0413.1
2012

Section 5.1:

Problem: #24: Determine whether the following pair of functions are linearly independent or linearly dependent on the real line.

$f(x) = \sin^2 x \quad , \quad g(x) = 1 - \cos 2x$

A1: Using the identity $\cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$ we get

$\cos(2x) = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$

$\Rightarrow 1 - \cos(2x) = 2\sin^2 x$

$\Rightarrow g(x) = 2f(x)$

$\therefore$ g and f are linearly dependent.

A2: You can also take the Wronskian

$$W = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \begin{vmatrix} \sin^2 x & 1 - \cos 2x \\ 2\sin x \cdot \cos x & + 2\sin 2x \end{vmatrix}$$

$\sin(2x) = 2\sin x \cos x$

$$= \begin{vmatrix} \sin^2 x & 2\sin^2 x \\ 2\sin x \cos x & 4\sin x \cos x \end{vmatrix} = 0.$$

$\therefore$ f and g are linearly dependent.

Section 5.2

Problem 10: Use the wronskian to prove that

$f(x) = e^{x}$, $g(x) = e^{2x}$, $h(x) = e^{3x}$ are

linearly indep. on the real line.

$$W = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= e^{x}\left(18e^{5x} - 12e^{5x}\right) - e^{2x}\left(9e^{4x} - 3e^{4x}\right) + e^{3x}\left(4e^{3x} - 2e^{3x}\right)$$

$$= 6e^{6x} - 6e^{6x} + 2e^{6x} = 2e^{6x} > 0 \text{ for all } x$$

$\Rightarrow$ f, g, h are linearly indep. on the real line.