0427.1

Hw 12:

Section 5.5 ~~#23~~ #43 (a) Write

$\cos 3x + i\sin 3x = e^{3ix} = (\cos x + i\sin x)^3$

by Euler's formula, expand, and equate real and imaginary parts to derive the identities:

$\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$

$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

Answer:

$\cos 3x + i\sin 3x = e^{3ix} = (\cos x + i\sin x)^3$

$= \cos^3 x + 3\cos^2 x\, i\sin x + 3\cos x\,(i\sin x)^2 + (i\sin x)^3$

$= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x$

$= \cos^3 x - 3\cos x\sin^2 x + i(3\cos^2 x\sin x - \sin^3 x)$

$\Rightarrow \cos 3x = \cos^3 x - 3\cos x\sin^2 x = \cos^3 x - 3\cos x(1-\cos^2 x)$

$= \cos^3 x + 3\cos^3 x - 3\cos x = 4\cos^3 x - 3\cos x$

$\Rightarrow \cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$

and

$\sin 3x = 3\cos^2 x\sin x - \sin^3 x = 3(1-\sin^2 x)\sin x - \sin^3 x$

$= 3\sin x - 3\sin^3 x - \sin^3 x = 3\sin x - 4\sin^3 x$

$\Rightarrow \sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

(b) Use the result of part (a) to find a general solution of ~~the form~~

$y'' + 4y = \cos^3 x$

Answer: let's first find ~~the~~ a particular solution:

$y'' + 4y = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$

For the $\frac{3}{4}\cos x$ term, the guess is:

$y_p = a\cos x + b\sin x$

$\Rightarrow y_p' = -a\sin x + b\cos x \qquad y_p'' = -a\cos x - b\sin x$

$y_p'' + 4y_p = -a\cos x - b\sin x + 4(a\cos x + b\sin x)$

$= (-a+4a)\cos x + 3b\sin x = 3a\cos x + 3b\sin x = \frac{3}{4}\cos x$

$\Rightarrow a = \frac{1}{4}, \quad b = 0 \qquad y_p = \frac{1}{4}\cos x$

For the $\frac{1}{4}\cos 3x$, the guess is (we know $b=0$, as above)

$y_p = a\cos 3x + \cancel{b\sin 3x}$

$y_p' = -3a\sin 3x \qquad y_p'' = -9a\cos 3x$

$y_p'' + 4y_p = -9a\cos 3x + 4a\cos 3x = -5a\cos 3x = \frac{1}{4}\cos x$

$\Rightarrow a = -\frac{1}{20} \quad \Rightarrow y_p = -\frac{1}{20}\cos 3x$

By the principle of superposition, a particular solution to $y'' + 4y = \cos^3 x$ is

$y_p = \frac{1}{4}\cos x - \frac{1}{20}\cos 3x$

~~Homog. soln~~ Solution to the homog. system: $y'' + 4y = 0$

$r^2 + 4 = 0$ ← char. eqn. $\qquad r = \pm 2i$

$\Rightarrow y_c = c_1\cos 2x + c_2\sin 2x$ ← doesn't coincide with the particular solution

$\Rightarrow$ The general solution is:

$y(x) = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\cos x - \frac{1}{20}\cos 3x.$

Section 5.6 #1.

$$\begin{cases} x'' + 9x = 10\cos 2t \\ x(0) = x'(0) = 0. \end{cases}$$

Homog. eqn: $x'' + 9x = 0$

Char. eqn: $r^2 + 9 = 0$ $r = \pm 3i$

$x_c(t) = c_1 \cos 3t + c_2 \sin 3t$.

Particular solution:

guess: $x_p = a \cdot \cos 2t + b \sin 2t$

$x_p' = -2a \sin 2t + 2b\cos 2t$, $x_p'' = -4a\cos 2t - 4b\sin 2t$

$x_p'' + 9x_p = -4a\cos 2t - 4b\sin 2t + 9a\cos 2t + 9b\sin 2t$

$= 5a\cos 2t + 5b\sin 2t = 10\cos 2t$

$\Rightarrow a = 2,\ b = 0$ $\Rightarrow x_p = 2\cos 2t$

$x(t) = x_c + x_p = c_1\cos 3t + c_2 \sin 3t + 2\cos 2t$

Initial conditions:

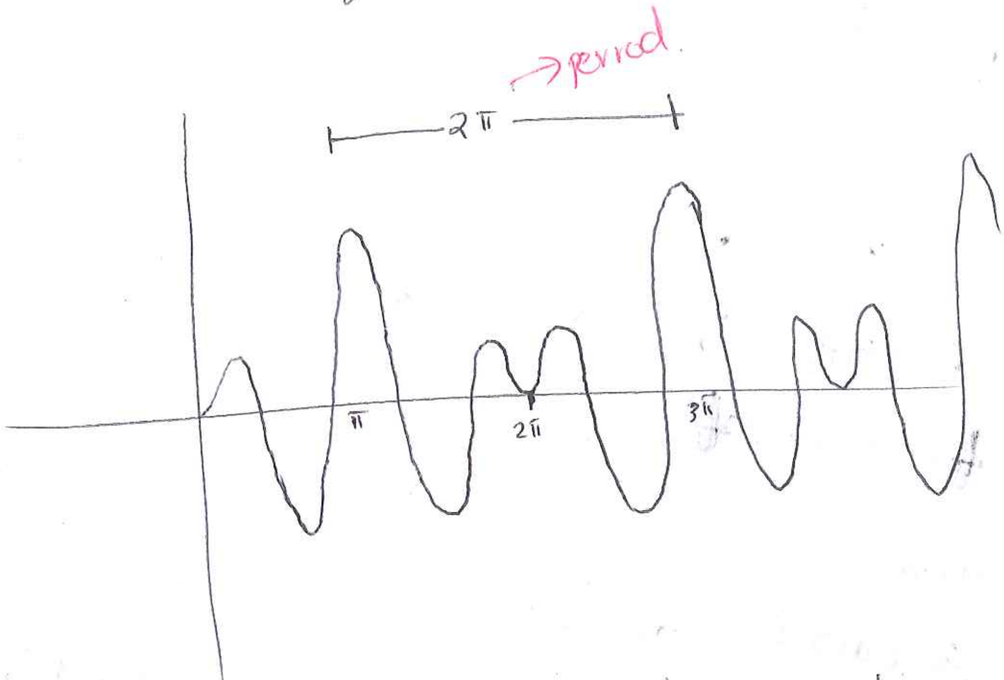
$x(0) = c_1 + 2 = 0$ $\Rightarrow c_1 = -2$

$x'(t) = -3c_1 \sin 3t + 3c_2\cos 3t - 4\sin t$

$x'(0) = 3c_2 = 0 \Rightarrow c_2 = 0$

$\therefore x(t) = -2\cos 3t + 2\cos 2t$

The following figure shows the graph of $x(t)$



From the graph we can easily see that $x(t)$ is 2π-periodic.

Another way to ~~see~~ compute the period is as follows:

$\cos(3t)$ is $2\pi/3$ periodic

$\cos(2t)$ is $2\pi/2 = \pi$ periodic

The least common (whole) multiple of $2\pi/3$ and π is 2π

$\Rightarrow x(t)$ is 2π-periodic.