03 02 2012

Hw 5:

Problem 32, Section 3.3

Show that the 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is row equivalent to the 2x2 identity matrix provided that $ad - bc \neq 0$.

Proof: Assume that $ad - bc \neq 0$.

Case 1: $a \neq 0, c \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[a \cdot R_2]{c \cdot R_1} \begin{bmatrix} ac & bc \\ ac & ad \end{bmatrix} \xrightarrow{(-1)R_1 + R_2} \begin{bmatrix} ac & bc \\ 0 & ad-bc \end{bmatrix}$$

$$\xrightarrow{\frac{1}{c}R_1} \begin{bmatrix} a & b \\ 0 & ad-bc \end{bmatrix} \xrightarrow[\text{since } ad-bc\neq 0]{\frac{1}{ad-bc} R_2} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(-b)R_2 + R_1} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{a}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}.$$

Case 2: $a = 0 \Rightarrow 0 - bc \neq 0 \Rightarrow b \neq 0, c \neq 0$

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \xrightarrow{\text{swap}(R_1, R_2)} \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \xrightarrow{\frac{1}{b}R_2} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(-d)R_2 + R_1} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{c}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}.$$

Case 3: $c = 0 \Rightarrow ad - 0 \neq 0 \Rightarrow a \neq 0, d \neq 0$

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \xrightarrow{\frac{1}{d}R_2} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \xrightarrow{(-b)R_2 + R_1} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{a}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$