HW 2 Sample Homework solutions.

L0208.1 2012

Section 1.3

Problem #27 (a) Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \le c \\ (x-c)^2 & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 2\sqrt{y}$ for all x.

Solution For $x < c$ $\frac{dy}{dx} = 0 = 2\sqrt{y}$ since $y(x) = 0$ for $x < c$

For $x > c$, $y(x) = (x-c)^2$ $\frac{dy}{dx} = 2(x-c)$, and

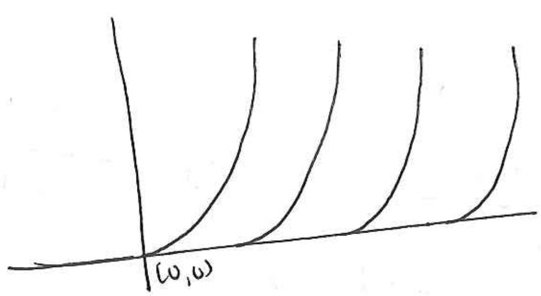
$2\sqrt{y} = 2\sqrt{(x-c)^2} = 2(x-c)$ since $x - c \geq 0$

$\therefore$ At $x = c$, y has 1 derivative and $\left.\frac{dy}{dx}\right|_{x=c} = 0 = 2\sqrt{y(c)}$

$\therefore \frac{dy}{dx} = 2\sqrt{y}$ for all x.

Construct a figure illustrating the fact that the initial value problem $y' = 2\sqrt{y}$, $y(0) = 0$ has infinitely many solutions.

Solution: ~~For~~ The graph of the function above for different positive values of c looks like:



and each of them satisfies the D.E and $y(0) = 0$.

(b) ~~Solution~~: For what values of b does the initial value problem $y' = 2\sqrt{y}$, $y(0) = b$ have (i) no solution, (ii) a unique solution that is defined for all x.

Solution:

If $b < 0$, then the initial value problem $y' = 2\sqrt{y}$, $y(0) = b$ has no solution, because the square root of a negative number would be involved.

If $b > 0$ we get a unique solution curve through $(0, b)$ defined for all x by following a parabola.

But starting at $(0, 0)$ we can follow the positive x-axis to the point $(c, 0)$ and then branching off on the parabola $(x-c)^2$.

Section 1.4 Problem #10 Find general solutions of

$$(1+x)^2 \frac{dy}{dx} = (1+y)^2$$

Separation of variables: $\frac{1}{(1+y)^2}\frac{dy}{dx} = \frac{1}{(1+x)^2}$

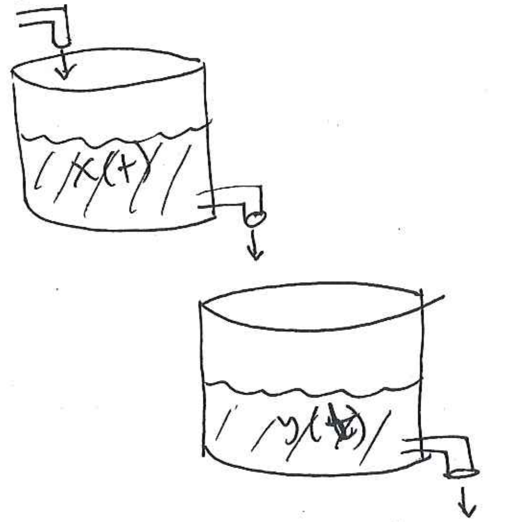
$$\Rightarrow \int \frac{1}{(1+y)^2}\,dy = \int \frac{1}{(1+x)^2}\,dx = -\frac{1}{1+x} + C$$

$$\Rightarrow \frac{-1}{1+y} = -\frac{1}{1+x} + C = \frac{-1 + C(1+x)}{1+x} \Rightarrow 1+y = \frac{-1-x}{-1+C(1+x)}$$

$$\Rightarrow y = \frac{\cancel{x} - C(1+x) - \cancel{1} - x}{-1 + C(1+x)} = \frac{x + C(1+x)}{1 - C(1+x)}$$

$$\therefore y = \frac{x + C(1+x)}{x - C(1+x)}$$

1.5 No. 38 Consider a cascade of two tanks shown in the figure below, with $V_1 = 100$ (gal) and $V_2 = 200$ (gal) the volume of brine in the two tanks. Each tank also initially contains 50 lb of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank 1.



(a) Find the amount $x(t)$ of salt in tank 1 at time t.

Answer:

r_i = rate of inflow into tank 1 = 5 gal/min

C_i = concentration of inflow = 0 (pure water)

r_o = rate of outflow = 5 gal/min

Initially the volume is 100 gal, and the rate of output is the same as the rate of input. Then the volume is constant in time.

According to equation 18 in page 54,

$$\frac{dx}{dt} = r_i C_i - \frac{r_o}{V_1} x = 0 - \frac{5}{100} x = -\frac{1}{20} x$$

$$\Rightarrow \ln x(t) = -\frac{1}{20} t + C \qquad x(t) = e^{-\frac{1}{20}t + C}$$

Initial condition: $x(0) = 50$ lb $\Rightarrow e^C = 50 \qquad C = \ln 50$

$$\Rightarrow x(t) = 50 e^{-\frac{1}{20}t}$$

(b) Suppose that $y(t)$ is the amount of ~~solute~~ salt in tank 2 at time t. Show first that

$$\frac{dy}{dx} = \frac{5x}{100} - \frac{5y}{200}$$

and then solve for y using the function $x(t)$ found in part (a)

Answer:

The output of tank 1 becomes the input of tank 2

Then, the concentration of the brine coming into tank 2 is

$$C_{ii} := \frac{x(t)}{V_1} = \frac{x(t)}{100}$$

and the rate is the same $r_{ii} = r_1 = 5\,gal/min$.

The output has the same rate too, $r_o = 5\,gal/min$.

Using equation (18) from the book again, applied to y, we get:

$$\frac{dy}{dt} = C_{ii} r_{ii} - \frac{r_o}{V_2} y(t) = \frac{x}{100} 5 - \frac{5}{200} y$$

$$= \frac{5x}{100} - \frac{5y}{200}$$

Then $\frac{dy}{dt} = \frac{1}{20} \cdot 50 e^{-\frac{1}{20}t} - \frac{1}{40} y = \frac{5}{2} e^{-\frac{1}{20}t} - \frac{1}{40} y$

$\Rightarrow \frac{dy}{dt} + \frac{1}{40} y = \frac{5}{2} e^{-\frac{1}{20}t} \longrightarrow$ Linear

Integrating factor: $\rho(t) = e^{\int \frac{1}{40} dt} = e^{\frac{1}{40}t}$

$\Rightarrow \frac{d}{dt}\left[e^{\frac{1}{40}t} y\right] = \frac{5}{2} e^{-\frac{1}{20}t} e^{\frac{1}{40}t} = \frac{5}{2} e^{-\frac{1}{40}t}$ Integrate both sides:

$\Rightarrow e^{\frac{1}{40}t} y = \frac{5}{2}(-40) e^{-\frac{1}{40}t} + C = -100 e^{-\frac{1}{40}t} + C \Rightarrow y = -100 e^{-\frac{1}{20}t} + C \cdot e^{-\frac{1}{40}t}$

Initial condition, $y(0) = 50\,lb \Rightarrow -100 + C = 50 \Rightarrow C = 150$

$\therefore y(t) = -100 e^{-\frac{1}{20}t} + 150 e^{-\frac{1}{40}t}$

(c) Find the maximum amount of salt in tank 2: $\frac{dy}{dt} = 5e^{-\frac{1}{20}t} - \frac{15}{4} e^{-\frac{1}{40}t} = 0$

$\Leftrightarrow 5e^{-\frac{1}{20}t} = \frac{15}{4} e^{-\frac{1}{40}t} \Leftrightarrow \frac{4}{3} = e^{\frac{1}{40}t} \Leftrightarrow t = 40 \ln(4/3) \approx 11.51$ min

To see [illegible] is a maximum: Since, there is only 1 critical point, just need to [illegible] both sides of crit. pt. $\frac{dy}{dt}\Big|$ $= 0.036330$, $\frac{dy}{dt}\Big|$ $= -0.0$ [illegible]