On a discrete number operator and its eigenvectors associated with the 5D discrete Fourier transform

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1 Introduction

2 Raising and lowering difference operators for eigenvectors of the 5D DFT

3 Eigenvalues and eigenvectors of the discrete number operator

4 Eigenvectors of the number operator $\mathcal{N}^{(5)}$ versus the Hermite functions $\psi_n(x)$

5 Concluding remarks

6 References
Abstract

We construct an explicit form of a difference analogue of the quantum number operator in terms of the raising and lowering operators that govern eigenvectors of the 5D discrete (finite) Fourier transform. Eigenvalues of this difference operator are represented by distinct nonnegative numbers so that it can be used to systematically classify, in complete analogy with the case of the continuous classical Fourier transform, eigenvectors of the 5D discrete Fourier transform, thus resolving the ambiguity caused by the well-known degeneracy of the eigenvalues of the discrete Fourier transform.
We are to begin by recalling first a few well-known facts about the classical Fourier transform (FT) and its finite analogue, discrete Fourier transform (DFT). It is known that the Hermite functions

$$\psi_n(x) := c_n^{-1} H_n(x) \exp \left( -\frac{x^2}{2} \right), \quad c_n = \sqrt{\sqrt{\pi} 2^n n!},$$

where $H_n(x)$ are the classical Hermite polynomials, represent an important explicit example of an orthonormal and complete system in the Hilbert space $L^2(\mathbb{R}, dx)$ of square-integrable functions on the full real line $x \in \mathbb{R}$. It is further well known that the functions $\psi_n(x)$ possess the simple transformation property with respect to the Fourier transform: they are eigenfunctions of the Fourier transform, associated with the eigenvalues $i^n$,

$$\mathcal{F} \psi_n(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} \psi_n(y) dy = i^n \psi_n(x).$$
The question is then can be posed whether there is a way of deriving the eigenfunctions of the Fourier transform, which does not presuppose a knowledge of the analytic formula to be proved.

Since mutually commuting operators have the same set of eigenfunctions, one may solve this problem by defining such a self-adjoint differential operator with simple spectrum of distinct eigenvalues, that commutes with the FT operator $\mathcal{F}$. Then the eigenfunctions of that differential operator can be found by solving a corresponding to this case differential equation and they will be at same time the eigenfunctions of the $\mathcal{F}$. So in this way one reduces a problem of finding eigenfunctions of the FT operator $\mathcal{F}$ to one of solving some differential equation.
To illustrate how to find such differential operator let us start with the first-order differential operator \( \frac{d}{dx} \) and evaluate its action on the Fourier integral transform:

\[
\frac{d}{dx} \int_{\mathbb{R}} e^{ixy} f(y) \, dy = i \int_{\mathbb{R}} e^{ixy} y f(y) \, dy ,
\]

where \( f(x) \in L^2(\mathbb{R}, dx) \). Consequently, from the right side of the above relation one deduces that the next step should be to evaluate

\[
x \int_{\mathbb{R}} e^{ixy} f(y) \, dy = -i \int_{\mathbb{R}} \left( \frac{d e^{ixy}}{dy} \right) f(y) \, dy
\]

\[
= i \int_{\mathbb{R}} e^{ixy} \frac{df(y)}{dy} \, dy ,
\]

upon integrating by parts the middle term in it.
From these two relations it thus follows that
\[
\left( x \pm \frac{d}{dx} \right) \int_{\mathbb{R}} e^{ixy} f(y) \, dy = \pm i \int_{\mathbb{R}} e^{ixy} \left( y \pm \frac{d}{dy} \right) f(y) \, dy.
\]

In the operator form these identities can be written as intertwining relations
\[
a \mathcal{F} = i \mathcal{F} a, \quad a^\dagger \mathcal{F} = -i \mathcal{F} a^\dagger,
\]
where \(a := \frac{1}{\sqrt{2}} \left( x + \frac{d}{dx} \right)\) and \(a^\dagger := \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} \right)\) are the lowering and raising first-order differential operators, which obey the standard Heisenberg commutation relation
\[
[a, a^\dagger] := aa^\dagger - a^\dagger a \equiv \left[ \frac{d}{dx} , x \right] = \mathbf{1}.
\]
The final step for finding the desired differential operator is actually revealed by the intertwining relations because one readily concludes that

\[ a^\dagger a \mathcal{F} = i a^\dagger \mathcal{F} a = \mathcal{F} a^\dagger a \]

on account of both identities in the intertwining relations. Consequently, the self-adjoint second-order differential number operator \( \mathcal{N} := a^\dagger a \) does commute with the FT operator \( \mathcal{F} \) and it only remains to resolve the eigenproblem \( \mathcal{N} f_n(x) = \lambda_n f_n(x) \) for this operator \( \mathcal{N} \). It is not difficult to show then that the eigenfunctions of the number operator \( \mathcal{N} \) are the Hermite functions \( \psi_n(x) \) (up to the arbitrariness in the choice of a normalization constant factor), whereas the corresponding eigenvalues are \( \lambda_n = n, \ n = 0, 1, 2, \ldots \).
Turning to the discrete Fourier transform $\Phi^{(N)}$, we recall that it is based on $N$ points and represented by the $N \times N$ unitary symmetric matrix with matrix elements

$$
\Phi^{(N)}_{m, n} = \frac{1}{\sqrt{N}} \exp \left( \frac{2\pi i}{N} mn \right) \equiv \frac{1}{\sqrt{N}} q^{mn},
$$

where $q := e^{\frac{2\pi i}{N}}$ and $m, n \in \{0, 1, \ldots, N-1\}$. Given a vector $\vec{v}$ with components $\{v_k\}_{k=0}^{N-1}$, one can compute another vector $\vec{u}$ with components

$$
u_m = \sum_{n=0}^{N-1} \Phi^{(N)}_{m, n} v_n,
$$

referred to as the discrete (finite) Fourier transform of the vector $\vec{v}$. 
Those vectors $\vec{f}_k$, which are solutions of the standard equations

$$
\sum_{n=0}^{N-1} \Phi_{m,n}^{(N)} (\vec{f}_k)_n = \lambda_k (\vec{f}_k)_m \quad k \in \{0, 1, \ldots, N-1\},
$$

then represent eigenvectors of the DFT operator $\Phi^{(N)}$, associated with the eigenvalues $\lambda_k$.

Since the fourth power of $\Phi^{(N)}$ is the unit matrix, the only four distinct eigenvalues among $\lambda_k$’s are $\pm 1$ and $\pm i$. 
Although there exists a plethora of discussion in the literature on eigenvectors of the DFT, the problem of deriving eigenvectors of DFT analytically still remains to be solved. Recently, M. Atakishiyeva and N. Atakishiyev have proposed a strategy for resolving this problem by constructing a self-adjoint difference operator $N^{(N)}$ (with distinct nonnegative eigenvalues) in terms of the difference raising and lowering operators, which are defined by the intertwining relations

$$b_N \Phi^{(N)} = i \Phi^{(N)} b_N, \quad b_N^T \Phi^{(N)} = -i \Phi^{(N)} b_N^T.$$ 

The ability to solve a difference equation for eigenvectors of this discrete number operator $N^{(N)}$, which commutes with the DFT operator $\Phi^{(N)}$, then enables one to define an analytical form of the desired set of eigenvectors for the latter operator.
An important aspect to observe at this point is that although the idea of making use an analogy with the continuous case for deriving eigenvectors of the DFT is not new (see, for example, [3, 4]), it seems, however, that there never was consistent attempt to find out how symmetry properties of the continuous Fourier transform might best be transferred to the discrete case.

The limited aim of this presentation is to restrict our attention to the 5D DFT and give a detailed account of how one can solve the eigenproblem for the discrete number operator $N^{(5)}$ by using the difference raising and lowering operators that govern eigenvectors of the 5D discrete Fourier transform $\Phi^{(5)}$. 
The motivation for selecting this special dimension $N = 5$ of the general discrete Fourier transform $\Phi^{(N)}$ is twofold. First, this dimension is large enough to contain a multiple eigenvalue and therefore one has to handle the same degeneracy problem as in the more general case. Second, this dimension is small enough in order to have calculational advantages that appear in the process of resolving the eigenproblem for the discrete number operator $\mathcal{N}^{(5)}$. We hope that this study will deepen our understanding of the case with an arbitrary ND discrete Fourier transform and help us to provide some rigorous proofs, still needed for general values of $N$. 
We recall that the 5D discrete (finite) Fourier transform (DFT) is traditionally represented by an $5 \times 5$ unitary symmetric matrix $\Phi^{(5)} = \left( \Phi^{(5)}_{m,n} \right)$ with elements

$$\Phi^{(5)}_{m,n} = \frac{1}{\sqrt{5}} \exp \left( \frac{2\pi i}{5} m n \right) \equiv \frac{1}{\sqrt{5}} q^{m n},$$

where $q = e^{\frac{2\pi i}{5}}$ is the 5th root of unity and indices $m, n$ run in the interval $\{0, 1, 2, 3, 4\}$. So the matrix form of $\Phi^{(5)}$ is

$$\left( \Phi^{(5)}_{m,n} \right) = \frac{1}{\sqrt{5}} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & q & q^2 & q^3 & q^4 \\
1 & q^2 & q^4 & q & q^3 \\
1 & q^3 & q & q^4 & q^2 \\
1 & q^4 & q^3 & q^2 & q
\end{pmatrix}.$$
In the above-mentioned paper by Atakishiyeva and Atakishiyev it was shown how to construct the difference lowering and raising operators $b_5$ and $b_5^T$ for eigenvectors of the DFT operator $\Phi^{(5)}$, which satisfy ‘proper’ intertwining relations with the $\Phi^{(5)}$ of the form
\[
 b_5 \Phi^{(5)} = i \Phi^{(5)} b_5, \quad b_5^T \Phi^{(5)} = -i \Phi^{(5)} b_5^T,
\]
where $b_5^T$ is the matrix transpose of $b_5$. Let us draw attention here to these intertwining relations, which evidently imply that if a vector $\vec{f}_k$ is the eigenvector of the DFT operator $\Phi^{(5)}$, associated with the eigenvalue $i^k$, $0 \leq k \leq 3$, then the vectors $b_5^T \vec{f}_k$ and $b_5 \vec{f}_k$ are also the eigenvectors of the same operator $\Phi^{(5)}$, associated with the eigenvalues $i^{k+1}$ and $i^{k-1}$, respectively.
But note carefully that this does not necessarily mean that those vectors $b_5 T f_k$ and $b_5 f_k$ essentially coincide (to within constant factors) with the eigenvectors $f_{k+1}$ and $f_{k-1}$ of the DFT operator $\Phi^{(5)}$, respectively, as it does happen to be the case with the Fourier transform operator $\mathcal{F}$. Later we detail the action of the operators $b_5$ and $b_5^T$ on eigenvectors of the DFT operator $\Phi^{(5)}$. The operators $b_5$ and $b_5^T$ are explicitly given as

$$b_5 := c \left( 2 S + T^{(+)} - T^{(-)} \right),$$

$$b_5^T := c \left( 2 S - T^{(+)} + T^{(-)} \right),$$

where the operator $S$ represents the diagonal matrix with elements $S_{kl} := \sin(k\theta)\delta_{kl}$, $\theta := 2\pi/5$, $0 \leq k, l \leq 4$ and a pair of the shift operators $T^{(\pm)}$ are defined as $T^{(\pm)}_{kl} := \delta_{k\pm1,l}$ with $\delta_{-1,l} \equiv \delta_{4,l}$ and $\delta_{5,l} \equiv \delta_{0,l}$. 

$c = \frac{1}{4} \sqrt{\frac{5}{\pi}},$
We also display the matrix form of the lowering and raising operators $b_5$ and $b_5^T$, respectively:

$$
(b_5)_{m,m'} = c \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & -1 \\
-1 & 2 \sin \theta & 1 & 0 & 0 \\
0 & -1 & 2 \sin 2\theta & 1 & 0 \\
0 & 0 & -1 & -2 \sin 2\theta & 1 \\
1 & 0 & 0 & -1 & -2 \sin \theta
\end{pmatrix},
$$

$$
(b_5^T)_{m,m'} = c \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 1 \\
1 & 2 \sin \theta & -1 & 0 & 0 \\
0 & 1 & 2 \sin 2\theta & -1 & 0 \\
0 & 0 & 1 & -2 \sin 2\theta & -1 \\
-1 & 0 & 0 & 1 & -2 \sin \theta
\end{pmatrix}.
$$
Note that the determinants of both matrices are equal to 0, $\det(b_5) = \det(b_5^T) = 0$; therefore they are not invertible. Observe also that both of these matrices are of ‘almost’ tridiagonal form: they have $\pm 1$ elements in the upper-right and lower-left corners but otherwise are tridiagonal. Since those $\pm 1$ elements can be regarded as cyclic extensions of the subdiagonal and the superdiagonal elements, these type of matrices are referred to as extended-tridiagonal matrices [Clary and Mugler]. Moreover, another confirmation of the ‘cyclic’ properties of the operators $b_5$ and $b_5^T$ is revealed by the identities

$\left(b_5\right)^5 + 5 c^4 \tau b_5 = 0$, $\left(b_5^T\right)^5 + 5 c^4 \tau b_5^T = 0$,

where $\tau$ is the golden ratio, $\tau := (\sqrt{5} + 1)/2 = -2 \cos 2\theta$. 
This particular irrational number $\tau$ is known to turn out frequently in geometry, particularly in figures with pentagonal symmetry; so it is not surprising that it appears here as well. Since the successive powers of $\tau$ obey the Fibonacci recurrence $\tau^{n+1} = \tau^n + \tau^{n-1}$, $n \geq 0$, this characteristic property of the golden ratio allows any polynomial in $\tau$ to be reduced to a linear expression in $\tau$.

In the sequel, it proves therefore convenient to parametrize the operators $b_5$ and $b_5^T$ in terms of the golden ratio $\tau$ and its conjugate $\tau^{-1} := (\sqrt{5} - 1)/2 = 2 \cos \theta = \tau - 1$. Taking into account that $2 \sin \theta = \kappa \tau^{1/2}$ and $2 \sin 2\theta = \kappa \tau^{-1/2}$, where $\kappa := (5)^{1/4}$, one rewrites matrix elements of the operators $b_5$ and $b_5^T$ as
On a discrete number operator and its eigenvectors associated with the 5D discrete Fourier transform

Raising and lowering difference operators for eigenvectors of the 5D DFT

\[
\begin{pmatrix}
(b_5^T)_{m,m'}
\end{pmatrix} = c
\begin{pmatrix}
0 & 1 & 0 & 0 & -1 \\
-1 & \kappa T^{1/2} & 1 & 0 & 0 \\
0 & -1 & \kappa T^{-1/2} & 1 & 0 \\
0 & 0 & -1 & -\kappa T^{-1/2} & 1 \\
1 & 0 & 0 & -1 & -\kappa T^{1/2}
\end{pmatrix},
\]

\[
\begin{pmatrix}
(b_5)_{m,m'}
\end{pmatrix} = c
\begin{pmatrix}
0 & -1 & 0 & 0 & 1 \\
1 & \kappa T^{1/2} & -1 & 0 & 0 \\
0 & 1 & \kappa T^{-1/2} & -1 & 0 \\
0 & 0 & 1 & -\kappa T^{-1/2} & -1 \\
-1 & 0 & 0 & 1 & -\kappa T^{1/2}
\end{pmatrix}.
\]
From definition of the lowering $b_5$ and raising $b_5^T$ operators it follows that their commutator

$$\mathcal{K} := [b_5, b_5^T] = b_5 b_5^T - b_5^T b_5$$

is equal to

$$\mathcal{K} = 4 c^2 \left[ T^{(+)} - T^{(-)}, S \right].$$

Its explicit matrix form in terms of the golden ratio $\tau$ is

$$\left( (\mathcal{K})_{m,m'} \right) = 2 \kappa \sqrt{\tau} c^2 \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & \tau - 2 & 0 & 0 \\ 0 & \tau - 2 & 0 & 2(1 - \tau) & 0 \\ 0 & 0 & 2(1 - \tau) & 0 & \tau - 2 \\ 1 & 0 & 0 & \tau - 2 & 0 \end{pmatrix}. $$

To compare the commutator $\mathcal{K}$ with the continuous case recall that the lowering and raising first-order differential operators $a$ and $a^\dagger$ obey the Heisenberg commutation relation $[a, a^\dagger] = I$. 
We close this section by emphasizing that it was intuitively understood much earlier that probably the extended-tridiagonal type matrices lie at the core of the adequate description of eigenvectors of the general ND discrete Fourier transform [Dickinson and Steiglitz, Clary and Mugler]. But it seems that this particular band structure was imprecisely attributed to those operators with distinct eigenvalues, which 
commute with the associated DFT operators and can be therefore used for the unambiguous classification of the eigenvectors of the latter ones. Only recently it has become clear that it is the lowering $b_N$ and raising $b_T^N$ operators for the eigenvectors of the DFT operator $\Phi^{(N)}$ that are of the extended-tridiagonal type. Defined by the standard intertwining relations

$$b_N \Phi^{(N)} = i \Phi^{(N)} b_N, \quad b_T^N \Phi^{(N)} = -i \Phi^{(N)} b_T^N,$$

these operators $b_N$ and $b_T^N$ do not commute with the $\Phi^{(N)}$. 
Nevertheless, from the same defining identities for the $b_N$ and $b^T_N$, it follows at once that their product $b^T_N b_N$ does commute with the DFT operator $\Phi^{(N)}$.

Moreover, although the operator $\mathcal{N}^{(N)} := b^T_N b_N$ is not of the extended-tridiagonal type, it turns out to be quite sufficient for finding explicit forms of all mutually orthogonal eigenvectors of the DFT operator $\Phi^{(N)}$ in a systematic and unambiguous way. As we shall see in the next section, this briefly outlined above algebraic approach to solving the eigenproblem for the operator $\mathcal{N}^{(N)}$ can be effectively employed in the particular case of the 5D DFT.
Let us study in detail a discrete number operator $\mathcal{N}^{(5)}$, whose matrix elements are defined as

$$
\left( \mathcal{N}^{(5)} \right)_{m,m'} = \left( b_5^T b_5 \right)_{m,m'}
$$

As a product of a matrix and its transpose, this defining matrix is symmetric and all of its eigenvalues are nonnegative. Moreover, since the determinant of this matrix is equal to zero, at least one of the eigenvalues should have zero value as well; but this lowest eigenvalue turns out to be unique and all eigenvalues of this matrix are actually distinct.

$$
= c^2 \begin{pmatrix}
2 & -\kappa\tau^{1/2} & -1 & -1 & -\kappa\tau^{1/2} \\
-\kappa\tau^{1/2} & 4 + \tau & \kappa\tau^{-3/2} & -1 & -1 \\
-1 & \kappa\tau^{-3/2} & 5 - \tau & 2\kappa\tau^{-1/2} & -1 \\
-1 & -1 & 2\kappa\tau^{-1/2} & 5 - \tau & \kappa\tau^{-3/2} \\
-\kappa\tau^{1/2} & -1 & -1 & \kappa\tau^{-3/2} & 4 + \tau
\end{pmatrix}.
$$
Before entering into further details about explicit forms of the eigenvalues and eigenvectors of the operator $\mathcal{N}^{(5)}$, we may recall first the following important facts, associated with this eigenproblem.

In this particular case under study, when the operator $\mathcal{N}^{(5)}$ is just represented by a $5 \times 5$ matrix, one can use some computer program in order to evaluate the eigenvalues and eigenvectors of the discrete number operator $\mathcal{N}^{(5)}$.
For instance, this is what one gets by using *Mathematica*:
Eigenvalues of the $N^{(5)}$ are $c^2 \nu_k$, $0 \leq k \leq 4$, where $\nu_k$'s, arranged in the descending order, are given by

$$
\nu_4 = \frac{1}{2} \left( 15 - \sqrt{5} \right) + \sqrt{5 - 2\sqrt{5}} = \kappa \left[ \kappa (3\tau - 2) + \tau^{-3/2} \right],
$$

$$
\nu_3 = \frac{1}{2} \left( 5 + \sqrt{5} \right) + \sqrt{5 + 2\sqrt{5}} = \kappa \left( \kappa + \tau^{1/2} \right) \tau,
$$

$$
\nu_2 = \frac{1}{2} \left( 15 - \sqrt{5} \right) - \sqrt{5 - 2\sqrt{5}} = \kappa \left[ \kappa (3\tau - 2) - \tau^{-3/2} \right],
$$

$$
\nu_1 = \frac{1}{2} \left( 5 + \sqrt{5} \right) - \sqrt{5 + 2\sqrt{5}} = \kappa \left( \kappa - \tau^{1/2} \right) \tau,
$$

$\nu_0 = 0$;
Eigenvectors \( \vec{y}_k \) of \( \mathcal{N}^{(5)} \), associated with these eigenvalues \( \nu_k \), 
\( 0 \leq k \leq 4 \), have the following components

\[
\left( \vec{y}_4 \right)_{k=0}^4 = \left\{ 0, \kappa + \tau^{\frac{1}{2}}, \tau^{-\frac{1}{2}}, -\tau^{-\frac{1}{2}}, -\kappa - \tau^{\frac{1}{2}} \right\},
\]

\[
\left( \vec{y}_3 \right)_{k=0}^4 = \left\{ 2(1 - \tau), 1, 1, 1, 1 \right\},
\]

\[
\left( \vec{y}_2 \right)_{k=0}^4 = \left\{ 2, -\left( \tau + 2\kappa \tau^{\frac{1}{2}} \right), 2\kappa \tau^{\frac{1}{2}} + 3\tau - 2, 2\kappa \tau^{\frac{1}{2}} + 3\tau - 2, -\left( \tau + 2\kappa \tau^{\frac{1}{2}} \right) \right\},
\]

\[
\left( \vec{y}_1 \right)_{k=0}^4 = \left\{ 0, \tau^{\frac{1}{2}} - \kappa, \tau^{-\frac{1}{2}}, -\tau^{-\frac{1}{2}}, \kappa - \tau^{\frac{1}{2}} \right\},
\]

\[
\left( \vec{y}_0 \right)_{k=0}^4 = \left\{ 2\tau + \kappa \tau^{\frac{1}{2}}, 1 + \kappa \tau^{-\frac{1}{2}}, 1, 1, 1 + \kappa \tau^{-\frac{1}{2}} \right\}.
\]
Since the discrete number operator $\mathcal{N}^{(5)}$ commutes with the DFT operator $\Phi^{(5)}$, the above eigenvectors of the $\mathcal{N}^{(5)}$ are at the same time eigenvectors of the $\Phi^{(5)}$: two of them, $\vec{y}_0$ and $\vec{y}_2$, are associated with the same eigenvalue $i^0 = 1$ of the $\Phi^{(5)}$, while the eigenvectors $\vec{y}_4$, $\vec{y}_3$ and $\vec{y}_1$ correspond to the eigenvalues $i$, $i^2 = -1$ and $i^3 = -i$ of the $\Phi^{(5)}$, respectively.

Obviously, these multiplicities corresponding to the eigenvalues $i^k$, $0 \leq k \leq 3$, of the 5D DFT operator $\Phi^{(5)}$, are the particular $N = 5$ cases of the general explicit expressions for the multiplicities $m_k(i^k)$ of the eigenvalues of the ND DFT,

$$m_0(1) = \left[ \frac{N}{4} \right] + 1,$$

$$m_1(i) = \left[ \frac{N + 1}{4} \right],$$

$$m_2(-1) = \left[ \frac{N + 2}{4} \right],$$

$$m_3(-i) = \left[ \frac{N + 3}{4} \right] - 1,$$

where the symbol $[X]$ stands for the greatest integer in $X$. 


It is important also to realize that the eigenvectors $\vec{y}_k$, $0 \leq k \leq 4$, are distinct from those eigenvectors of the DFT operator $\Phi^{(5)}$, which have appeared before in the literature. For instance, Matveev evaluated explicit forms of the eigenvectors of the ND DFT operator $\Phi^{(N)}$ for the values of $N$ from $N = 2$ to $N = 8$ by combining the technique of spectral projectors for the operator $\Phi^{(N)}$ with the Gramm–Schmidt orthogonalization algorithm. In particular, Matveev’s eigenvectors $\vec{v}_n$, $1 \leq n \leq 5$, of the operator $\Phi^{(5)}$ are interrelated with the eigenvectors $\vec{y}_k$, $0 \leq k \leq 4$, in the following way: the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are some linear combinations of the $\vec{y}_0$ and $\vec{y}_2$, whereas the eigenvectors $\vec{v}_3$, $\vec{v}_4$, and $\vec{v}_5$ coincide with the $\vec{y}_3$, $\vec{y}_4$, and $\vec{y}_1$, respectively, up to normalization by constant factors:

$$\vec{v}_3 = \vec{y}_3, \quad \vec{v}_4 = \frac{\kappa}{2} \vec{y}_4, \quad \vec{v}_5 = -\frac{\kappa}{2} \vec{y}_1.$$
Finally, the incentive for making those extended comments, given above, is just to emphasize that this set of eigenvectors of the discrete number operator $\mathcal{N}^{(5)}$, produced by *Mathematica*, is still ambiguous until a rule is given for ordering those eigenvectors and associated eigenvalues of the operator $\mathcal{N}^{(5)}$.

We return now to a study of the eigenvectors and eigenvalues of the operator $\mathcal{N}^{(5)}$ in a systematic algebraic way. Since the lowest eigenvalue of the $\mathcal{N}^{(5)}$ is 0, its lowest eigenvector $\vec{f}_0$ is defined as

$$\mathcal{N}^{(5)} \vec{f}_0 = 0.$$

Moreover, an explicit form of the same eigenvector $\vec{f}_0$ can be found from the simpler equation

$$b_5 \vec{f}_0 = 0.$$
Since the symmetric matrix \((\mathcal{N}^{(5)})_{m,m'}\) clearly exhibits additional symmetry among the entries of all antidiagonals, it is evident that all eigenvectors of the operator \(\mathcal{N}^{(5)}\) must be either ‘even’ or ‘odd’ with respect to that particular reflection symmetry about the subantidiagonal \(\{(\mathcal{N}^{(5)})_{5-k,k}\}_{k=1}^{4}\) of the matrix \((\mathcal{N}^{(5)})_{m,m'}\), that is,

\[
\left( \vec{f}_n \right)_k = (-1)^n \left( \vec{f}_n \right)_{5-k}, \quad 0 \leq n, k \leq 4.
\]

This means that we should look for a lowest eigenvector \(\vec{f}_0\), whose componentwise structure is of the form

\[
\left( \vec{f}_0 \right)_{k=0} = (\alpha, \beta, \gamma, \gamma, \beta).
\]

Employing an explicit form of the matrix \(b_5\), one obtains only two linearly-independent equations for the three unknowns \(\alpha, \beta,\) and \(\gamma\):

\[
\alpha = \beta \kappa \tau^{1/2} + \gamma, \quad \beta = \gamma (1 + \kappa \tau^{-1/2}).
\]
Taking into account that $\kappa^2 = 2\tau - 1$, one concludes that $\alpha = \gamma (2\tau + \kappa \tau^{1/2})$ and the lowest eigenvector $\vec{f}_0$ with the components

$$\left( \vec{f}_0 \right)_{k=0}^4 = \gamma \left( 2\tau + \kappa \tau^{1/2}, 1 + \kappa \tau^{-1/2}, 1, 1, 1 + \kappa \tau^{-1/2} \right)$$

is thus determined by its defining equation up to normalization by a constant $\gamma$. Notice that $\vec{f}_0 = \gamma \vec{y}_0$, thus the $\vec{f}_0$ is also the eigenvector of the DFT operator $\Phi^{(5)}$ corresponding to the eigenvalue $i^0 = 1$.

Also, to normalize the lowest eigenvector $\vec{f}_0$ to have length one, it suffices to choose normalization constant as $\gamma = \gamma_0 \equiv (\nu_2 \nu_3)^{-1/2}$, and we shall employ in what follows the same notation $\vec{f}_0$ for the unit-length lowest eigenvector with the components

$$\left( \vec{f}_0 \right)_{k=0}^4 = \gamma_0 \left( 2\tau + \kappa \tau^{1/2}, 1 + \kappa \tau^{-1/2}, 1, 1, 1 + \kappa \tau^{-1/2} \right).$$
On a discrete number operator and its eigenvectors associated with the 5D discrete Fourier transform

In order to find next eigenvectors of the number operator \( N^{(5)} \), we first define 4 vectors of the form

\[
\vec{f}_k := \frac{d_k}{c_k} \left( b_5^T \right)^k \vec{f}_0, \quad 1 \leq k \leq 4,
\]

where \( \vec{f}_0 \) is the lowest eigenvector of the \( N^{(5)} \) and \( d_k \)'s are some normalization scalar factors. Since \( \vec{f}_0 \) is the eigenvector of the DFT operator \( \Phi^{(5)} \) also and it corresponds to the eigenvalue \( i^0 = 1 \), from the second intertwining relation it follows at once that all vectors \( \vec{f}_k, 1 \leq k \leq 4 \), are, in effect, the eigenvectors of the \( \Phi^{(5)} \),

\[
\Phi^{(5)} \vec{f}_k = \frac{d_k}{c_k} \Phi^{(5)} \left( b_5^T \right)^k \vec{f}_0 = i \frac{d_k}{c_k} b_5^T \Phi^{(5)} \left( b_5^T \right)^{k-1} \vec{f}_0 = \ldots
\]

\[
= i^k \frac{d_k}{c_k} \left( b_5^T \right)^k \Phi^{(5)} \vec{f}_0 = i^k \frac{d_k}{c_k} \left( b_5^T \right)^k \vec{f}_0 = i^k \vec{f}_k,
\]

corresponding to the eigenvalues \( i^k \), respectively.
Moreover, it actually turns out that all vectors $\vec{f}_k$, $0 \leq k \leq 4$, are at the same time the eigenvectors of the number operator $\mathcal{N}^{(5)}$. Indeed, since the operators $\Phi^{(5)}$ and $\mathcal{N}^{(5)}$ commute, one checks easily that

$$\Phi^{(5)} \mathcal{N}^{(5)} \vec{f}_k = \mathcal{N}^{(5)} \Phi^{(5)} \vec{f}_k = i^k \mathcal{N}^{(5)} \vec{f}_k$$

is valid for all integer values of $k$ between 0 and 4. This means that for any integer $k \in [0, 4]$ both vectors $\vec{f}_k$ and $\mathcal{N}^{(5)} \vec{f}_k$ are associated with the same eigenvalues $i^k$ of the 5D DFT operator $\Phi^{(5)}$. Consequently, three vectors $\vec{f}_k$, $1 \leq k \leq 3$, do represent the eigenvectors of the number operator $\mathcal{N}^{(5)}$ because the corresponding multiplicities $m(i^k) = 1$ for those integer values of $1 \leq k \leq 3$. 
As for the last vector $\vec{f}_4$, one readily verifies that

$$\mathcal{N}^{(5)} \vec{f}_4 \simeq b_5^T b_5 \left( b_5^T \right)^{4} \vec{f}_0 \simeq b_5^T b_5 b_5^T \vec{f}_3 \simeq b_5^T \left( \mathcal{N}^{(5)} + \mathcal{K} \right) \vec{f}_3$$

$$\simeq b_5^T \vec{f}_3 \simeq \vec{f}_4,$$

where the symbol $\text{A} \simeq \text{B}$ indicates that $\text{A}$ is equal to $\text{B}$ multiplied by a nonzero scalar constant factor and we employed the fact that the vector $\vec{f}_3$ is an eigenvector of the operator $\mathcal{K}$ also, despite the noncommutativity of the operators $\mathcal{K}$ and $\mathcal{N}^{(5)}$. 
Since the vector $\vec{f}_0$ has been already defined as the lowest eigenvector of the operator $\mathcal{N}^{(5)}$, one does conclude that the 5 orthonormal vectors $\vec{f}_k$, $0 \leq k \leq 4$, explicitly given as

$$\vec{f}_0 = \frac{1}{\sqrt{\nu_2 \nu_3}} \vec{y}_0, \quad \vec{f}_1 = \frac{1}{2} \sqrt{\frac{\tau}{\nu_3}} \vec{y}_4, \quad \vec{f}_2 = \frac{\sqrt{\tau}}{2\kappa} \vec{y}_3,$$

$$\vec{f}_3 = \frac{1}{2} \sqrt{\frac{\tau}{\nu_1}} \vec{y}_1, \quad \vec{f}_4 = \frac{1}{2\sqrt{\nu_2 \nu_3}} \vec{y}_2,$$

do represent the desired set of the eigenvectors for the number operator $\mathcal{N}^{(5)}$,

$$\mathcal{N}^{(5)} \vec{f}_k = \lambda_k \vec{f}_k, \quad 0 \leq k \leq 4,$$

associated with the eigenvalues

$$\lambda_0 = 0, \quad \lambda_1 = c^2 \nu_4, \quad \lambda_2 = c^2 \nu_3, \quad \lambda_3 = c^2 \nu_1, \quad \lambda_4 = c^2 \nu_2,$$

respectively.
The explicit analytical form of the spectrum of the discrete number operator $\mathcal{N}^{(5)}$ can be thus represented as

$$\lambda_k = c^2 \left[ 5(1 - \delta_{k,0}) + 4 \left( (\tau - 1) \sin k\theta + \cos k\theta \right) \sin 2k\theta \right],$$

where $\theta = 2\pi/5$ and $0 \leq k \leq 4$.

Our first graph compares these eigenvalues $\lambda_k$, $0 \leq k \leq 4$, of the discrete number operator $\mathcal{N}^{(5)}$ and the first 5 eigenvalues of the standard quantum number operator $\mathbf{N} = \mathbf{a}^{\dagger}\mathbf{a}$. 
On a discrete number operator and its eigenvectors associated with the 5D discrete Fourier transform.
The next step is to clarify how these eigenvectors transform under the action of the raising $b^T_5$ and lowering $b_5$ difference operators and then to compare them with the behaviour of their continuous counterparts $\psi_n(x)$, which satisfy the well-known relations

$$a\psi_n(x) = \sqrt{n}\psi_{n-1}(x), \quad a^\dagger\psi_n(x) = \sqrt{n+1}\psi_{n+1}(x).$$

But observe first that from definition of vectors $\vec{f}_k$ it follows at once that

$$\vec{f}_{k+1} = \frac{d_{k+1}}{c d_k} b^T_5 \vec{f}_k, \quad 0 \leq k \leq 3, \quad d_0 = 1.$$
Therefore the required relations can be explicitly written for each appropriate value of the index \( k \) as

\[
\mathbf{b}_5^T \vec{f}_0 = \frac{4 \kappa c^2}{\sqrt{\tau \lambda_4}} \vec{f}_1 \equiv \eta \sqrt{\lambda_1} \vec{f}_1,
\]

\[
\mathbf{b}_5^T \vec{f}_k = \sqrt{\lambda_{k+1}} \vec{f}_{k+1}, \quad 1 \leq k \leq 3,
\]

where

\[
\eta := \frac{4 \kappa}{\sqrt{\tau \nu_2 \nu_4}} \equiv \frac{4}{\sqrt{5 \tau + 21}}.
\]
It remains only to evaluate the last identity

$$ b_5^T \vec{f}_4 = \frac{1}{4c^4\kappa^2} (b_5^T)^5 \vec{f}_0 = -\frac{5\tau}{4\kappa^2} b_5^T \vec{f}_0 = -\sqrt{1 - \eta^2} \lambda_1 \vec{f}_1, $$

which is a direct consequence of the second ‘cyclic’ identity and the relation

$$ \vec{f}_4 = \frac{1}{4c^4\kappa^2} (b_5^T)^4 \vec{f}_0, $$

readily obtained by the successive use of all entries in the above chain of relations for vectors $\vec{f}_k$. The last formulas are thus discrete analogues of those, which are collected in the second identity of the continuous case.
As for the action of the lowering difference operator $b_5$, the situation here is slightly different. The point is that already at the first step one evaluates that

$$b_5 \vec{f}_1 = \delta b_5 b_5^T \vec{f}_0 = \delta \left( K + N^{(5)} \right) \vec{f}_0 = \delta K \vec{f}_0 = \delta (\alpha \vec{f}_0 + \beta \vec{f}_4),$$

where $\delta := \left( \eta \sqrt{\lambda_1} \right)^{-1}$ and coefficients $\alpha$ and $\beta$ can be explicitly defined by the following easy algebra:

$$\langle \vec{f}_0, b_5 \vec{f}_1 \rangle = \delta \alpha = \langle b_5^T \vec{f}_0, \vec{f}_1 \rangle = \delta^{-1} \langle \vec{f}_1, \vec{f}_1 \rangle = \delta^{-1} = \eta \sqrt{\lambda_1},$$

$$\langle \vec{f}_4, b_5 \vec{f}_1 \rangle = \delta \beta = \langle b_5^T \vec{f}_4, \vec{f}_1 \rangle = -\sqrt{(1 - \eta^2) \lambda_1} \langle \vec{f}_1, \vec{f}_1 \rangle = -\sqrt{(1 - \eta^2) \lambda_1}. $$
From last two relations one thus concludes that

$$\alpha = \eta^2 \lambda_1, \quad \beta = -\eta \sqrt{1 - \eta^2} \lambda_1,$$

and the action of lowering operator $b_5$ on the eigenvector $\vec{f}_1$ now explicitly reads

$$b_5 \vec{f}_1 = \sqrt{\lambda_1} \left[ \eta \vec{f}_0 - \sqrt{1 - \eta^2} \vec{f}_4 \right].$$

The evaluation of the action of the lowering operator $b_5$ on the remaining three eigenvectors $\vec{f}_n$, $n = 2, 3, 4$, requires less efforts for the following reason. As we have already remarked above, three vectors $\vec{f}_1, \vec{f}_2$ and $\vec{f}_3$, which are associated with the eigenvalues $i^k (k = 1, 2, 3)$ with multiplicities 1, are actually common eigenvectors of the number operator $\mathcal{N}^{(5)}$ and the operator $\mathcal{K}$ (although these operators do not commute).
Moreover, the explicit form of corresponding eigenvalues of the operator $\mathcal{K}$,

$$\mathcal{K} \vec{f}_n = (\lambda_{n+1} - \lambda_n) \vec{f}_n, \quad n = 1, 2, 3,$$

is a direct consequence of the evident intertwining relation $\mathcal{N}^{(5)} b_5^T = b_5^T (\mathcal{N}^{(5)} + \mathcal{K})$. Therefore, for $2 \leq k \leq 4$ one readily derives that

$$b_5 \vec{f}_k = \frac{1}{\sqrt{\lambda_k}} b_5 b_5^T \vec{f}_{k-1} = \frac{1}{\sqrt{\lambda_k}} \left( \mathcal{K} + \mathcal{N}^{(5)} \right) \vec{f}_{k-1} = \sqrt{\lambda_k} \vec{f}_{k-1}.$$
We are now in a position to write down all matrix elements of the raising and lowering operators $b^T_5$ and $b_5$ in the basis, built over the eigenvectors $\vec{f}_n$, $0 \leq n \leq 4$:

\[
\left( \left( \vec{f}_k, b^T_5 \vec{f}_l \right) \right) = \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\eta \sqrt{\lambda_1} & 0 & 0 & 0 & -\sqrt{1 - \eta^2} \lambda_1 \\
0 & \sqrt{\lambda_2} & 0 & 0 & 0 \\
0 & 0 & \sqrt{\lambda_3} & 0 & 0 \\
0 & 0 & 0 & \sqrt{\lambda_4} & 0 \\
\end{pmatrix},
\]

\[
\left( \left( \vec{f}_k, b_5 \vec{f}_l \right) \right) = \\
\begin{pmatrix}
0 & \eta \sqrt{\lambda_1} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\lambda_2} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\lambda_3} & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\lambda_4} & 0 \\
0 & -\sqrt{1 - \eta^2} \lambda_1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]
To close this section, we point out that as a consistency check one may verify that both of these matrix realizations of the raising and lowering operators $b^T_5$ and $b_5$ in the $\vec{f}_n$-basis, respectively, do possess the same ‘cyclic’ properties as indicated in identities, displayed in Section 2. Indeed, a direct computation of the 5th power of the matrix $b_5$ shows that

$$\left( \left( \vec{f}_k, b_5 \vec{f}_l \right) \right)^5 + \left[ (1 - \eta^2) \lambda_1 \lambda_2 \lambda_3 \lambda_4 \right]^{1/2} \left( \left( \vec{f}_k, b_5 \vec{f}_l \right) \right) = 0,$$

where

$$\left[ (1 - \eta^2) \lambda_1 \lambda_2 \lambda_3 \lambda_4 \right]^{1/2} = c^4 \left[ (1 - \eta^2) \nu_1 \nu_2 \nu_3 \nu_4 \right]^{1/2} = 5 c^4 \tau,$$

upon using definitions of the eigenvalues $\nu_k$, $1 \leq k \leq 4$, given at the beginning of Section 3. Finally, since the operator $b^T_5$ is the matrix transpose of $b_5$, the former one has the same ‘cyclic’ property as the latter.
From the outset the ND discrete Fourier transform $\Phi^{(N)}$ was conceived as a finite (discrete) analogue of the Fourier transform $\mathcal{F}$ and the N eigenvectors $\vec{f}_k$, $0 \leq k \leq N - 1$, of the former transform operator were therefore required to converge to the corresponding Hermite functions $\psi_n(x)$ in the limit as $N \to \infty$. So the question is: how the eigenvectors $\vec{f}_k$, $0 \leq k \leq N - 1$, of a ND DFT $\Phi^{(N)}$ with a fixed integer value of N can be related to the Hermite functions $\psi_n(x)$, $0 \leq n < \infty$?

Note first that a ND discrete Fourier transform is actually a discrete (finite) image of the $N$-dimensional subspace of the infinite dimensional Hilbert space $L^2(\mathbb{R}, dx)$, spanned by the first N basis functions $\psi_n(x)$, $0 \leq n \leq N - 1$, in this space, rather than of the whole Hilbert space $L^2(\mathbb{R}, dx)$ itself.
Consequently, one should find out how the eigenvectors $\vec{f}_k$ match the first $N$ Hermite functions $\psi_n(x), 0 \leq n \leq N - 1$. In this regard, it is important to take into account the following fundamental properties of the Hermite functions $\psi_n(x)$: $\psi_n(-x) = (-1)^n \psi_n(x)$ and each function $\psi_n(x)$ has exactly $n$ alternations in its sign. As we have already remarked above, the eigenvectors $\vec{f}_k$ do exhibit the same type of the reflection symmetry as

$$\psi_n(-x) = (-1)^n \psi_n(x)$$

in the continuous case, so that it remains only to verify that each eigenvector $\vec{f}_k$ has the same number of alternations in its components $(\vec{f}_k)_l, 0 \leq l \leq N - 1$, as a Hermite function $\psi_n(x)$, associated with it.
But the careful examination of the eigenvectors $\vec{f}_n$ under study indicates that their components are not appropriately structured in order to enable one to match them with the first 5 Hermite functions $\psi_n(x)$. It has been then realized that one actually needs to rearrange components of the eigenvectors $\vec{f}_k$ and introduce another set of centered vectors $\vec{f}_k^{(c)}$ with the components

$$\left(\vec{f}_k^{(c)}\right)_{l=-2}^2,$$

defined as

$$\left(\vec{f}_k^{(c)}\right)_{l-2} = \left((U)_{l,m}\right)\left(\vec{f}_k\right)_m, \quad 0 \leq k, l, m \leq 4,$$

where $U$ is the unitary operator, $UU^T = U^T U = I$, with the matrix elements

$$\left((U)_{m,m'}\right) = \begin{pmatrix}0 & 0 & 0 & 1 & 0 \\0 & 0 & 0 & 0 & 0 \\1 & 0 & 0 & 0 & 0 \\0 & 1 & 0 & 0 & 0 \\0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$
Their explicit componentwise forms are

\[
\left( f^c_0 \right)^2_{l=-2} = \frac{1}{\sqrt{\nu_2 \nu_3}} \left\{ 1, 1 + \kappa \tau^{-1/2}, 2\tau + \kappa \tau^{1/2}, 1 + \kappa \tau^{-1/2}, 1 \right\},
\]

\[
\left( f^c_1 \right)^2_{l=-2} = \frac{1}{2} \sqrt{\frac{\tau}{\nu_3}} \left\{ -\tau^{-1/2}, -\kappa - \tau^{1/2}, 0, \kappa + \tau^{1/2}, \tau^{-1/2} \right\},
\]

\[
\left( f^c_2 \right)^2_{l=-2} = \frac{\sqrt{\tau}}{2\kappa} \left\{ 1, 1, 2(1-\tau), 1, 1 \right\},
\]

\[
\left( f^c_3 \right)^2_{l=-2} = \frac{1}{2} \sqrt{\frac{\tau}{\nu_1}} \left\{ -\tau^{-1/2}, \kappa - \tau^{1/2}, 0, \tau^{1/2} - \kappa, \tau^{-1/2} \right\},
\]

\[
\left( f^c_4 \right)^2_{l=-2} = \frac{1}{2\sqrt{\nu_2 \nu_3}} \left\{ 2\kappa \tau^{1/2} + 3\tau - 2, -\left( \tau + 2\kappa \tau^{1/2} \right), 2, II, I \right\}.
\]

In next 5 graphs we compare the centered vectors \( f^c_n \) and the first Hermite functions \( \psi_n(x) \), \( 0 \leq n \leq 4 \), respectively.
On a discrete number operator and its eigenvectors associated with the 5D discrete Fourier transform

Eigenvectors of the number operator $N^{(5)}$ versus the Hermite functions $\psi_{\pi_1}(x)$

$\epsilon = \sqrt{\frac{2\pi}{5}}$
On a discrete number operator and its eigenvectors associated with the 5D discrete Fourier transform

Eigenvectors of the number operator $\mathcal{N}^{(5)}$ versus the Hermite functions $\psi_{n}(x)$
On a discrete number operator and its eigenvectors associated with the 5D discrete Fourier transform

Eigenvectors of the number operator $\mathcal{N}^{(5)}$ versus the Hermite functions $\psi_{n_l}(x)$

$$\epsilon = \sqrt{\frac{2\pi}{5}}$$
On a discrete number operator and its eigenvectors associated with the 5D discrete Fourier transform

Eigenvectors of the number operator $\mathcal{N}^{(5)}$ versus the Hermite functions $\psi_{n_1}(x)$

$\psi_3(x)$

$x$

$\varepsilon = \sqrt{\frac{2\pi}{5}}$
On a discrete number operator and its eigenvectors associated with the 5D discrete Fourier transform.

Eigenvectors of the number operator $N^{(5)}$ versus the Hermite functions $\psi_n(x)$. 

\[ \varepsilon = \sqrt{\frac{2\pi}{5}} \]
It is to be emphasized that thus introduced centered vectors $\vec{f}_n^{(c)}$ are actually eigenvectors of the \textit{centered discrete number operator} $\mathcal{N}^{(5; c)}$, 

\[ \mathcal{N}^{(5; c)} := \mathbf{U} \mathcal{N}^{(5)} \mathbf{U}^T, \]

associated with the same eigenvalues $\lambda_n$: $\mathcal{N}^{(5; c)} \vec{f}_n^{(c)} = \lambda_n \vec{f}_n^{(c)}$. Matrix elements of this operator $\mathcal{N}^{(5; c)}$ are explicitly given as 

\[
\left( \mathcal{N}^{(5; c)} \right)_{m,m'} = \left( (b_5^{(c)})^T b_5^{(c)} \right)_{m,m'}
\]

\[
= c^2 \begin{pmatrix}
5 - \tau & \kappa \tau^{-3/2} & -1 & -1 & 2 \kappa \tau^{-1/2} \\
\kappa \tau^{-3/2} & 4 + \tau & -\kappa \tau^{1/2} & -1 & -1 \\
-1 & -\kappa \tau^{1/2} & 2 & -\kappa \tau^{1/2} & -1 \\
-1 & -1 & -\kappa \tau^{1/2} & 4 + \tau & \kappa \tau^{-3/2} \\
2 \kappa \tau^{-1/2} & -1 & -1 & \kappa \tau^{-3/2} & 5 - \tau
\end{pmatrix},
\]

where the centered lowering and raising operators $b_5^{(c)}$ and $(b_5^{(c)})^T$ are defined through the operators $b_5$ and $b_5^T$, respectively, by the same similarity transformation with the operator $\mathbf{U}$. 
Furthermore, the centered vectors $\vec{f}_{n}^{(c)}$ turn out to be also eigenvectors of the centered discrete Fourier transform operator $\Phi^{(5; c)}$,

$$\Phi^{(5; c)} := U \Phi^{(5)} U^T,$$

corresponding to the respective eigenvalues $i^n$. The eigenproblem for the centered DFT operator $\Phi^{(5; c)}$ can be thus written in the matrix form as

$$\sum_{n=-2}^{2} \Phi^{(5; c)}_{m, n} (\vec{f}_{k}^{(c)})_n = \lambda_k (\vec{f}_{k}^{(c)})_m, \quad 0 \leq k \leq 4, \quad -2 \leq m \leq 2,$$

and associated matrix for this eigenproblem has elements

$$\left( \Phi^{(5; c)}_{m, n} \right) = \frac{1}{\sqrt{5}} \begin{pmatrix} q^4 & q^2 & 1 & q^3 & q \\ q^2 & q & 1 & q^4 & q^3 \\ 1 & 1 & 1 & 1 & 1 \\ q^3 & q^4 & 1 & q & q^2 \\ q & q^3 & 1 & q^2 & q^4 \end{pmatrix}.$$
A word of explanation regarding the present results is in order at this point, but let us recall first the following. Square matrices $A$ and $B$ are said to be similar, if there is a nonsingular matrix $C$ (which is referred to as a transforming matrix of $B$ to $A$) such that $A = C B C^{-1}$. In the case when the transforming matrix $C$ is a unitary matrix $U$, $U U^\dagger = I$, then $B$ is unitarily similar to $A$ (see, for example, [Lancaster and Tismenetsky]).

Taking that into account note that in the process of establishing how the eigenvectors $\vec{f}_n$ are related to the Hermite functions $\psi_n(x)$, we actually arrived at another form of the eigenproblem for the 5D DFT operator, which appear to be different from the traditional one. This form of DFT is also well known, but less frequently used. However, some authors have even suggested that it is better not to use the standard Fourier matrix that represents a discretization of the Fourier transform, but rather to use a ‘centered’ version of it [Clary and Mugler 2].
But it is now clear that $\Phi^{(5)}$ and $\Phi^{(5;c)}$ are actually unitarily similar, with the transforming matrix $U$, introduced above, and they represent therefore the same linear transformation after a change of basis.

Nonetheless, it is true that there is considerable merit in working with the non-standard 5D DFT operator $\Phi^{(5;c)}$ because it explicitly displays all those symmetry properties in the DFT eigenproblem, which are so characteristic of the continuous Fourier transform. As a direct consequence of the appealing feature of this form, the matrix form of $\Phi^{(5;c)}$ clearly exhibits its remarkable symmetry: the matrix $\Phi^{(5;c)}$ is a centrosymmetric matrix [Dickinson and Steiglitz].
This means that it is reproduced by the similarity transformation with the transforming matrix $J$,

$$\Phi^{(5; c)} = J \Phi^{(5; c)} J,$$

where $J$ is the $5 \times 5$ matrix with ones on the antidiagonal,

$$\left( J_{m, n} \right) = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Finally, the operator $\mathcal{N}^{(5; c)}$, which commutes with $\Phi^{(5; c)}$, is also of the centrosymmetric type.
To summarize, we have constructed an explicit form of a difference analogue of the quantum number operator in terms of the raising and lowering operators that govern eigenvectors of the 5D discrete (finite) Fourier transform. The main algebraic properties of this operator have been examined in detail. The eigenvalues of this discrete number operator are represented by distinct nonnegative numbers so that this operator has been used to systematically classify, in complete analogy with the case of the continuous classical Fourier transform, eigenvectors of the 5D DFT, thus resolving the ambiguity caused by the well-known degeneracy of the eigenvalues of the discrete Fourier transform. We hope that this particular knowledge will help us to extend our results to the case of an arbitrary ND discrete Fourier transform.

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On a discrete number operator and its eigenvectors associated with the 5D discrete Fourier transform

References

Carlitz L 1959 Some cyclotomic matrices  Acta Arithmetica 5 293-308

Auslander L and Tolimieri R 1979 Is computing with the finite Fourier transform pure or applied mathematics? Bull. Amer. Math. Soc. Vol.1 847–897


References

Terras A 1999 Fourier Analysis on Finite Groups and Applications (Cambridge University Press: Cambridge)


Matveev V B 2001 Intertwining relations between the Fourier transform and discrete Fourier transform, the related functional identities and beyond Inverse Problems 17 633–657
On a discrete number operator and its eigenvectors associated with the 5D discrete Fourier transform

References

Atakishiyev N M 2006 On q-extensions of Mehta’s eigenvectors of the finite Fourier transform Int. J. Mod. Phys. A 21 4993–5006


Koshy T 2001 Fibonacci and Lucas Numbers with Applications (New York: Wiley)

Dunlap R A 2006 The Golden Ratio and Fibonacci Numbers (Singapore: World Scientific)