07/6.1
2017

Tarea 6 Fundamentos de Matemáticas para Materiales

Problema 1: Evaluar las siguientes integrales de fracciones polinomiales.

- $\int \frac{2x^5-10x^3-2x^2+10}{x^2-5}\,dx$

R: El grado del polinomio del numerador es mayor que el del denominador. Entonces dividimos:

$$\begin{array}{l|l} 2x^5-10x^3-2x^2+10 & x^2-5 \\ \hline 2x^5-10x^3 & 2x^3-2 \\ \quad -2x^2+10 & \\ \quad -2x^2+10 & \\ \quad 0 & \end{array}$$

$$\Rightarrow \int \frac{2x^5-10x^3-2x^2+10}{x^2-5}\,dx = \int (2x^3-2)\,dx = \frac{2x^4}{4} - 2x + C = \frac{x^4}{2} - 2x + C$$

- $\int \frac{1}{x^2+3}\,dx = \int \frac{1}{3\left(\frac{x^2}{3}+1\right)}\,dx = \frac{1}{3}\int \frac{1}{1+\left(\frac{x}{\sqrt{3}}\right)^2}\,dx = \frac{1}{3}\sqrt{3}\arctan\left(\frac{x}{\sqrt{3}}\right)+C$

- $\int \frac{5x-2}{x(x-2)}\,dx$

R: Dado que el grado del polinomio del denominador es el mayor, expresamos la fracción como suma de funciones simples

$$\frac{5x-2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} = \frac{A(x-2)+Bx}{x(x-2)} = \frac{(A+B)x-2A}{}$$

$$\Rightarrow \begin{array}{l} -2=-2A \\ A+B=5 \end{array} \Rightarrow \begin{array}{l} A=1 \\ B=4 \end{array}$$

$$\Rightarrow \int \frac{5x-2}{x(x-2)}\,dx = \ln|x| + 4\ln|x-2| + C$$

- $\int \frac{2x^2+1}{(x-3)^2}\,dx$ Como los grados son el mismo, se dividen

$$\begin{array}{l|l} 2x^2+1 & x^2-6x+9 \\ \hline 2x^2-12x+18 & 2 \\ \quad 12x-17 & \end{array}$$

$\Rightarrow 2x^2+1 = 2(x^2-6x+9) + (12x-17)$

$\Rightarrow \int \frac{2x^2+1}{x^2-6x+9}\, dx = \int 2\, dx + \int \frac{12x-17}{x^2-6x+9}\, dx$

Ahora el segundo término tiene un polinomio de grado mayor y se puede escribir como sumas simples:

$$\frac{12x-17}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} = \frac{A(x-3)+B}{(x-3)^2}$$

$\Rightarrow 12x-17 = Ax-3A+B$

$\Rightarrow A=12, \quad -36+B=-17 \quad \Rightarrow B=-17+36=19$

$$\Rightarrow \int \frac{2x^2+1}{(x-3)^2} = 2x + 12\ln|x-3| + 19\frac{(x-3)^{-1}}{-1} + C$$

$$= 2x + 12\ln|x-3| - \frac{19}{x-3} + C$$

$$\int \frac{x^2}{x^2+3}\, dx = \int \frac{x^2+3-3}{x^2+3}\, dx = \int 1\, dx - 3\int \frac{1}{x^2+3}\, dx$$

$$= x - \cancel{3}\int \frac{1}{1+(x/\sqrt{3})^2}\, dx = x - \sqrt{3}\arctan(x/\sqrt{3}) + C$$

$$\int \frac{x^5-x^2}{x^3-2x}\, dx$$

Como el grado del numerador es mayor, dividimos

$$\begin{array}{l|l}
x^5 \qquad -x^2 & x^3-2x \\ \cline{2-2}
\underline{x^5-2x^3} & x^2+2 \\
2x^3-x^2 & \\
\underline{2x^3 \qquad -4x} & \\
-x^2+4x &
\end{array}$$

$$\Rightarrow \frac{x^5-x^2}{x^3-2x} = x^2+2 + \frac{4x-x^2}{x^3-2x}$$

$$\frac{x(4-x)}{x(x^2-2)} = \frac{4-x}{(x-\sqrt{2})(x+\sqrt{2})} = \frac{A}{x-\sqrt{2}} + \frac{B}{x+\sqrt{2}} = \frac{A(x+\sqrt{2})+B(x-\sqrt{2})}{(x^2-2)}$$

$$\Rightarrow \quad A+B=-1$$
$$\sqrt{2}A-\sqrt{2}B = 4 \Rightarrow A-B = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$\Rightarrow 2A = -1+2\sqrt{2} \Rightarrow A = -\tfrac{1}{2}+\sqrt{2} = \sqrt{2}-\tfrac{1}{2}$$
$$B = -1-\sqrt{2}+\tfrac{1}{2} = -\sqrt{2}-\tfrac{1}{2}.$$

$$\Rightarrow \int \frac{x^5-x^2}{x^3-2x}\,dx = \int\left[x^2+2 + \frac{\sqrt{2}-\frac{1}{2}}{x-\sqrt{2}} - \frac{\sqrt{2}+\frac{1}{2}}{x+\sqrt{2}}\right]dx$$

$$= \frac{x^3}{3}+2x + \left(\sqrt{2}-\tfrac{1}{2}\right)\ln|x-\sqrt{2}| - \left(\sqrt{2}+\tfrac{1}{2}\right)\ln|x+\sqrt{2}| + C.$$

— o —

$$\int \frac{x^5+x^3+x}{x^4+x^2+1}\,dx = \int \frac{x(x^4+x^2+1)}{x^4+x^2+1}\,dx = \int x\,dx = \frac{x^2}{2}+C$$

— o —

$$\int \frac{x+1}{x^3+x^2-6x}\,dx$$

R: Factorizar el denominador:
$$x^3+x^2-6x = x(x^2+x-6) = x(x+3)(x-2).$$

$\Rightarrow$ Como el denominador tiene grado mayor:

$$\frac{x+1}{x^3+x^2-6x} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2} = \frac{A(x+3)(x-2)+Bx(x-2)+Cx(x+3)}{x(x+3)(x-2)}$$

$$\Rightarrow x+1 = A(x^2+x-6)+B(x^2-2x)+C(x^2+3x)$$
$$= x^2(A+B+C)+x(A-2B+3C)-6A$$

$$\Rightarrow A = -1/6, \quad B+C = -A = 1/6$$
$$-2B+3C = 1-A = 1+1/6 = \tfrac{7}{6}$$

$$5C = \cancel{3}\ \tfrac{2}{6}+\tfrac{7}{6} = 9/6 = 3/2 \Rightarrow C = 3/10. \Rightarrow B = 1/6 - \tfrac{3}{10} = \frac{5-9}{30} = \frac{-4}{30} = \frac{-2}{15}$$

$$\Rightarrow \int\frac{x+1}{x^3+x^2-6x}\,dx = -\tfrac{1}{6}\ln|x| - \tfrac{2}{15}\ln|x+3| + \tfrac{3}{10}\ln|x-2| + C.$$

$$\int \frac{1}{x^2+2x+2}\,dx$$

R: Las raices son: $x = \dfrac{-2 \pm \sqrt{4-4\cdot1\cdot2}}{2} = -1 \pm i$

$$\Rightarrow \int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx = \arctan(x+1) + C.$$

— o —

Problema 2: Encuentra las soluciones generales de las ecuaciones diferenciales ordinarias de segundo orden.

(a) $10u''(x) - 3u'(x) + u(x) = 0$

Buscar soluciones de la forma $u(x) = e^{\lambda x}$

$$\Rightarrow 10\lambda^2 - 3\lambda + 1 = 0 \Rightarrow \lambda = \frac{3 \pm \sqrt{9-4\cdot10\cdot1}}{20} = \frac{3 \pm \sqrt{-31}}{20}$$

$$= \frac{3}{20} \pm \frac{\sqrt{31}}{20} i$$

$$\Rightarrow u(x) = A e^{\frac{3}{20}x} \cos\left(\frac{\sqrt{31}}{20}x\right) + B e^{\frac{3}{20}x} \operatorname{sen}\left(\frac{\sqrt{31}}{20}x\right).$$

(b) $u''(x) - u(x) = 0 \Rightarrow$ Polinomio característico es $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\Rightarrow u(x) = Ae^{x} + Be^{-x}$

(c) $u''(x) - 6u'(x) - 2u(x) = 0 \Rightarrow \lambda^2 - 6\lambda - 2 = 0$

$$\Rightarrow \lambda = \frac{6 \pm \sqrt{36-4\cdot1\cdot(-2)}}{2} = 3 \pm \frac{\sqrt{44}}{2} = 3 \pm \sqrt{11}$$

$$\Rightarrow u(x) = Ae^{(3+\sqrt{11})x} + Be^{(3-\sqrt{11})x}$$

(d) $u''(x) - 4u' + 5u(x) = 0 \Rightarrow \lambda^2 - 4\lambda + 5 = 0$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16-4\cdot1\cdot(5)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

$$\Rightarrow u(x) = Ae^{2x}\cos x + Be^{2x} \operatorname{sen} x.$$