Polinomios ortogonales: Una introducción a la teoría de transformaciones espectrales

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Orthogonal polynomials in $\mathbb{R}$

Given a nontrivial probability measure μ supported on some infinite subset E of the real line, a (unique) sequence of orthonormal polynomials $\{p_n\}_{n\geqslant 0}$ can be defined as

$$\int_E p_m(x)p_n(x)d\mu(x) = \delta_{m,n}, \quad n, m \geqslant 0, \tag{1}$$

where

$$p_n(x) = \gamma_n x^n + \zeta_n x^{n-1} + \text{lower degree terms}, \tag{2}$$

with $\gamma_n > 0$, $n \geqslant 0$.

Classical orthogonal polynomials:

- Jacobi $d\mu(x) = (1-x)^\alpha(1+x)^\beta dx$ in $[-1, 1]$. (Tchebychev, Gegenbauer, Legendre)
- Laguerre $d\mu(x) = x^\alpha e^{-x}dx$ in $\mathbb{R}_+$.
- Hermite $d\mu(x) = e^{-x^2}dx$ in $\mathbb{R}$.

Some applications

OP appear in a wide range of applications such as:

- Approximation theory
- Integrable systems
- Numerical integration
- Signal theory
- Image processing
- Etc, etc, etc.

Three term recurrence relation

Starting from $p_0(x) = 1$ and $p_{-1}(x) = 0$, $\{p_n\}_{n\geqslant 0}$ satisfies

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geqslant 0, \tag{3}$$

where

$$a_n = \int_E xp_{n-1}(x)p_n(x)d\mu(x) = \frac{\gamma_{n-1}}{\gamma_n} > 0, \quad n \geqslant 1,$$

and

$$b_n = \int_E xp_n^2(x)d\mu(x) = \frac{\zeta_n}{\gamma_n} - \frac{\zeta_{n+1}}{\gamma_{n+1}}, \quad n \geqslant 0.$$

Favard's theorem: Given any sequences $\{a_n\}_{n\geqslant 1}$, $\{b_n\}_{n\geqslant 0}$ of real numbers, the polynomials constructed with (3) are orthogonal with respect to some measure $d\mu(x)$.

The monic Jacobi matrix

On the other hand, the monic OP with respect to μ are given by $P_n(x) = p_n(x)/\gamma_n$, $n \geqslant 0$. In such a case, (3) becomes

$$P_{n+1}(x) = (x - b_n)P_n(x) - d_n P_{n-1}(x), \quad n \geqslant 0, \tag{4}$$

with $d_n = a_n^2$, and has the matrix representation

$$xP(x) = \mathbf{J}P(x),$$

where

$$\mathbf{J} = \begin{pmatrix} b_0 & 1 & 0 & 0 & \cdots \\ d_1 & b_1 & 1 & 0 & \cdots \\ 0 & d_2 & b_2 & 1 & \ddots \\ 0 & 0 & d_3 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

is known as monic Jacobi matrix.

The LU factorization of J

Notice that $P_n(0) \neq 0$, $n \geqslant 1 \iff \mathbf{J}$ has a unique LU factorization, where $\mathbf{L}$ and $\mathbf{U}$ are bidiagonal matrices

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ l_1 & 1 & 0 & 0 & \cdots \\ 0 & l_2 & 1 & 0 & \ddots \\ 0 & 0 & l_3 & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} u_1 & 1 & 0 & 0 & \cdots \\ 0 & u_2 & 1 & 0 & \cdots \\ 0 & 0 & u_3 & 1 & \ddots \\ 0 & 0 & 0 & u_4 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{5}$$

where

$$l_1 = \frac{d_1}{b_0}, \quad l_n = \frac{d_n}{b_{n-1} - l_{n-1}}, \quad n \geqslant 2, \tag{6}$$

$$u_1 = b_0, \quad u_n = b_{n-1} - l_{n-1}, \quad n \geqslant 2. \tag{7}$$

Darboux transformations

Darboux transformation without parameter

$$\mathbf{J} = \mathbf{LU}, \quad \mathbf{J}_p := \mathbf{UL}$$

Darboux transformation (not unique)

$$\mathbf{J} = \mathbf{UL}, \quad \mathbf{J}_d := \mathbf{LU}$$

Notice that $\mathbf{J}_p$ and $\mathbf{J}_d$ are again tridiagonal matrices with ones as entries on the upper diagonal and, according to Favard's theorem, they are monic Jacobi matrices associated with some nontrivial measure $\tilde{\mu}$.

Canonical spectral transformations on $\mathbb{R}$

Christoffel transformation ($\mathcal{R}_C$)

$$d\tilde{\mu} = (x-\beta)d\mu, \quad \beta \notin supp(\mu).$$

Uvarov transformation ($\mathcal{U}_U$)

$$d\tilde{\mu} = d\mu + M_r\delta(x-\beta), \quad M_r \in \mathbb{R}.$$

Geronimus transformation ($\mathcal{R}_G$)

$$d\tilde{\mu} = \frac{d\mu}{x-\beta} + M_r\delta(x-\beta), \quad \beta \notin supp(\mu), M_r \in \mathbb{R}.$$

Proposition

$$\begin{aligned} \mathcal{R}_C \circ \mathcal{R}_G &= \mathcal{I} \quad \text{Identity transformation} \\ \mathcal{R}_G \circ \mathcal{R}_C &= \mathcal{R}_U \end{aligned}$$

LST and Stieltjes functions

The Stieltjes function associated with μ is

$$S(x) = \int_E \frac{d\mu(t)}{x-t} = \sum_{k=0}^{\infty} \frac{\mu_k}{x^{k+1}},$$

where $\mu_k = \int_E x^k d\mu(x)$ are the moments of μ. It has been shown that the previous transformations can be expressed as

$$\widetilde{S}(x) = \frac{A(x)S(x) + B(x)}{D(x)}, \tag{8}$$

where $\widetilde{S}(x)$ is the Stieltjes function associated with $\tilde{\mu}$, and $A(x)$, $B(x)$, $D(x)$ are polynomials in the variable x, which are known. Furthermore,

Proposition (Zhedanov, 97)

All transformations of the form (8) can be obtained as a composition of Christoffel and Geronimus transformations.

Rational spectral transformations

Associated polynomials

From a OPS $\{P_n\}_{n\geqslant 0}$, define the monic associated polynomials or order k, $\{P_n^{(k)}\}_{n\geqslant 0}$, by the shifted recurrence relation

$$P_{n+1}^{(k)}(x) = (x - b_{n+k})P_n^{(k)}(x) - d_{n+k}P_{n-1}^{(k)}(x), \quad n \geqslant 0,$$

i.e. removing the first k rows and columns of $\mathbf{J}$.

Anti-associated polynomials

If we "push" the first k rows and columns of $\mathbf{J}$, and introduce new coefficients $b_{-i} \quad (i = k, k-1, ..., 1)$ and $d_{-i} \quad (i = k-1, k-2, ..., 0)$, then the anti-associated polynomials of order k are defined by

$$P_{n+1}^{(-k)}(x) = (x - \tilde{b}_{n+k})P_n^{(-k)}(x) - \tilde{d}_{n+k}P_{n-1}^{(-k)}(x), \quad n \geqslant 0,$$

where $\{\tilde{b}_i\}_{i\geqslant 0} = \{b_{-i}\}_{i=k}^{1} \bigcup \{b_i\}_{i\geqslant 0}$ and $\{\tilde{d}_i\}_{i\geqslant 1} = \{d_{-i}\}_{i=k-1}^{0} \bigcup \{d_i\}_{i\geqslant 1}$.

RST and Stieltjes functions

It has been shown that the previous transformations can be expressed as

$$\widetilde{S}(x) = \frac{A(x)S(x) + B(x)}{C(x)S(x) + D(x)}, \tag{9}$$

where $\widetilde{S}(x)$ is the transformed Stieltjes function, and $A(x)$, $B(x)$, $C(x)$, $D(x)$ are polynomials in the variable x, which are known. Furthermore,

Proposition (Zhedanov, 97)

All transformations of the form (9) can be obtained as a combination of Christoffel, Geronimus, associated and anti-associated transformations.

ST and Jacobi matrices

Question

Can we express $\mathcal{R}_C$, $\mathcal{R}_U$, and $\mathcal{R}_G$ in terms of the corresponding monic Jacobi matrices?

Proposition

Let $\mathbf{J}$ be the monic Jacobi matrix associated with μ, and $\beta \in \mathbb{R}$ such that $P_n(\beta) \neq 0$, $n \geqslant 1$. Then,

$$\mathbf{J} - \beta\mathbf{I} = \mathbf{L}\mathbf{U}, \quad \tilde{\mathbf{J}} := \mathbf{U}\mathbf{L} + \beta\mathbf{I},$$

then $\tilde{\mathbf{J}}$ is the monic Jacobi matrix associated with $d\tilde{\mu} = (x-\beta)d\mu$, i.e. the Christoffel transformation.

Christoffel transformation

Proposition

Let μ and $\mathbf{J}$ be as before. Consider the following transformations

$$
\begin{aligned}
\mathbf{C}_1 &:= \mathbf{J} - \beta_1 \mathbf{I} = \mathbf{L_1 U_1}, & \tilde{\mathbf{C}}_1 &:= \mathbf{U_1 L_1} + \beta_1 \mathbf{I}, \\
\mathbf{C}_2 &:= \tilde{\mathbf{C}}_1 - \beta_2 \mathbf{I} = \mathbf{L_2 U_2}, & \tilde{\mathbf{C}}_2 &:= \mathbf{U_2 L_2} + \beta_2 \mathbf{I}, \\
&\vdots \\
\mathbf{C}_m &:= \tilde{\mathbf{C}}_{m-1} - \beta_m \mathbf{I} = \mathbf{L_m U_m}, & \tilde{\mathbf{C}}_m &:= \mathbf{U_m L_m} + \beta_m \mathbf{I},
\end{aligned}
$$

with $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$. If $\{P_{n,i}\}$ is the MOPS associated with $\tilde{\mathbf{C}}_i$, $1 \leqslant i \leqslant m-1$, and assuming that $P_n(\beta) \neq 0$, $P_{n,i}(\beta_{i+1}) \neq 0$, $n \geqslant 1$, $1 \leqslant i \leqslant m-1$, then $\tilde{\mathbf{C}}_m$ is the monic Jacobi matrix associated with the measure

$$
d\tilde{\mu} = (x-\beta_1)(x-\beta_2)\ldots(x-\beta_m)d\mu.
$$

Uvarov transformation

Proposition

Let $\mathbf{J}_0$ *be the monic Jacobi matrix associated with* μ. *Consider*

$$\begin{aligned} \mathbf{J}_0 - \beta\mathbf{I} &= \mathbf{L_1U_1}, & \mathbf{J}_1 &:= \mathbf{U_1L_1}, \\ \mathbf{J}_1 &= \mathbf{U_2L_2}, & \mathbf{J}_2 &:= \mathbf{L_2U_2} + \beta\mathbf{I}. \end{aligned}$$

Then $\mathbf{J}_2$ *is the monic Jacobi matrix associated with the measure*

$$d\tilde{\mu} = d\mu + M_r\delta(x-\beta),$$

i.e. the Uvarov transformation of μ, *where*

$$M_r = \frac{\mu_0(b_0 - \beta - s)}{s},$$

with $\mu_0 = \int_E d\mu(x)$ *and* s *is the free parameter associated with the* UL *factorization of* $\mathbf{J}_1$.

Geronimus transformation

Proposition

Let $\mathbf{J}_1$ be the monic Jacobi matrix associated with $\hat{\mu}$. Suppose there exists μ s.t. $d\hat{\mu} = (x-\beta)d\mu$. If

$$\mathbf{J}_1 - \beta \mathbf{I} = \mathbf{U_1 L_1}, \quad \mathbf{J}_2 := \mathbf{L_1 U_1} + \beta \mathbf{I},$$

then $\mathbf{J}_2$ is the monic Jacobi matrix associated with

$$d\tilde{\mu} = \frac{d\hat{\mu}}{x-\beta} + M_r \delta(x-\beta),$$

i.e. the Geronimus transformation of $\hat{\mu}$, where $M_r = \frac{\int_E d\hat{\mu}}{s}$ and s is the free parameter associated with the UL factorization of $\mathbf{J}_1$.

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Measures on $\mathbb{T}$ and Toeplitz matrices

If σ is a nontrivial positive Borel measure supported on the unit circle, then we can consider the inner product

$$\langle p, q\rangle = \int_{\mathbb{T}} p(z)\overline{q(z)}d\sigma(z),$$

The moments are defined by $c_n := \langle 1, z^n\rangle = \int_{\mathbb{T}} \overline{z^n} d\sigma(z), \quad n \in \mathbb{Z}$.
Notice that we have

$$c_n := \langle 1, z^n\rangle = \int_{\mathbb{T}} \overline{z^n} d\sigma(z) = \int_{\mathbb{T}} z^{-n} d\sigma(z) = \left\langle z^{-n}, 1\right\rangle = \overline{\left\langle 1, z^{-n}\right\rangle} = \bar{c}_{-n},$$

and thus the Gram matrix in terms of the standard basis $\{1, z, z^2, \ldots\}$ is the Toeplitz matrix

$$\mathbf{T} = \begin{pmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \tag{10}$$

Orthogonal polynomials on $\mathbb{T}$

We can apply G-S to get a sequence $\{\varphi_n\}_{n\geqslant 0}$, where $\varphi(z)$ has the form

$$\varphi(z) = \kappa_n z^n + \text{lower order terms.}$$

We have $\Phi_n(z) = \varphi_n(z)/\kappa_n$, satisfying

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad (11)$$

$$\Phi_{n+1}(z) = \left(1 - |\Phi_{n+1}(0)|^2\right) z\Phi_n(z) + \Phi_{n+1}(0)\Phi_{n+1}^*(z), \quad (12)$$

$$\Phi_n^*(z) = z^n \bar{\Phi}_n(z^{-1}) \text{ (reversed polynomial),}$$

$$\{\Phi_n(0)\}_{n\geqslant 1} \text{ (Verblunsky, Schur, reflection parameters).}$$

$$|\Phi_n(0)| < 1, \quad n \geqslant 1.$$

Furthermore, if $\mathbf{k}_n = ||\Phi_n||^2 = \kappa_n^{-2}$, then

$$\mathbf{k}_n = (1 - |\Phi_n(0)|^2)\mathbf{k}_{n-1}$$

Hessenberg matrices

The multiplication operator with respect to $\{\varphi_n\}_{n\geqslant 0}$ is represented in a matrix form by

$$z\varphi(z) = \mathbf{H}_\varphi \varphi(z), \tag{13}$$

where $\varphi(z) = [\varphi_0(z), \varphi_1(z), \ldots, \varphi_n(z), \ldots]^t$ and $\mathbf{H}_\varphi$ is a lower Hessenberg matrix whose entries are

$$h_{n,j} = \begin{cases} \frac{\kappa_n}{\kappa_{n+1}} & \text{if} \quad j = n+1, \\ -\frac{\kappa_j}{\kappa_n}\Phi_{n+1}(0)\overline{\Phi_j(0)} & \text{if} \quad j \leqslant n, \\ 0 & \text{if} \quad j > n+1. \end{cases} \tag{14}$$

Notice that $\mathbf{H}_\varphi$ is defined in terms of $\{\Phi_n(0)\}_{n\geqslant 1}$.

Hessenberg matrices (cont.)

Proposition

$\mathbf{H}_\varphi$ *satisfies*

(i) $\mathbf{H}_\varphi \mathbf{H}_\varphi^* = \mathbf{I}$,

(ii) $\mathbf{H}_\varphi^* \mathbf{H}_\varphi = \mathbf{I} - \lambda_\infty(0)\varphi(0)\varphi(0)^*$,

where $\mathbf{I}$ *is the semi-infinite identity matrix and* $\lambda_\infty(0) = \prod_{n=0}^{\infty}(1 - |\Phi_{n+1}(0)|^2)$.

Remark

$\mathbf{H}_\varphi$ is unitary $\iff \sum_{n=0}^{\infty} |\Phi_n(0)|^2 = +\infty \iff \log \sigma' \notin L^1\left(\frac{d\theta}{2\pi}\right)$ ($\sigma \notin$ Szegő class).

Remark

In the monic case, $\mathbf{H}_\Phi$ has as entries

$$h_{n,j} = \begin{cases} 1 & \text{if} \quad j = n+1, \\ -\frac{\mathbf{k}_n}{\mathbf{k}_j}\Phi_{n+1}(0)\overline{\Phi_j(0)} & \text{if} \quad j \leqslant n, \\ 0 & \text{if} \quad j > n+1. \end{cases} \tag{15}$$

Canonical spectral transformations on $\mathbb{T}$

Christoffel transformation ($\mathcal{F}_C$)

$$d\tilde{\sigma} = |z-\alpha|^2 d\sigma, \quad \alpha \in \mathbb{C}.$$

Uvarov transformation ($\mathcal{F}_U$)

$$d\tilde{\sigma} = d\sigma + M_c\delta(z-\alpha) + \overline{M}_c\delta(z-\bar{\alpha}^{-1}), \quad \alpha \in \mathbb{C} \setminus \{0\}, \quad M_c \in \mathbb{C}.$$

Geronimus transformation ($\mathcal{F}_G$)

$$d\tilde{\sigma} = \frac{d\sigma}{|z-\alpha|^2} + M_c\delta(z-\alpha) + \bar{M}_c\delta(z-\bar{\alpha}^{-1}), \quad \alpha \in \mathbb{C} \setminus \{0\}, \quad M_c \in \mathbb{C}.$$

Proposition

$$\begin{aligned} \mathcal{F}_C \circ \mathcal{F}_G &= \mathcal{I} \quad \text{Identity transformation} \\ \mathcal{F}_G \circ \mathcal{F}_C &= \mathcal{F}_U \end{aligned}$$

ST and Carathéodory functions

Define

$$F(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k} z^k,$$

In the positive definite case, $F(z)$ is analytic, $\Re\mathfrak{e}[F(z)] > 0$ in $\mathbb{D}$, and

$$F(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} d\sigma(w).$$

The previous transformations can be expressed as

$$\widetilde{F}(z) = \frac{A(z)F(z) + B(z)}{D(z)}, \qquad (16)$$

where $\widetilde{F}(z)$ is associated with $\tilde{\sigma}$ and $A(z), B(z), D(z)$ are known polynomials in z.

Rational spectral transformations

Associated polynomials

Denote by $\{\Phi_n^{(N)}\}_{n\geqslant 0}$ the associated polynomials of order N, defined by

$$\Phi_{n+1}^{(N)}(z) = z\Phi_n^{(N)}(z) + \Phi_{n+N+1}(0)(\Phi_n^{(N)})^*(z), \quad n \geqslant 0,$$

i.e. the first N coefficients are removed.

Anti-associated polynomials

Let $v_1, v_2, \ldots, v_N \in \mathbb{C}$ with $|v_j| < 1$, $1 \leqslant j \leqslant N$. Define $\{\hat{\Phi}_n(0)\}_{n\geqslant 1} = \{v_j\}_{j=1}^N \bigcup \{\Phi_j(0)\}_{j=1}^{\infty}$. Then, the polynomials

$$\Phi_{n+1}^{(-N)}(z) = z\Phi_n^{(-N)}(z) + \hat{\Phi}_{n+1}(0)(\Phi_n^{(-N)})^*(z), \quad n \geqslant 0,$$

are called anti-associated polynomials of order N.

RST and Carathéodory functions

Aleksandrov transformation

Define $\{\Phi_n^{\lambda}(0)\}_{n\geqslant 1}$, where $\Phi_n^{\lambda}(0) = \lambda\Phi_n(0)$, with $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then,

$$\Phi_{n+1}^{\lambda}(z) = z\Phi_n^{\lambda}(z) + \Phi_{n+1}^{\lambda}(0)(\Phi_n^{\lambda})^*(z),$$

are called Aleksandrov polynomials.

These transformations can be expressed as

$$\widetilde{F}(z) = \frac{A(z)F(z) + B(z)}{C(z)F(z) + D(z)}, \tag{17}$$

where $\widetilde{F}(z)$ is the transformed Carathéodory function and $A(z), B(z), C(z), D(z)$ are known polynomials in z.

ST and Hessenberg matrices

Question

Can we express $\mathcal{F}_C$, $\mathcal{F}_U$, and $\mathcal{F}_G$ in terms of the corresponding Hessenberg matrices?

Christoffel transformation

Let $d\sigma_C = |z-\alpha|^2 d\sigma$, and $\{\psi_n\}_{n\geqslant 0}$ its OPS. The relation between both families of polynomials is

$$(z-\alpha)\psi_n(z) = \sqrt{\frac{K_n(\alpha,\alpha)}{K_{n+1}(\alpha,\alpha)}}\varphi_{n+1}(z) - \sum_{j=0}^{n} \frac{\varphi_{n+1}(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{K_{n+1}(\alpha,\alpha)K_n(\alpha,\alpha)}}\varphi_j(z), \tag{18}$$

where $K_n(z,y) = \sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(y)}$.

In matrix form

$$(z-\alpha)\psi(z) = \mathbf{M}_C\varphi(z), \tag{19}$$

where $\mathbf{M}_C$ has entries

$$m_{i,j} = \begin{cases} -\frac{\varphi_{i+1}(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{K_{i+1}(\alpha,\alpha)K_i(\alpha,\alpha)}}, & \text{if} \quad j \leqslant i, \\ \sqrt{\frac{K_i(\alpha,\alpha)}{K_{i+1}(\alpha,\alpha)}}, & \text{if} \quad j = i+1, \\ 0, & \text{if} \quad j > i+1. \end{cases} \tag{20}$$

Christoffel transformation

Proposition

$\mathbf{M}_C$ *satisfies*

(*i*) $\mathbf{M}_C \mathbf{M}_C^* = \mathbf{I}$.

(*ii*) $\mathbf{M}_C^* \mathbf{M}_C = \mathbf{I} - \lambda_\infty(\alpha)\varphi(\alpha)\varphi(\alpha)^*$,

Proposition

Let $\mathbf{M}_{C_n}$ *be the* $n \times n$ *principal submatrix of* $\mathbf{M}_C$. *Then,*

(*i*) $\mathbf{M}_{C_n} \mathbf{M}_{C_n}^* = \mathbf{I}_n - \frac{K_{n-1}(\alpha,\alpha)}{K_n(\alpha,\alpha)} e_n e_n^*$, *where* $e_n = [0, \ldots, 0, 1]^t \in \mathbb{C}^{(n,1)}$.

(*ii*) $\mathbf{M}_{C_n}^* \mathbf{M}_{C_n} = \mathbf{I}_n - \frac{1}{K_n(\alpha,\alpha)} \varphi^{(n)}(\alpha)\varphi^{(n)*}(\alpha)$, *where* $\varphi^{(n)}(\alpha) = [\varphi_0(\alpha), \varphi_1(\alpha), \ldots, \varphi_{n-1}(\alpha)]^t$

Christoffel transformation (cont.)

Furthermore, if $\mathbf{L}_{\varphi\psi}$ is the lower triangular matrix such that $\varphi(z) = \mathbf{L}_{\varphi\psi}\psi(z)$, then

Proposition

We have

$$\mathbf{H}_{\varphi} - \alpha\mathbf{I} = \mathbf{L}_{\varphi\psi}\mathbf{M}_{C}, \quad (21)$$

$$\mathbf{H}_{\psi} - \alpha\mathbf{I} = \mathbf{M}_{C}\mathbf{L}_{\varphi\psi}. \quad (22)$$

An "almost" QR factorization appears, since $(\mathbf{M}_C)_n$ is a quasi-unitary matrix, i.e. its first $n-1$ rows constitute an orthonormal set, and the last row is orthogonal with respect to this set, but is not normalized.

Uvarov transformation

Let σ_U be the Uvarov transformation of σ. If we assume $\{\upsilon_n\}_{n\geqslant 0}$ is its associated OPS, and define by $\mathbf{H}_\upsilon$ its corresponding Hessenberg matrix, then

Proposition

$$\mathbf{H}_\varphi - \alpha\mathbf{I} = \mathbf{L}_{\varphi\psi}\mathbf{M}_C, \quad (23)$$

$$\mathbf{H}_\upsilon - \alpha\mathbf{I} = \mathbf{L}_U\mathbf{M}_U, \quad (24)$$

where $\mathbf{L}_U = \mathbf{L}_{\upsilon\varphi}\mathbf{L}_{\varphi\psi}$, $\mathbf{M}_U = \mathbf{M}_C\mathbf{L}_{\upsilon\varphi}^{-1}$, *and* $\mathbf{L}$ *are the matrices of change of bases for the orthonormal polynomial families denoted by their subindices.*

Geronimus transformation

Let σ_G be the Geronimus transformation of σ. If $\{G_n\}_{n\geqslant 0}$ is its OPS and $\mathbf{M}_G$ a Hessenberg matrix such that

$$(z-\alpha)\Phi(z) = \mathbf{M}_G G(z).$$

Then we get

Proposition

Let $\mathbf{L}_G$ *be such that* $G(z) = \mathbf{L}_G \Phi(z)$ *and denote by* $\mathbf{H}_G$ *the Hessenberg matrix associated with* $\{G_n\}_{n\geqslant 0}$*. Then,*

$$\mathbf{H}_\Phi - \alpha \mathbf{I} = \mathbf{M}_G \mathbf{L}_G \tag{25}$$

and

$$\mathbf{H}_G - \alpha \mathbf{I} = \mathbf{L}_G \mathbf{M}_G. \tag{26}$$

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Laurent polynomials space

Let $\Lambda_{(k,l)}$ be $span\{z^j\}_{j=k}^{l}$, $k \leqslant l$, and $P_{(k,l)}$ the orthogonal projection over $\Lambda_{(k,l)}$ with respect to a bilinear functional $\mathcal{L}$. Set

$$\Lambda^{(n)} = \begin{cases} \Lambda_{(-k,k)} & n = 2k, \\ \Lambda_{(-k,k+1)} & n = 2k+1, \end{cases}$$

and let $P^{(n)}$ be the orthogonal projection over $\Lambda^{(n)}$. Furthermore, define

$$\chi_n^{(0)} = \begin{cases} z^{-k} & n = 2k, \\ z^{k+1} & n = 2k+1. \end{cases}$$

Applying Gram-Schmidt, we obtain the CMV basis from

$$\chi_n = (1 - P^{(n-1)})\chi_n^{(0)}.$$

The CMV basis

$\{\chi_n\}_{n\geqslant 0}$ can be expressed in terms of $\{\Phi_n(z)\}_{n\geqslant 0}$ as follows

$$\begin{aligned} \chi_{2n}(z) &= z^{-n}\Phi_{2n}^*(z), \quad n \geqslant 0, \\ \chi_{2n-1}(z) &= z^{-n+1}\Phi_{2n-1}(z), \quad n \geqslant 1, \end{aligned}$$

and satisfies the following recurrence relations

$$\begin{aligned} z\chi_0 &= -\Phi_1(0)\chi_0 + \rho_0\chi_1, \\ z\begin{pmatrix}\chi_{2n-1}\\ \chi_{2n}\end{pmatrix} &= \widehat{\Xi}_{2n-1}^T\begin{pmatrix}\chi_{2n-2}\\ \chi_{2n-1}\end{pmatrix} + \Xi_{2n}\begin{pmatrix}\chi_{2n}\\ \chi_{2n+1}\end{pmatrix}, \quad n \geqslant 1, \end{aligned}$$

with

$$\Xi_n := \begin{pmatrix} \dfrac{-\rho_{n-1}\Phi_{n+1}(0)}{-\Phi_n(0)\Phi_{n+1}(0)} & \dfrac{\rho_{n-1}\rho_n}{\Phi_n(0)\rho_n} \end{pmatrix}, \quad \widehat{\Xi}_n := \begin{pmatrix} \dfrac{-\hat{\rho}_{n-1}\Phi_{n+1}(0)}{-\Phi_n(0)\Phi_{n+1}(0)} & \dfrac{\hat{\rho}_{n-1}\hat{\rho}_n}{\Phi_n(0)\hat{\rho}_n} \end{pmatrix},$$

where $\rho_n = |1 - |\Phi_{n+1}(0)|^2|^{1/2}$ and $\hat{\rho}_n = \varsigma_n\rho_n$, with $\varsigma_n = sign(1 - |\Phi_n|^2)$.

A five diagonal matrix

Thus, the five diagonal matrix $\mathcal{C}$ of CMV representation is defined as

$$\mathcal{C}_{i,j} = \left\langle \chi_i, z\chi_j \right\rangle_{\mathcal{L}},$$

in such a way that

$$\mathcal{C} = \begin{pmatrix} -\Phi_1(0) & -\Phi_2(0)\hat{\rho}_0 & \hat{\rho}_1\hat{\rho}_0 & 0 & 0 & \dots \\ \rho_0 & -\Phi_2(0)\overline{\Phi_1(0)} & \overline{\Phi_1(0)}\hat{\rho}_1 & 0 & 0 & \dots \\ 0 & -\Phi_3(0)\rho_1 & -\Phi_3(0)\overline{\Phi_2(0)} & -\Phi_4(0)\hat{\rho}_2 & \hat{\rho}_3\hat{\rho}_2 & \dots \\ 0 & \rho_2\rho_1 & \overline{\Phi_2(0)}\rho_2 & -\Phi_4(0)\overline{\Phi_3(0)} & \overline{\Phi_3(0)}\hat{\rho}_3 & \dots \\ 0 & 0 & 0 & -\Phi_5(0)\rho_3 & -\Phi_5(0)\overline{\Phi_4(0)} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

CMV factorization

Furthermore,

$$\mathcal{C} = \mathcal{W}\mathcal{M},$$

where

$$\mathcal{M} = \begin{pmatrix} 1 & & & \\ & \Theta_1 & & \\ & & \Theta_3 & \\ & & & \ddots \end{pmatrix},$$

$$\mathcal{W} = \begin{pmatrix} \Theta_0 & & & \\ & \Theta_2 & & \\ & & \Theta_4 & \\ & & & \ddots \end{pmatrix},$$

with

$$\Theta_j = \begin{pmatrix} -\Phi_{j+1}(0) & \dfrac{\rho_j}{\Phi_{j+1}(0)} \\ \hat{\rho}_j & \end{pmatrix}.$$

ST and CMV matrices

Open question

Can we express $\mathcal{F}_C$, $\mathcal{F}_U$, and $\mathcal{F}_G$ in terms of the corresponding CMV matrices?

Partial answer: Yes (Cantero-Marcellán-Velázquez, 2015)

Contents

1 Polinomios ortogonales en la recta y matrices de Jacobi

2 Ortogonalidad en la circunferencia unidad y matrices de Hessenberg

3 La representación CMV

4 Algunas generalizaciones

Matrix orthogonal polynomials

A matrix polynomial has the form $P(x) = A_n z^n + \ldots A_0$, where A_i are $q \times q$ matrices.

A matrix inner product can be defined by

$$\int_E P(x) d\mu(x) Q^T(x),$$

where $d\mu(x)$ is a $q \times q$ symmetric matrix of measures with support in $E \in \mathbb{R}$. Orthogonality is defined by

$$\int_E P_n(x) d\mu(x) P_m^T(x) = \delta_{n.m} C_n,$$

where C_n is a nonsingular matrix.

Spectral transformation for matrix polynomials

- Christoffel transformation (Marcellán, Mañas - 2015)
- Uvarov transformation (Marcellán, Piñar, 2000s)
- Geronimus transformation (Marcellán, LG - 2015)

Other perturbations studied by Choque, Domínguez de la Iglesia, LG.

¡Gracias por su atención!