MATH 322 - SEC 001, SPRING 2013. HOMEWORK 1

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Due : Friday, February 8

Please show all your work and/or justify your answers for full credit.

Problem 1: (*Problem 1.2.7 from textbook*) Consider conservation of thermal energy

$$\frac{d}{dt}\int_a^b e dx = \phi(a,t) - \phi(b,t) + \int_a^b Q dx,$$

for any segment of a one-dimensional rod $a \le x \le b$. By using the fundamental theorem of calculus,

$$\frac{\partial}{\partial b}\int_a^b f(x)dx = f(b),$$

derive the heat equation

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q.$$

Problem 2: (*Problem 1.2.9 of textbook*) Consider a thin one-dimensional rod without sources of thermal energy whose lateral surface area is not unsalted.

(a) Assume that the heat energy flowing out of the lateral sides per unit surface area per unit time is $w(x,t)$. Derive the partial differential equation for the temperature $u(x,t)$.

(b) Assume that $w(x,t)$ is proportional to the temperature difference between the rod $u(x,t)$ and a known outside temperature $\gamma(x,t)$. Derive that

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}\left[u(x,t) - \gamma(x,t)\right]h(x), \tag{0.0.1}$$

where $h(x)$ is a positive x-dependent proportionality, P is the lateral perimeter, and A is the cross-sectional area.

(c) Compare equation (0.0.1) to the equation for a one-dimensional rod whose lateral surfaces are insulated, but with the heat sources.

(d) Specialize (0.0.1) to a rod of circular cross section with constant thermal properties and $0°$ outside temperature.

(e) Consider the assumptions in part (d). Suppose that the temperature in the rod is uniform [i.e., $u(x,t) = u(t)$]. Determine $u(t)$ if initially $u(0) = u_0$.

Problem 3: (*Problem 1.3.2 of textbook*)Two one-dimensional rods of different materials joined at $x = x_0$ are said to be in perfect thermal contact if the temperature is continuous at $x = x_0$:

$$u(x_0^-, t) = u(x_0^+, t),$$

and no heat energy is lost at $x = x_0$ (i.e., the heat energy flowing out of one flows into the other). What mathematical equation represents the latter condition at $x = x_0$? Under what special conditions is $\frac{\partial u}{\partial x}$ continuous at $x = x_0$?

Problem 4: Consider the diffusion equation for $0 \le x \le 2\pi$ with Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial t} & = & k\frac{\partial^2 u}{\partial x^2} \\ u(0,t) & = & 0 \\ u(2\pi,t) & = & 0 \end{cases}$$

where $k > 0$ is a positive constant. Show that $u_{\text{steady}}(x) = 0$ is the only steady-state solution satisfying the boundary conditions above. Find all solutions of the form

$$u(x,t) = \phi(t)\sin(x)$$

Prove that in all cases,

$$\lim_{t\to\infty} u(x,t) = 0 = u_{\text{steady}}(x).$$

Problem 5:(*Problem 1.4.1 of textbook*) Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:

(a) $Q = 0,\ u(0) = 0,\ u(L) = T$

(d) $Q = 0,\ u(0) = T,\ \frac{\partial u}{\partial x}(L) = \alpha$

(f) $\frac{Q}{K_0} = x^2,\ u(0) = T,\ \frac{\partial u}{\partial x}(L) = 0$

(h) $Q = 0,\ \frac{\partial u}{\partial x}(0) - [u(0) - T] = 0,\ \frac{\partial u}{\partial x}(L) = \alpha$

Problem 6: (*Problem 1.4.7 of textbook*) For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

(a)
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1,\ u(x,0) = f(x),\ \frac{\partial u}{\partial x}(0,t) = 1,\ \frac{\partial u}{\partial x}(L,t) = \beta$$

(b)
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ u(x,0) = f(x),\ \frac{\partial u}{\partial x}(0,t) = 1,\ \frac{\partial u}{\partial x}(L,t) = \beta$$

(c)
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x - \beta,\ u(x,0) = f(x),\ \frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0.$$