\int_0^L \left(\frac{\sigma}{p}\right)^{1/2} dx\right) = \frac{-\,(\sigma p)^{-1/4}(L)\sqrt{\lambda}\left(\frac{\sigma}{p}\right)^{1/2}(L)}{\left((\sigma p)^{-1/4}\right)'(L) + h\,(\sigma p)^{-1/4}(L)}.$$