A Matrix Interpretation of Coherent Pairs of Linear Functionals

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1 Preliminaries and Notation

2 \((M, N)\)-coherent pair of order \((m, k)\)

3 A Matrix Interpretation of Coherence
\[ \mathbb{P} \] will denote the linear space of polynomials with complex coefficients.

\[ \langle \mathcal{U}, p(x) \rangle \] will denote the image of \( p \in \mathbb{P} \) by the linear functional \( \mathcal{U} \).

\( r(x)\mathcal{U} \) is the linear functional defined by \( \langle r(x)\mathcal{U}, p(x) \rangle = \langle \mathcal{U}, r(x)p(x) \rangle \), \( r, p \in \mathbb{P} \).

The Dirac delta linear functional at \( a \in \mathbb{C} \) is defined by \( \langle \delta_a, p(x) \rangle = p(a) \).

\( D\mathcal{U} \) will denote its (distributional) derivative defined by

\[ \langle D\mathcal{U}, p(x) \rangle = -\langle \mathcal{U}, p'(x) \rangle, \quad p \in \mathbb{P}. \]

If \( P_{n+m}(x) \) is a monic polynomial of degree \( n + m \), then \( P_n^{[m]}(x) \) will denote the monic polynomial of degree \( n \)

\[ P_n^{[m]}(x) = \frac{P_{n+m}(x)}{(n+1)_m}, \quad n, m \geq 0, \]

where \( (\alpha)_n \) is the Pochhammer symbol

\[ (\alpha)_0 = 1, \quad (\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1), \quad n \geq 1. \]
ORTHOGONAL POLYNOMIALS

A linear functional $\mathcal{U}$ is said to be regular if and only if the following equivalent statements hold

- The principal leading submatrices $H_n$ of its Hankel matrix associated with its sequence of moments $\{u_n\}_{n \geq 0}$ satisfy $\det(H_n) \neq 0$, $n \geq 0$, where

\[
H_n = \left[ u_{i+j} \right]_{i,j=0}^n = \begin{bmatrix}
  u_0 & u_1 & \cdots & u_n \\
  u_1 & u_2 & \cdots & u_{n+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_n & u_{n+1} & \cdots & u_{2n}
\end{bmatrix}, \quad u_n = \langle \mathcal{U}, x^n \rangle, \quad n \geq 0.
\]

- There exists a (unique) sequence of monic polynomials $\{P_n(x)\}_{n \geq 0}$ such that

\[
\deg P_n = n, \quad \langle \mathcal{U}, P_n(x)P_m(x) \rangle = k_n \delta_{n,m}, \quad k_n \neq 0, \ n, m \geq 0.
\]

$\{P_n(x)\}_{n \geq 0}$ is called the sequence of monic orthogonal polynomials (SMOP) with respect to $\mathcal{U}$.

When $\det(H_n) > 0$, $n \geq 0$, $\mathcal{U}$ is said to be positive definite. In this case, there exists a positive Borel measure $\mu$ supported on an infinite subset $E$ of the real line such that

\[
\langle \mathcal{U}, p(x) \rangle = \int_E p(x) d\mu(x), \quad p \in \mathbb{P}.
\]
**Theorem.** (Favard Theorem). A sequence of monic polynomials \( \{P_n(x)\}_{n \geq 0} \) is a SMOP with respect to a regular linear functional \( \mathcal{U} \) if and only if there exist \( \{\alpha_n^P\}_{n \geq 0}, \{\beta_n^P\}_{n \geq 0} \subset \mathbb{C}, \beta_n^P \neq 0, n \geq 2 \), such that they satisfy a three-term recurrence relation (TTRR)

\[
P_n(x) = (x - \alpha_n^P) P_{n-1}(x) - \beta_n^P P_{n-2}(x), \quad n \geq 2,
\]

\[
\alpha_1^P P_0(x) = x - \alpha_1^P, \quad P_0(x) = 1.
\]

Moreover, \( \mathcal{U} \) is positive definite if and only if \( \alpha_n^P \) is real and \( \beta_{n+1}^P > 0 \), for \( n \geq 1 \).

**Remark.** This TTRR can be written in a matrix form as

\[
x p(x) = J_P p(x),
\]

where

\[
p(x) = \begin{bmatrix} P_0(x), & P_1(x), & \cdots \end{bmatrix}^T, \quad J_P = \begin{bmatrix} \alpha_1^P & 1 & 0 & 0 & \cdots \\ \beta_2^P & \alpha_2^P & 1 & 0 & \cdots \\ 0 & \beta_3^P & \alpha_3^P & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},
\]

where the semi-infinite tridiagonal matrix \( J_P \) is said to be the *monic Jacobi matrix* associated with \( \mathcal{U} \).
Semiclassical and Classical Linear Functionals

- A regular linear functional $U$ (and its SMOP) is said to be *semiclassical* if there exist a monic polynomial $\sigma(x)$ and a polynomial $\tau(x)$, with $\deg \tau \geq 1$, such that

$$D(\sigma(x)U) = \tau(x)U.$$  

(1.1)

The *class of $U$* is $\min \max \{\deg \sigma - 2, \deg \tau - 1\}$, where the minimum is taken among all pairs $(\sigma, \tau)$ such that (1.1) holds.

- When $U$ is semiclassical of class zero, i.e., $\deg \sigma \leq 2$ and $\deg \tau = 1$, $U$ (and its SMOP) is said to be *classical*.

**Proposition.** Let $\{P_n(x)\}_{n \geq 0}$ be the SMOP with respect to a regular linear functional $U$. The following statements are equivalent

- $U$ is a classical linear functional given by (1.1).
- $\{P_n^{[m]}(x)\}_{n \geq 0}$ is the SMOP with respect to the linear functional $U^{[m]} = \sigma^m(x)U$.

Besides, $\{P_n^{[m]}(x)\}_{n \geq 0}$ is a classical SMOP of the same kind as $\{P_n(x)\}_{n \geq 0}$ since $U^{[m]}$ satisfies

$$D\left(\sigma(x)U^{[m]}\right) = (\tau(x) + m\sigma'(x))U^{[m]}.$$
Preliminaries and Notation

(M, N)-coherent pair of order (m, k)

A Matrix Interpretation of Coherence

A pair of positive Borel measures \((\mu_0, \mu_1)\) supported on the real line is a *coherent pair* if their corresponding SMOP \(\{P_n(x)\}_{n \geq 0}\) and \(\{Q_n(x)\}_{n \geq 0}\) satisfy

\[
\frac{P'_{n+1}(x)}{n+1} + a_n \frac{P'_n(x)}{n} = Q_n(x), \quad a_n \neq 0, \quad n \geq 1.
\]

This condition of coherence arose as a sufficient condition for the *algebraic property*

\[
P_{n+1}(x) + \frac{n+1}{n} a_n P_n(x) = S_{n+1}(x; \lambda) + c_{n,\lambda} S_n(x; \lambda), \quad n \geq 1,
\]

where \(\{c_{n,\lambda}\}_{n \geq 1}\) are rational functions in \(\lambda > 0\) and \(\{S_n(x; \lambda)\}_{n \geq 0}\) is the SMOP associated with the Sobolev inner product

\[
\langle p(x), r(x) \rangle_\lambda = \int_{\mathbb{R}} p(x)r(x) d\mu_0(x) + \lambda \int_{\mathbb{R}} p'(x)r'(x) d\mu_1(x), \quad \lambda > 0,
\]

where \(p(x)\) and \(r(x)\) are polynomials with real coefficients.

For us: \((\mathcal{U}, \mathcal{V})\) is a \((1, 0)\)-coherent pair.
When $\mu_0$ and $\mu_1$ constitute a coherent pair, they stated an **algorithm** to compute the Fourier-Sobolev coefficients $\{f_n(\lambda)/s_n(\lambda)\}_{n \geq 0}$ with

$$
f_n(\lambda) = \langle f(x), S_n(x; \lambda) \rangle_\lambda, \quad \text{and} \quad s_n(\lambda) = \langle S_n(x; \lambda), S_n(x; \lambda) \rangle_\lambda, \quad n \geq 0,
$$
of the Fourier expansion

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f_n(\lambda)}{s_n(\lambda)} S_n(x; \lambda),$$

for a function $f(x)$ in the Sobolev space

$$W^{1,2}[I, \mu_0, \mu_1] = \{ f : I \to \mathbb{R} | f \in L^2_{\mu_0}(I), f' \in L^2_{\mu_1}(I) \},$$

where $I$ is a open interval of the real line.

**Remark.** This algorithm does not need the explicit expressions of the Sobolev orthogonal polynomials $S_n(x; \lambda)$, $n \geq 0$. 


- He determined all \((1, 0)\)-coherent pairs of regular linear functionals.
- If \((\mathcal{U}, \mathcal{V})\) is a \((1, 0)\)-coherent pair, then at least one of the functionals must be classical (i.e., Laguerre, Jacobi, or Bessel). Besides, \(\sigma(x)\mathcal{U} = \rho(x)\mathcal{V}\) holds, where \(\deg \sigma \leq 2\), \(\deg \rho = 1\).


A pair of regular linear functionals \((\mathcal{U}, \mathcal{V})\) is a *generalized coherent pair* (for us, \((1, 1)\)-coherent pair) if their respective SMOP \(\{P_n(x)\}_{n \geq 0}\) and \(\{Q_n(x)\}_{n \geq 0}\) satisfy

\[
Q_n(x) + b_{n,1}Q_{n-1}(x) = \frac{P'_{n+1}(x)}{n+1} + a_{n,1} \frac{P'_n(x)}{n}, \quad a_{n,1} \neq 0, \; n \geq 1.
\]

- They determined all \((1, 1)\)-coherent pairs of regular linear functionals.
- If \((\mathcal{U}, \mathcal{V})\) is a \((1, 1)\)-coherent pair, then at least one of the functionals must be semiclassical of class at most 1. Besides, \(\sigma(x)\mathcal{U} = \rho(x)\mathcal{V}\) holds, where \(\deg \sigma \leq 3\), \(\deg \rho = 1\).
Two regular linear functionals $\mathcal{U}$ and $\mathcal{V}$ constitute a

\[(M, N)\text{-coherent pair of order } (m, k),\]

if their corresponding SMOP \(\{P_n(x)\}_{n \geq 0}\) and \(\{Q_n(x)\}_{n \geq 0}\), satisfy

\[
P_n^{[m]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m]}(x) = Q_n^{[k]}(x) + \sum_{i=1}^{N} b_{i,n} Q_{n-i}^{[k]}(x), \quad n \geq 0,
\]

where \(M, N, m, k \in \mathbb{Z}^+ \cup \{0\}, \{a_{i,n}\}_{n \geq 0}, \{b_{i,n}\}_{n \geq 0} \subset \mathbb{C}, a_{M,n} \neq 0 \text{ if } n \geq M, b_{N,n} \neq 0 \text{ if } n \geq N, \text{ and } a_{i,n} = b_{i,n} = 0 \text{ when } i > n.

- When \(k = 0\), \((\mathcal{U}, \mathcal{V})\) is called a \((M, N)\text{-coherent pair of order } m\).
- When \((m, k) = (1, 0)\), \((\mathcal{U}, \mathcal{V})\) said to be a \((M, N)\text{-coherent pair}\.\)
Theorem. Let \((U, V)\) be a \((M, N)\)-coherent pair of order \((m, k)\) given as above with \(m \geq k\). Then, there exist polynomials \(\phi_{M+k+n}(x), \psi_{N+m+n}(x), \varphi(x),\) and \(\rho(x)\), \(\deg \phi_{M+k+n}(x) = M + k + n, \deg \psi_{N+m+n} = N + m + n,\) such that

\[D^{m-k}[\phi_{M+k+n}(x)V] = \psi_{N+m+n}(x)U, \quad n \geq 0,\]
\[\varphi(x)U = \rho(x)V.\]

Moreover,

(i) If \(m = k\), then \(U\) is a semiclassical linear functional if and only if so is \(V\).

(ii) If \(m > k\), then \(U\) and \(V\) are semiclassical linear functionals.

When \(U, V\) are positive definite linear functionals, then their corresponding positive Borel measures \(\mu_0, \mu_1\) are related by

\[d\mu_0(x) = \frac{\rho(x)}{\varphi(x)}d\mu_1(x) + \sum_{\ell=1}^{\deg \varphi} \eta_{\ell,\varphi}\delta_{x_\ell,\varphi} \quad \text{or} \quad d\mu_1(x) = \frac{\varphi(x)}{\rho(x)}d\mu_0(x) + \sum_{\ell=1}^{\deg \rho} \eta_{\ell,\rho}\delta_{x_\ell,\rho}.\]
Theorem. Let \((\mu_0, \mu_1)\) be a \((M, N)\)-coherent pair of order \(m\) given as above. Then

\[
P_{n+m}(x) + \sum_{i=1}^{M} \frac{(n+1)_m}{(n-i+1)_m} a_{i,n} P_{n-i+m}(x) = S_{n+m}(x; \lambda) + \sum_{j=1}^{K} c_{j,n}(\lambda) S_{n-j+m}(x; \lambda), \quad n \geq 0.
\]

\(S_n(x; \lambda) = P_n(x), \quad n < m,\) where \(K = \max\{M, N\}, \) \(\{c_{j,n}(\lambda)\}_{n \geq 0}, 1 \leq j \leq K,\) are rational functions in \(\lambda > 0,\) and \(\{S_n(x; \lambda)\}_{n \geq 0}\) is the SMOP associated with the Sobolev inner product

\[
\langle p(x), r(x) \rangle_\lambda = \int_{\mathbb{R}} p(x) r(x) d\mu_0 + \lambda \int_{\mathbb{R}} p^{(m)}(x) r^{(m)}(x) d\mu_1, \quad p, r \in P, \lambda > 0, \quad m \in \mathbb{Z}^+,
\]

where \(p(x)\) and \(r(x)\) are polynomials with real coefficients.

Conversely, if (2.1) holds, then there exist complex numbers \(\{b_{j,n}\}_{n \geq 0}, 1 \leq j \leq K,\) such that \((\mu_0, \mu_1)\) is a \((M, K)\)-coherent pair of order \(m\) given by

\[
P^{[m]}_n(x) + \sum_{i=1}^{M} a_{i,n} P^{[m]}_{n-i}(x) = Q_n(x) + \sum_{j=1}^{K} b_{j,n} Q_{n-j}(x), \quad n \geq 0,
\]
**Computation of the Fourier-Sobolev Coefficients**

For a function $f(x)$ in the weighted Sobolev linear space

$$W^{m,2}[I, \mu_0, \mu_1] = \left\{ g : I \rightarrow \mathbb{R} \mid g \in L^2_{\mu_0}(I), g^{(m)} \in L^2_{\mu_1}(I) \right\},$$

we obtain an algorithm to compute the coefficients in the algebraic property (2.1)

$$\{c_{j,n}(\lambda)\}_{n \geq 0}, \quad 1 \leq j \leq K,$$

as well as the Fourier-Sobolev coefficients

$$\{f_n/s_n\}_{n \geq 0}$$

such that

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f_n}{s_n} S_n(x; \lambda),$$

where

$$f_n = \langle f(x), S_n(x; \lambda) \rangle_{\lambda} \quad \text{and} \quad s_n = \|S_n\|_{\lambda}^2, \quad n \geq 0,$$

when $(\mu_0, \mu_1)$ is a $(M, N)$-coherent pair of order $m$, as follows.
Remark.

- The scheme of the computation of Sobolev OP norms \( \{s_{m+n}\}_{n \geq 0} \) and \( \{c_{j,n+j}(\lambda)\}_{n \geq 0}, 1 \leq j \leq K \), is given by the successive decreasing diagonals of the following matrix

\[
\begin{pmatrix}
  s_m & s_{m+1} & s_{m+2} & \cdots \\
  0 & c_{1,1}(\lambda) & c_{1,2}(\lambda) & c_{1,3}(\lambda) & \cdots \\
  0 & 0 & c_{2,2}(\lambda) & c_{2,3}(\lambda) & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \cdots \\
  0 & \cdots & \cdots & \cdots & 0 & c_{K,K}(\lambda) & c_{K,K+1}(\lambda) & c_{K,K+2}(\lambda) & \cdots \\
  \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  f_m & f_{m+1} & f_{m+2} & f_{m+3} & \cdots & f_{m+K} & f_{m+K+1} & f_{m+K+2} & \cdots
\end{pmatrix}^{(K+1) \times \infty}
\]

- The computation of the Fourier-Sobolev coefficients \( \{f_n/s_n\}_{n \geq 0} \) does not need to know explicitly the Sobolev SMOP \( \{S_n(x; \lambda)\}_{n \geq 0} \), when \( (\mu_0, \mu_1) \) is a \((M, N)\)-coherent pair of order \( m \). However, to get the Fourier-Sobolev series, we can recursively compute the Sobolev SMOP using the algebraic property (2.1).
Let us consider the Jacobi weight

$$d\mu^{\alpha,\beta}(x) = (1 - x)^\alpha (1 + x)^\beta \chi_{(-1,1)}(x)dx, \quad \alpha, \beta > -1.$$ 

Its SMOP $$\{\hat{P}_n^{(\alpha,\beta)}\}_{n \geq 0}$$ satisfies, for $$\alpha > 2, \beta > 3$$, (J. Petronilho, 2006)

$$\frac{\left(\hat{P}_{n+3}^{(\alpha-3,\beta-4)}(x)\right)'''}{(n+1)_3} + a_{1,n} \frac{\left(\hat{P}_{n+2}^{(\alpha-3,\beta-4)}(x)\right)'''}{(n)_3} + a_{2,n} \frac{\left(\hat{P}_{n+1}^{(\alpha-3,\beta-4)}(x)\right)'''}{(n-1)_3} = \hat{P}_n^{(\alpha-2,\beta)}(x) + b_{1,n} \hat{P}_{n-1}^{(\alpha-2,\beta)}(x), \quad n \geq 0,$$

$$b_{1,n} = \frac{2n(n + \alpha - 2)}{(2n + \alpha + \beta - 3)(2n + \alpha + \beta - 2)}, \quad a_{1,n} = -\frac{4n(n + \beta - 1)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta - 3)},$$

$$a_{2,n} = \frac{4n(n - 1)(n + \beta - 2)(n + \beta - 1)}{(2n + \alpha + \beta - 4)(2n + \alpha + \beta - 3)^2(2n + \alpha + \beta - 2)}.$$

Thus, if $$\alpha > 2, \beta > 3$$, the measures

$$d\mu_0 = d\mu^{\alpha-3,\beta-4} \quad \text{and} \quad d\mu_1 = d\mu^{\alpha-2,\beta}$$

form a $$(2, 1)$$-coherent pair of order 3, with

$$P_n(x) = \hat{P}_n^{(\alpha-3,\beta-4)} \quad \text{and} \quad Q_n(x) = \hat{P}_n^{(\alpha-2,\beta)}.$$
With the help of MAPLE, we applied the algorithm to the function $f : [-1, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) = e^{-3(x - \frac{1}{10})^2} \sin(10x),$$

in order to compute its Fourier-Sobolev coefficients with respect to the Sobolev SMOP associated with the inner product

$$\langle g(x), h(x) \rangle_\lambda = \int_{-\infty}^{\infty} g(x)h(x)d\mu_0 + \lambda \int_{-\infty}^{\infty} g'''(x)h'''(x)d\mu_1,$$

defined by the $(2, 1)$-coherent pair of order 3,

$$(\mu_0, \mu_1) \equiv (\mu^{1,1}, \mu^{2,5}), \quad \text{i.e.,} \quad (\alpha, \beta) = (4, 5).$$

This is possible because $f \in L^2_{\mu_2}(-1, 1)$ and $f''' \in L^2_{\mu_1}(-1, 1)$.

Let us see that the approximations for $f(x)$ and its derivatives $f'(x), f''(x), f'''(x)$, given by the partial sums of the Fourier-Sobolev series and its derivatives are better than the corresponding approximations given by the Fourier-Jacobi series and its derivatives.
**Figure:** $f(x)$ and the partial sums of degree 30 of its Fourier-Jacobi and Fourier-Sobolev series in $[-1, 1]$, $[0.9, 0.98]$, $[-1, -0.98]$, and $[0.98, 1]$. ($\lambda = 0.1$)
**Figure:** $f'(x)$ and the derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$ in $[-1, 1]$, $[0.9, 0.98]$, $[-1, -0.98]$, and $[0.98, 1]$. ($\lambda = 0.1$)
**Figure:** $f''(x)$ and the second derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$ in $[-1, 1]$, $[0.9, 0.98]$, $[-1, -0.98]$, and $[0.98, 1]$. ($\lambda = 0.1$)
**Figure:** $f'''(x)$ and the third derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$ in $[-1, 1]$, $[0.9, 0.98]$, $[-1, -0.98]$, and $[0.98, 1]$. ($\lambda = 0.1$)
Table: Errors of the approximations of $f(x)$ ($i = 0$) and its derivatives ($i = 1, 2, 3$) with the partial sums of degree 30 of the Fourier-Jacobi ($\ell = J$) and Fourier-Sobolev ($\ell = S$) series of $f(x)$ and their derivatives

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\varepsilon_{J,L^2}^{(i)}$</th>
<th>$\varepsilon_{S,L^2}^{(i)}$</th>
<th>$\varepsilon_{J,\mu_0}^{(i)}$</th>
<th>$\varepsilon_{S,\mu_0}^{(i)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1.05 \times 10^{-7}$</td>
<td>$1.33 \times 10^{-10}$</td>
<td>$1.92 \times 10^{-9}$</td>
<td>$1.85 \times 10^{-11}$</td>
</tr>
<tr>
<td>1</td>
<td>$2.76 \times 10^{-3}$</td>
<td>$2.61 \times 10^{-7}$</td>
<td>$1.87 \times 10^{-5}$</td>
<td>$3.93 \times 10^{-9}$</td>
</tr>
<tr>
<td>2</td>
<td>$8.99 \times 10$</td>
<td>$2.35 \times 10^{-3}$</td>
<td>$6.45 \times 10^{-1}$</td>
<td>$2.15 \times 10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>$1.65 \times 10^6$</td>
<td>$2.93 \times 10$</td>
<td>$1.47 \times 10^4$</td>
<td>$3.05 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

$$
\varepsilon_{\ell,L^2}^{(i)} = \|f^{(i)} - S_{30,\ell}^{(i)}(f)\|_{L^2}^2 = \int_{-1}^{1} |f^{(i)}(x) - S_{30,\ell}^{(i)}(x;f)|^2 dx,
$$

$$
\varepsilon_{\ell,\mu_0}^{(i)} = \|f^{(i)} - S_{30,\ell}^{(i)}(f)\|_{\mu_0}^2 = \int_{-1}^{1} \left(f^{(i)}(x) - S_{30,\ell}^{(i)}(x;f)\right)^2 (1 - x^2) dx,
$$

$$
E_{\ell,\lambda}^{(i)} = \|f^{(i)} - S_{30,\ell}^{(i)}(f)\|_{\lambda}^2 = \int_{-1}^{1} \left(f^{(i)}(x) - S_{30,\ell}^{(i)}(x;f)\right)^2 (1 - x^2) dx
$$

$$
+ (0.1) \int_{-1}^{1} \left(f^{(3+i)}(x) - S_{30,\ell}^{(3+i)}(x;f)\right)^2 (1 - x)^2 (1 + x)^5 dx.
$$
In the literature

- The following particular cases was analyzed
  - $(1, 0)$, $(1, 1)$, $(2, 0)$, and $(M + N, 2M + 1)$-coherent pairs,
  - $(1, 1)$ and $(M, N)$-coherent pairs of order $0$,
  - $(M, N)$-coherent pairs of order $(k + 1, k)$ and $(k, k)$,
  - $(1, 0)$-coherent pairs of order $2$,
  - $(1, 1)$-coherent pairs of order $m$.

In these situations, $\mathcal{U}$ and $\mathcal{V}$ must be semiclassical linear functionals and they are related by rational factor. Besides, in the case of $(1, 0)$-coherence, $\mathcal{U}$ or $\mathcal{V}$ must be classical.

- The algebraic property which relates coherent pairs and Sobolev orthogonal polynomials was obtained for
  - $(1, 0)$, $(2, 0)$, $(1, 1)$, $(M + 1, 0)$, and $(M, N)$-coherent pairs.

- The algorithm for the computation of the Fourier-Sobolev coefficients was given for
  - $(1, 0)$, $(2, 0)$, and $(M, N)$-coherent pairs.
1 Preliminaries and Notation

2 $(M, N)$-coherent pair of order $(m, k)$

3 A Matrix Interpretation of Coherence
Let us consider a $(M, N)$-coherent pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ given by

$$\frac{P'_{n+1}(x)}{n+1} + \sum_{i=1}^{M} a_{i,n} \frac{P'_{n-i+1}(x)}{n-i+1} = Q_n(x) + \sum_{i=1}^{N} b_{i,n} Q_{n-i}(x), \quad n \geq 0,$$

$$a_{M,n} \neq 0, \quad n \geq M, \quad b_{N,n} \neq 0, \quad n \geq N, \quad a_{i,n} = b_{i,n} = 0, \quad i > n,$$

i.e., writing in a matrix form,

$$\mathcal{A}_1 \mathbf{p}_1'(x) = \mathcal{B} \mathbf{q}(x) \quad \text{and} \quad \mathcal{A} \mathbf{p}'(x) = \mathcal{B} \mathbf{q}(x),$$

where

$$\mathbf{p}(x) = \begin{bmatrix} P_0(x), & P_1(x), & \cdots \end{bmatrix}^T, \quad \mathbf{p}_1(x) = \begin{bmatrix} P_1(x), & P_2(x), & \cdots \end{bmatrix}^T,$$

$$\mathbf{q}(x) = \begin{bmatrix} Q_0(x), & Q_1(x), & \cdots \end{bmatrix}^T,$$

and,

$$\begin{bmatrix}
0 & \cdots & 0 & \frac{a_{M,n}}{n-M+1} & \cdots & \frac{a_{2,n}}{n-1} & \frac{a_{1,n}}{n} & \frac{1}{n+1} & 0 & \cdots \\
\hline
\hline
n-M+1 & \text{zeros} & & & & & & \hline
\end{bmatrix},$$

$$\begin{bmatrix}
0 & \cdots & 0 & \frac{a_{M,n}}{n-M+1} & \cdots & \frac{a_{1,n}}{n} & \frac{1}{n+1} & 0 & \cdots \\
\hline
\hline
n-M & \text{zeros} & & & & & & \hline
\end{bmatrix},$$

$$\begin{bmatrix}
0 & \cdots & 0 & b_{N,n} & \cdots & b_{1,n} & 1 & 0 & \cdots \\
\hline
\hline
n-N & \text{zeros} & & & & & & \hline
\end{bmatrix},$$

are the corresponding $n$th rows, $n \geq 0$, of the lower Hessenberg matrix $\mathcal{A}$ and the nonsingular lower triangular matrices $\mathcal{A}_1$ and $\mathcal{B}$, respectively.
**Proposition.** If two SMOP \( \{P_n(x)\}_{n \geq 0} \) and \( \{Q_n(x)\}_{n \geq 0} \) are related by
\[
p'(x) = Mq(x),
\]
where \( M \) is a infinite matrix, then
\[
\mathcal{J}_P^2M - 2\mathcal{J}_P M\mathcal{J}_Q + M\mathcal{J}_Q^2 = 0,
\]
where \( \mathcal{J}_P \) and \( \mathcal{J}_Q \) are the Jacobi matrices associated with \( \{P_n(x)\}_{n \geq 0} \) and \( \{Q_n(x)\}_{n \geq 0} \), respectively.

**Corollary.** Let \((U, V)\) be a \((M, N)\)-coherent pair. Then
\[
\mathcal{J}_P^2 \begin{bmatrix} 0 \\ A_1^{-1}B \end{bmatrix} - 2\mathcal{J}_P \begin{bmatrix} 0 \\ A_1^{-1}B \end{bmatrix} \mathcal{J}_Q + \begin{bmatrix} 0 \\ A_1^{-1}B \end{bmatrix} \mathcal{J}_Q^2 = 0.
\]

**Corollary.** If \((U, V)\) is a \((M, N)\)-coherent pair such that \( A \) is a nonsingular matrix (e.g, when \( M = 1 \) and \( N \geq 0 \), \( A \) is a nonsingular upper bidiagonal matrix), then
\[
(M_P - M_Q)^2 = [M_P, M_Q],
\]
where \([M_P, M_Q]\) is the commutator of the matrices \( M_P \) and \( M_Q \) given by
\[
[M_P, M_Q] = M_P M_Q - M_Q M_P, \quad \text{and}, \quad M_P = A\mathcal{J}_P A^{-1}, \quad M_Q = B\mathcal{J}_Q B^{-1}.
\]
**Proposition.** Let \((U, V)\) be a \((M, N)\)-coherent pair of order \(m, m \geq 0\), of regular linear functionals given by

\[
\hat{A}\hat{p}(x) = Bq(x),
\]

where \(B\) and \(q(x)\) are given as above

\[
\hat{p}(x) = \begin{bmatrix} P_0^m(x), & P_1^m(x), & \cdots \end{bmatrix}^T,
\]

and \(\hat{A}\) is the nonsingular lower triangular matrix such that the entries of its main diagonal are all 1’s, and its \(n\)th row for \(n \geq 1\) (counting the rows from zero), is

\[
\begin{bmatrix} 0 & \cdots & 0 \underbrace{a_{M, n} & \cdots & a_{1, n} & 1 & 0 & \cdots} \end{bmatrix}.
\]

If \(U\) is a classical linear functional, then \(J_{P^m}\) and \(J_Q\), the monic Jacobi matrices associated with the SMOP \(\{P_n^m(x)\}_{n \geq 0}\) and \(\{Q_n(x)\}_{n \geq 0}\), respectively, are similar matrices satisfying

\[
\hat{A}J_{P^m}\hat{A}^{-1} = \mathcal{M}_{P^m} = \mathcal{M}_Q = B J_Q B^{-1}.
\]
If \((\mu_0, \mu_1)\) is a \((M, N)\)-coherent pair of order \(m\), \(m \geq 1\), of positive Borel measures, let us consider the Sobolev SMOP \(\{S_n(x; \lambda)\}_{n \geq 0}\) and the algebraic property (2.1)

\[
P_{n+m}(x) + \sum_{i=1}^{M} \tilde{a}_{i,n} P_{n-i+m}(x) = S_{n+m}(x; \lambda) + \sum_{j=1}^{K} c_{j,n}(\lambda) S_{n-j+m}(x; \lambda), \quad n \geq 0,
\]

\(S_n(x; \lambda) = P_n(x), \quad n \leq m\), where \(K = \max\{M, N\}\), \(c_{j,n}(\lambda) = 0, \quad n < j \leq K, \quad \tilde{a}_{i,n} = \frac{(n+1)m}{(n-i+1)m} a_{i,n}, \quad n \geq 0, \quad \tilde{a}_{i,n} = 0, \quad i > n, \quad \tilde{a}_{0,n} = 1, \quad n \geq 0\). We can express these relations as

\[
\tilde{A}p(x) = Cs(x; \lambda), \quad \text{where}
\]

\[
p(x) = \begin{bmatrix} P_0(x), & P_1(x), & \cdots \end{bmatrix}^T, \quad s(x; \lambda) = \begin{bmatrix} S_0(x; \lambda), & S_1(x; \lambda), & \cdots \end{bmatrix}^T,
\]

and \(\tilde{A}\) and \(C\) are nonsingular lower triangular matrices such that their first \(m\) rows are \(\begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots \end{bmatrix}\), and their \((n + m)\)th rows, \(n \geq 0\), are, respectively,

\[
\begin{bmatrix}
0 & \cdots & 0 \\
\overbrace{0}^{n-M+m \text{ zeros}} & \tilde{a}_{M,n} & \cdots & \tilde{a}_{1,n} & 1 & 0 & \cdots \\
0 & \cdots & 0 \\
\overbrace{c_{K,n}(\lambda) & \cdots & c_{1,n}(\lambda) & 1 & 0 & \cdots}^{n-K+m \text{ zeros}}
\end{bmatrix},
\]
\[ \tilde{A}p(x) = Cs(x; \lambda), \quad \text{where} \]

\[ p(x) = \begin{bmatrix} P_0(x), & P_1(x), & \cdots \end{bmatrix}^T, \quad s(x; \lambda) = \begin{bmatrix} S_0(x; \lambda), & S_1(x; \lambda), & \cdots \end{bmatrix}^T. \]

In this way, the matrix representation of the multiplication operator by \( x \) in terms of the basis \( \{S_n(x; \lambda)\}_{n \geq 0} \) is

\[ xs(x; \lambda) = \mathcal{H}_{S,P,\lambda} s(x; \lambda), \]

where \( \mathcal{H}_{S,P,\lambda} \) is the lower Hessenberg matrix

\[ \mathcal{H}_{S,P,\lambda} = C^{-1} \tilde{A} J_P \tilde{A}^{-1} C, \]

similar to the monic Jacobi matrix \( J_P \) associated with the SMOP \( \{P_n(x)\}_{n \geq 0} \).

Thanks for your attention!!
OPEN PROBLEMS

• For fixed values of $M$, $N$, $m$, $k$, and order $(m, k)$, to describe all $(M, N)$-coherent pairs, $(M, N)\text{-}D_\nu$-coherent pairs, and $(M, N)$-coherent pairs on the unit circle.

• To study asymptotic properties of the Sobolev polynomials $S_n(x; \lambda)$, $S_n(x; \lambda, \nu)$, and $S_n(z; \lambda)$, $n \geq 0$, orthogonal with respect to the inner products

\[
\langle p(x), r(x) \rangle_\lambda = \langle U, p(x)r(x) \rangle + \lambda \langle V, p^{(m)}(x)r^{(m)}(x) \rangle, \quad \lambda > 0, \ m \in \mathbb{Z}^+,
\]

\[
\langle p(x), r(x) \rangle_{\lambda, \nu} = \langle U, p(x)r(x) \rangle + \lambda \langle V, (D_\nu^m p)(x)(D_\nu^m r)(x) \rangle, \quad \lambda > 0, \ m \in \mathbb{Z}^+,
\]

\[
\langle p(z), r(z) \rangle_\lambda = \langle U, p(z)\overline{r}(1/z) \rangle + \lambda \langle V, p^{(m)}(z)\overline{r^{(m)}(1/z)} \rangle, \quad \lambda > 0, \ m \in \mathbb{Z}^+,
\]

respectively, when $(U, V)$ is a $(M, N)$-coherent pair of order $m$, a $(M, N)\text{-}D_\nu$-coherent pair of order $m$, and a $(M, N)$-coherent pair of order $m$ on the unit circle, respectively.

• To study the zeros of the Sobolev orthogonal polynomials $S_n(x; \lambda)$, $S_n(x; \lambda, \nu)$, and $S_n(z; \lambda)$, $n \geq 0$.

• To analyze the convergence of Fourier series expansions in terms of the Sobolev orthogonal polynomials $S_n(x; \lambda)$, $S_n(x; \lambda, \nu)$, and $S_n(z; \lambda)$, $n \geq 0$. 