Problem 1: 0221.1 2012

Practice exam:

$$\begin{cases} \frac{dy}{dx} = (e^{y+2} - 1) y (y-2) \\ y(0) = y_0 \end{cases}$$

(a) Sketch, roughy, ~~a~~ slope field and classify all the critical points.

Critical points:

$e^{y+2} - 1 = 0, \quad y = 0, \quad y = 2 \quad \Rightarrow \quad y = -2, \; y = 0, \; y = 2$

If $y < -2 \Rightarrow e^{y+2} - 1 < 0, \; y < 0, \; y - 2 < 0$

$\Rightarrow \frac{dy}{dx} = (<0)(<0)(<0) \Rightarrow \frac{dy}{dx} < 0$

If $-2 < y < 0 \Rightarrow y + 2 > 0 \Rightarrow e^{y+2} - 1 > 0$

Also, $y < 0, \; y - 2 < 0 \Rightarrow \frac{dy}{dx} > 0$

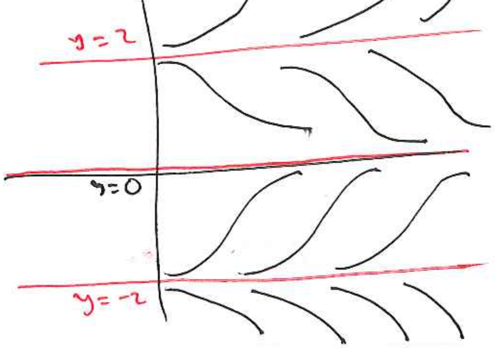
If ~~y~~ $0 < y < 2 \Rightarrow e^{y+2} - 1 > 0, \; y > 0, \; y - 2 \cancel{>} 0$ $<$

$\Rightarrow \frac{dy}{dx} < 0$

If $y > 2 \Rightarrow e^{y+2} - 1 > 0, \; y > 0, \; y - 2 > 0$

$\Rightarrow \frac{dy}{dx} > 0$

$\Rightarrow$ The slope fied looks like:



$\Rightarrow y = 2$ is unstable

$y = 0$ is stable

$y = -2$ is unstable.

(b) Determine (from your sketch), the ~~asy~~ asymptotic behavior for $y_0 = -1$ as $t \to \infty$.

$y_0 = -1 \in [-2, 0] \Rightarrow \lim_{t \to +\infty} y(t) = 0$, based on the sketch.

Problem 2 Solve

$$\begin{cases} (x+y)\,y' = x - y \\ y(1) = 0. \end{cases}$$

~~Let's try the substitution:~~

~~$v = x+y$~~

This problem can be solved in two ways:

As seen in class, we can try to transform this equation into one of the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ and try $v = y/x$

Divide by x:

$\left(1 + \frac{y}{x}\right)\frac{dy}{dx} = 1 - \frac{y}{x}$ Try the substitution $v = \frac{y}{x}$

$y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow (1+v)\left(v + x\frac{dv}{dx}\right) = 1 - v \Rightarrow x(1+v)\frac{dv}{dx} = 1 - v - v(1+v) = 1 - 2v - v^2$

$\Rightarrow \frac{1+v}{1-2v-v^2}\frac{dv}{dx} = \frac{1}{x}$ Integrate both sides:

$$\int \frac{1+v}{1-2v-v^2}\,dv = \ln|x| + C_1$$

Let's find the first integral

$$\int \frac{1+v}{1-2v-v^2}\,dv =$$

$w = 1+v \Rightarrow dw = dv, \quad 1-2v-v^2 = 1-2(w-1)-(w-1)^2 = \cancel{1}-\cancel{2w}+2-w^2+\cancel{2w}-\cancel{1}$

$= 2-w^2$

$$= \int \frac{w}{2-w^2}\,dw = \frac{1}{2}\int \frac{1}{2-w_2}\,dw_2 = -\frac{1}{2}\ln\left(|w^2-2|\right)$$

$w_2 = w^2 \qquad dw_2 = 2w\,dw \qquad = -\frac{1}{2}\ln|(1+v)^2-2|$

$\Rightarrow -\frac{1}{2}\ln|(1+v)^2-2| = \ln x + C_1 \Rightarrow \ln|(1+v)^2-2| = -2\ln x + C_2$

$|(1+v)^2-2| = x^{-2}\cdot C_3 \Rightarrow (1+v)^2 = 2 + C_3x^{-2}$ (C_3 could be pos. or neg.)

$v = -1 \pm \sqrt{2+C_3x^{-2}}, \quad v = \frac{y}{x}$

$\Rightarrow y = -x \pm \sqrt{2x^2+C_3}$

The other way is much simpler and requires a clever substitution $v = x+y$

$y = v-x \qquad \frac{dy}{dx} = \frac{dv}{dx} - 1$

$(x+y)\frac{dy}{dx} = x-y$

$\Rightarrow v\left(\frac{dv}{dx}-1\right) = x-(v-x) = 2x-v \Rightarrow v\frac{dv}{dx} - \cancel{v} = 2x - \cancel{v}$

$\Rightarrow v\frac{dv}{dx} = 2x$ Integrate w.r.t. x: $v = \pm\sqrt{2x^2+C_2}$, $v = x+y$

$\frac{v^2}{2} = x^2 + C_1 \Rightarrow v^2 = 2x^2 + C_2$

$\therefore y = -x \pm \sqrt{2x^2+C_2}$

$y(1) = -1 \pm \sqrt{2+C_2} = 0 \Rightarrow 2+C_2 = 1 \quad C_2 = -1$

$y(x) = -x + \sqrt{2x^2-1}$, Range of validity:

$2x^2-1 \geq 0 \iff x^2 \geq \frac{1}{2} \iff |x| \geq \frac{1}{\sqrt{2}}$

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Problem 3: Consider the initial value problem:

$$\begin{cases} \frac{dy}{dx} = -\frac{5}{2} x^4 y^3 \\ y(0) = -1 \end{cases}$$

(a) Find $y(x)$ explicitly. For what values of ~~t~~ x is the solution defined?

Separation of variables:

$$y^{-3} \frac{dy}{dx} = -\frac{5}{2} x^4 \quad \text{Integrate both sides:}$$

$$\frac{y^{-2}}{-2} = -\frac{5}{2} \frac{x^5}{5} + C = -\frac{1}{2} x^5 + C_2 \Rightarrow y^{-2} = x^5 + C_3$$

$$y^2 = \frac{1}{x^5 + C_3} \qquad y = \frac{1}{\pm\sqrt{x^5 + C_3}}$$

$$y(0) = \frac{1}{\pm\sqrt{C_3}} = -1 \Rightarrow C_3 = 1$$

$$y(x) = -\frac{1}{\sqrt{x^5 + 1}}$$

Defined where $x^5 > -1 \Leftrightarrow x > -1$.

Problem 4 Write the following system as $Ax=b$ and determine for what values of k the system has (i) a unique solution, (ii) no solution, and (iii) infinitely many solutions.

$$x_1 - x_2 + 2x_3 = 4$$
$$2x_1 + 3x_2 - x_3 = k$$
$$-2x_1 + x_2 - 3x_3 = 2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 & 4 \\ 2 & 3 & -1 & k \\ -2 & 1 & -3 & 2 \end{bmatrix} \xrightarrow{(-2)R_1+R_2} \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 5 & -5 & k-8 \\ -2 & 1 & -3 & 2 \end{bmatrix}$$

$$\xrightarrow{(+2)R_1+R_3} \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 5 & -5 & k-8 \\ 0 & -1 & 1 & 10 \end{bmatrix} \xrightarrow{+\frac{1}{5}R_2} \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & \frac{k-8}{5} \\ 0 & -1 & 1 & 10 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & \frac{k-8}{5} \\ 0 & 0 & 0 & 10+\frac{k-8}{5} \end{bmatrix}$$

Consistent if $10 + \frac{k-8}{5} = 0$ $\quad k-8 = -50 \quad k = -42$

$k = -42$ infinitely many sol.

$k \neq -42$ no solutions

Never a unique solution.