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MATH 115 - SEC 011, WINTER 2011. QUIZ 7
TIME LIMIT: 20 MINUTES

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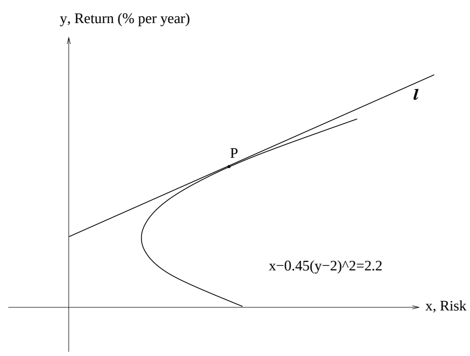
Good luck!

Problem 1. In Modern Portfolio Theory, a clients portfolio is structured in a way that balances risk and return. For a certain type of portfolio, the risk, x, and return, y, are related by the equation $x - 0.45(y-2)^2 = 2.2$. This curve is shown in the graph below. The point P represents a particular portfolio of this type with a risk of 3.8 units. The tangent line, l, through point P is also shown.



(a) Using implicit differentiation, find $\frac{dy}{dx}$, and the coordinates of the point(s) where the slope is undefined.

Taking derivatives both sides we get

$$\frac{d}{dx}\left(x - 0.45(y-2)^2\right) = 0$$

$$= 1 - 0.45 \cdot 2 \cdot (y-2)\frac{dy}{dx} = 1 - 0.9(y-2)\frac{dy}{dx} = 0$$

Then,

$$1 = 0.9(y-2)\frac{dy}{dx},$$

which implies

$$\frac{dy}{dx} = \frac{1}{0.9(y-2)}.$$

The derivative doesn't exist when the denominator vanishes, this is when $y = 2$. Substituting $y = 2$ in the equation

$$x - 0.45(y-2)^2 = 2.2 \tag{0.0.1}$$

we get $x = 2.2$. Therefore, the slope doesn't exist at $x = 2.2, y = 2$.

(b) The y-intercept of the tangent line for a given portfolio is called the Risk Free Rate of Return. Use your answer from (a) to find the Risk Free Rate of Return for this portfolio.

Let's find the equation of the tangent line at $x = 3.8$. Let's first find y at $x = 3.8$ from equation (0.0.1).

$$3.8 - 0.45(y-2)^2 = 2.2, \text{ so } -0.45(y-2)^2 = 2.2 - 3.8 = -1.6$$
$$(y-2)^2 = \frac{1.6}{0.45} \approx 3.55$$
$$y - 2 = \pm\sqrt{\frac{1.6}{0.45}}.$$

We will take the plus sign because we are interested on the upper part of the graph. So

$$y = 2 + \sqrt{\frac{1.6}{0.45}} \approx 3.89.$$

Therefor, the slope at $x = 3.8$ is

$$\frac{dy}{dx} = \frac{1}{0.9\sqrt{\frac{1.6}{0.45}}}$$

Using the fact that $x = 3.8, y = 2 + \sqrt{\frac{1.6}{0.45}}$ is on the line, and using the general form $f(a) + f'(a)(x - a)$ of the tangent line, we conclude that the equation for the tangent line is

$$y = 2 + \sqrt{\frac{1.6}{0.45}} + \frac{1}{0.9 \cdot \sqrt{\frac{1.6}{0.45}}} \cdot (x - 3.8)$$

The y-intercept is then obtained by substituting $x = 0$ in the equation above:

$$2 + \sqrt{\frac{1.6}{0.45}} - \frac{3.8}{0.9 \cdot \sqrt{\frac{1.6}{0.45}}} \approx 1.646446609$$

(c) Now, estimate the return of an optimal portfolio having a risk of 4 units by using your information from part (b). Would this be an overestimate or an underestimate? Why?

We can estimate the return of an optimal portfolio having risk 4 by just substituting $x = 4$ in the equation of the tangent line:

$$2 + \sqrt{\frac{1.6}{0.45}} + \frac{1}{0.9 \cdot \sqrt{\frac{1.6}{0.45}}} \cdot (4 - 3.8) \approx 4.003469213$$

Problem 2.

(1) Explain why the following equation has a solution near 0:

$$e^t = 0.2t + 1.098$$

One easy way is to plot the two curves in the calculator. Without a graph, we can also note that

$$e^t - 0.2t - 1.098 = \begin{cases} -0.098 < 0 \text{ at } t = 0 \\ 0.00549 > 0 \text{ at } t = 0.12 \end{cases}$$

So, since at $t = 0$ the difference is negative, and at $t = 0.12$ the difference is positive, then somewhere in between ([0,0.12]), the difference needs to be zero. As a result, the equation above has a solution near zero.

(2) Replace e^t by its linearization near 0. Solve the new equation to get an approximate solution to the original equation.

Let $f(t) = e^t$. Then $f'(t) = e^t$. So, $f(0) = 1$, $f'(0) = 1$. So, the local linearization of e^t near $t = 0$ is

$$f(0) + f'(0)(t - 0) = 1 + t$$

Replacing the local linearization in the equation above, we get

$$1 + t = 0.2t + 1.098,$$

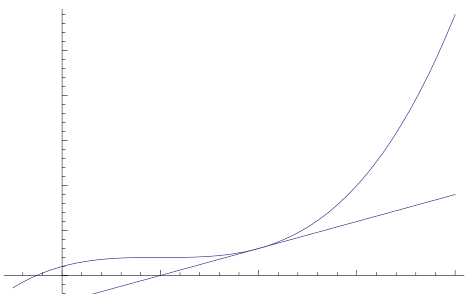
$$0.8t = 0.098,$$

$$t = \frac{0.098}{0.8} = 0.1225$$

Finding the intersection of e^t and $0.2t + 1.098$ in the calculator, we get $t = 0.11405199$ as the real answer. We notice that the approximation found above is good, considering that we found it at almost zero effort.

Problem 3.

(i) Graph $f(x) = x^3 - 3x^2 + 3x + 1$



(ii) Find and add to your sketch the local linearization of $f(x)$ at $x = 2$.

The derivative of $f(x)$ is

$$f'(x) = 3x^2 - 6x + 3,$$

so

$$f'(2) = 3 \cdot 4 - 6 \cdot 2 + 3 = 3,$$

and

$$f(2) = 8 - 3 \cdot 4 + 6 + 1 = 3$$

The local linearization of $f(x)$ at $x = 2$ is then

$$y = 3 + 3(x - 2) = 3x - 3$$

(iii) Compute and mark on your on sketch the true value of $f(1.5)$, the tangent line approximation to $f(1.5)$ and the error in the approximation.

$$f(1.5) = (1.5)^3 - 3(1.5)^2 + 3(1.5) + 1 = 2.125.$$

Using the equation from part (ii), we use the tangent line equation at $x = 2$ to approximate a value at $x = 1.5$, which gives

$$3 + 3 \cdot (1.5 - 2) = 1.5.$$

Comparing the real answer, and the approximation, we compute the error as

$$Error = 2.125 - 1.5 = 0.625$$