Problem #52, section 3.6

The square matrix A is called orthogonal provided that $A^T = A^{-1}$. Show that the determinant of such a matrix must be either $+1$ or -1

Proof:

Property 7 says that $|A^T| = A^{-1}$

We also know that if A is invertible, then

$$\cancel{|A^T| =} \quad |A^{-1}| = \frac{1}{|A|} \quad (\text{eqn. } 15, \text{page } 211)$$

$\Rightarrow$ Taking the determinant on both sides of eqn above, we get:

$$\cancel{|A^T| = 1} \quad |A^T| = |A^{-1}| \Rightarrow |A| = \frac{1}{|A|}$$

$$\Rightarrow |A|^2 = 1 \quad \Rightarrow |A| = 1 \text{ or } |A| = -1.$$

Check the following properties for 2×2 matrices: 0309.1 2012

Property 1: If the $n\times n$ matrix B is obtained from A by multiplying a single row (or column) of A by the constant k, then $\det B = k \det A$.

Proof:
Row: Without loss of generality, we can assume that the first row is the one multiplied by k.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

$$|B| = kad - c\cdot kb = k(ad-bc) = k|A|.$$

The proof for the column is similar.

Property 2: If the $n\times n$ matrix B is obtained from A by interchanging two rows (or two columns), then $\det B = -\det A$.

Proof: Without loss of generality (WLOG), assume that the two rows are interchanged

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$|B| = cb - ad = -(ad-bc) = -|A|.$$

Property 3: If two rows (or two colums) of the $n\times n$ matrix are identical, then $\det A = 0$.

WLOG: Two rows are identical

$$A = \begin{bmatrix} a & b \\ a & b \end{bmatrix} \Rightarrow |A| = ab - ab = 0.$$

Property 4: Suppose that the $n\times n$ matrices A_1, A_2 an B are identical except for their ith rows - that is, the other $n-1$ rows of the three matrices are identical - and that the ith rows of B is the sum of the ith rows of A_1 and A_2. Then $\det B = \det A_1 + \det A_2$.

Pf: WLOG assume $i=1$

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ c & d \end{bmatrix},\ A_2 = \begin{bmatrix} a_2 & b_2 \\ c & d \end{bmatrix},\ B = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c & d \end{bmatrix}$$

$$|B| = (a_1+a_2)d - c\cdot(b_1+b_2) = a_1 d - cb_1 + a_2 d - cb_2$$
$$= |A_1| + |A_2|. \ \blacksquare$$

Property 5: If the $n\times n$ matrix B is obtained by adding a constant multiple of one row (or column) of A to another row (or colum) of A, then $\det B = \det A$.

Proof: We can use properties 1, 3 and 4 to prove it, or do it directly.

Assume the first row $\times k$ is added to row 2 WLOG.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ B = \begin{bmatrix} a & b \\ c+ka & d+kb \end{bmatrix}$$

$$\Rightarrow |B| = a\cdot(d+kb) - (c+ka)b = ad - bc + kab - kab = |A|.$$

Property 6: The determinant of a triangular matrix is equal to the product of its diagonal elements.

If $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \Rightarrow |A| = ad - 0b = ad$

If $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \Rightarrow |A| = ad - c.0 = ad.$

Property 7: If A is a square matrix, then
$\det(A^T) = \det A$

Proof: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$|A^T| = ad - bc = |A|.$$