Homotopical Homology and Cohomology

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Preface

In the mid fifties, in a celebrated paper [13], Albrecht Dold and René Thom proved a theorem which roughly states that the homotopy groups of the infinite symmetric product of a pointed space are isomorphic to the reduced homology groups of the given space with $\mathbb{Z}$-coefficients. The infinite symmetric product of a space is the free H-space generated by the space. The proof of the theorem involves the concept of quasifibration. This theorem was the core of [2]. A few years later, M. McCord [35] proved a similar theorem, but using the free topological group generated by the space instead. Indeed this construction allowed to put as coefficients not only $\mathbb{Z}$, but any abelian group or any ring. These topological abelian groups, which are constructed in a functorial way, when applied to the spheres, yield the Eilenberg–Mac Lane spaces which are used to define cohomology. The aim of this book is to introduce algebraic topology by defining homology and cohomology from this point of view. This approach has been successfully used in algebraic geometry in order to apply the methods of algebraic topology to study geometric phenomena.

The book is intended to have advanced undergraduate level or basic graduate level, i.e. it can be used as a text book to give an alternative introduction to homology and cohomology. Furthermore, the book compares this viewpoint with the traditional one. Therefore singular homology and cohomology are presented, perhaps with less detail than in other books, but still explaining their construction. An explicit isomorphism is then given between the homotopical homology (and cohomology) and the singular homology (and cohomology).

One of the techniques used in the book is that of simplicial sets. So we have devoted a chapter to their study. Their close relatives, the simplicial complexes are briefly presented in a section of the introductory chapter.

Other chapters include fibrations and cofibrations and the higher homotopy groups, as well as homological algebra. In the introductory chapter, there are also sections on category theory,and on a convenient category of topological spaces, where we work. The book is intended, as far as possible, to be self-contained, but for a good understanding of the material, basic knowledge on point-set topology, as well as on group and module theory are required.

This electronic version of the book is preliminary. The book is still in process of being written, and many details are due.

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Contents

Introduction

We expect from the reader the knowledge of basic notions both in point-set and algebraic topology, mainly about the fundamental group and covering maps. Notions of categories, functors and natural transformations, as well as the concept of adjoint functors. We recall the main definitions.

We shall use the following basic categories: Sets and functions, denoted by $\mathfrak{Set}$, topological spaces and continuous maps, denoted by $\mathfrak{Top}$, as well as several (full) subcategories of $\mathfrak{Top}$.

Chapter 1 Basic concepts and notation

In this preliminary chapter we shall study the basic notions that will be needed in the book. After a short account on basic symbols and useful constructions, we start recalling the very basic concepts in category theory, like functors, natural transformations and adjointness. Then we continue with the topological setup of the text, namely the category of k-spaces, where we shall work. The simplicial complexes build up an important source of spaces. We devote the next section to them. The geometric realization of a simplicial complex is a CW-complex. We study CW-complexes in the last section.

1.1 Basic symbols

Throughout the text we shall use the following basic symbols, among others. The symbol $\approx$ between two topological spaces means that they are homeomorphic, $\simeq$ between continuous functions or topological spaces means that they are homotopic or homotopy equivalent, and $\cong$ between groups (abelian or nonabelian) means they are isomorphic. The symbol $\circ$ denotes composition of functions (maps, homomorphisms) and will be omitted ocasionally, if doing so does not lead to confusion. The term *map* invariably means a continuous function between topological spaces, and the term *function* is reserved either for functions between sets or for those maps whose codomain is $\mathbb{R}$ or $\mathbb{C}$.

And now a final note about some additional notation that will be used in the text. If X is a topological space and $A \subseteq X$ is a subspace, then in agreement with the special cases mentioned below we shall use the notation $\overset{\circ}{A}$ to denote the topological interior of A in X, and the notation ∂A to denote its topological boundary. $X \sqcup Y$ denotes the topological sum of X and Y. On the other hand, if V is a vector space provided with a scalar product (or Hermitian product, if the space is complex), which we usually denote by $\langle -, - \rangle$, then we use the notation $\| \cdot \|$ or $| \cdot |$ to denote the norms in V associated to the inner product, that is, $\|x\|$ or $|x| = \sqrt{\langle x, x \rangle}$. Likewise, if $U \subseteq V$ is a linear subspace, we use $U^\perp = \{x \in V \mid \langle x, a \rangle = 0 \text{ for all } a \in U\}$ to denote the *orthogonal complement* of U in V with respect to the inner product.

1.2 Some basic topological spaces

Euclidean spaces, various of its subspaces, and spaces derived from these will all play an important role for us.

$\mathbb{R}$ will represent the set (as well as the topological space and the real vector space) of *real numbers*. $\mathbb{R}^0$ will denote the singleton set (of only one point) $\{0\} \subset \mathbb{R}$. Frequently, we shall use the notation $*$ for an (arbitrary) singleton set. $\mathbb{R}^n$ will be the notation for *Euclidean space of dimension* n, or *Euclidean* n*-space*, such that

$$\mathbb{R}^n = \{x = (x_1, \dots, x_n) \mid x_i \in \mathbb{R}, \quad 1 \leq i \leq n\} .$$

Using the equality

$$((x_1, \dots, x_m), (y_1, \dots, y_n)) = (x_1, \dots, x_m, y_1, \dots, y_n)$$

we identify the Cartesian product $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$. Likewise, we identify $\mathbb{R}^n$ with the closed subspace $\mathbb{R}^n \times 0 \subset \mathbb{R}^{n+1}$. We give $\bigcup_{n=0}^{\infty} \mathbb{R}^n = \mathbb{R}^\infty$ the topology of the union (which is the colimit topology, as we shall see shortly). $\mathbb{R}^\infty$ consists, therefore, of infinite sequences of real numbers $(x_1, x_2, x_3, \dots)$ almost all of which are zero, that is to say, such that $x_k = 0$ for k sufficiently large. $\mathbb{R}^n$ is identified with the subspace of sequences $(x_1, \dots, x_n, 0, 0, \dots)$. The topology of $\mathbb{R}^\infty$ is such that a set $A \subset \mathbb{R}^\infty$ is closed if and only if $A \cap \mathbb{R}^n$ is closed in $\mathbb{R}^n$ for all n.

Topologically we identify the set (as well as the topological space and the complex vector space) $\mathbb{C}$ of *complex numbers* with $\mathbb{R}^2$ using the equality $x + \mathrm{i}y = (x, y)$, where i represents the imaginary unit, that is $\mathrm{i} = \sqrt{-1}$. Analogously with the real case, we have the *complex space of dimension* n, $\mathbb{C}^n = \{z = (z_1, \dots, z_n) \mid z_i \in \mathbb{C},\ 1 \leq i \leq n\}$, or *complex* n*-space*.

In $\mathbb{R}^n$ we define for every $x = (x_1, \dots, x_n)$ its *norm* by

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2}\,;$$

likewise, in $\mathbb{C}^n$ we define the *norm* by

$$|z| = \sqrt{z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n}\,,$$

where $\bar{z}$ denotes the *complex conjugate* $x - \mathrm{i}y$ of $z = x + \mathrm{i}y$. Up to the natural identification between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, it is an *exercise* to show that the two norms coincide.

For $n \geq 0$ we shall use from now on the following subspaces of Euclidean space:

$\mathbb{D}^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, the *unit disk* or the *unit ball* of dimension n.

$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\} = \partial\mathbb{D}^n$, the *unit sphere* of dimension $n - 1$.

$\overset{\circ}{\mathbb{D}}{}^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$, the *unit cell* of dimension n.

$I^n = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1,\ 1 \leq i \leq n\}$, the *unit cube* of dimension n.

$\partial I^n = \{x \in I^n \mid x_i = 0 \text{ or } 1 \text{ for some } i\}$, the *boundary* of I^n in $\mathbb{R}^n$.

$I = I^1 = [0,1] \subset \mathbb{R}$, the *unit interval.*

$\Delta^n = \{x \in \mathbb{R}^{n+1} \mid x_i \geq 0 \text{ and } \sum x_i = 1\}$, the *standard n-simplex.*

$\dot{\Delta}^n = \{x \in \Delta^n \mid x_i = 0 \text{ for at least one } i\}$, the *boundary of the standard n-simplex.*

Briefly, we usually call $\mathbb{D}^n$ the *unit n-disk*, $\mathbb{S}^{n-1}$ the *unit $(n-1)$-sphere*, $\mathring{\mathbb{D}}^n$ the *unit n-cell*, and I^n the *unit n-cube*. It is worth mentioning that all of the spaces just defined are connected (in fact, pathwise connected), except for $\mathbb{S}^0$, ∂I, and $\dot{\Delta}^1$, these being homeomorphic to each other, of course. The disks, the spheres, the cubes, and their boundaries also are compact (but not the cells, except for the 0-cell $\mathring{\mathbb{D}}^0 = *$).

The *group of two elements* $\mathbb{Z}_2 = \{-1, 1\}$ (which can also be seen as the quotient of the *group of the integers* $\mathbb{Z}$ modulo $2\mathbb{Z}$) acts on $\mathbb{S}^n$ by the *antipodal action*, that is, $(-1)x = -x \in \mathbb{S}^n$. The *orbit space* of the action, which is the result of identifying each $x \in \mathbb{S}^n$ with its antipode $-x$, is denoted by $\mathbb{RP}^n$ and is called *real projective space* of dimension n.

1.2.1 Exercise.

(a) Show that $\mathbb{S}^n$ is the one-point compactification of $\mathbb{R}^n$. (*Hint:* Use the stereographic projection. See [42].)

(b) Show that there is a homeomorphism $\mathbb{S}^n \approx I^n/\partial I^n$. (*Hint:* Show that the n-cube I^n is homeomorphic to n-ball $\mathbb{D}^n$, then prove that the n-cell $\mathring{\mathbb{D}}^n$ is homeomorphic to $\mathbb{R}^n$ and use the fact that the quotient $\mathbb{D}^n/\mathbb{S}^{n-1}$ is the one-point compactification of $\mathring{\mathbb{D}}^n$.)

The *infinite-dimensional sphere* $\mathbb{S}^\infty = \bigcup_{n=0}^\infty \mathbb{S}^n$, where the inclusion $\mathbb{S}^{n-1} \subset \mathbb{S}^n$ is defined by the inclusion $\mathbb{R}^n \subset \mathbb{R}^{n+1}$, is a subspace of $\mathbb{R}^\infty$. The action of $\mathbb{Z}_2$ in $\mathbb{S}^n$ induces an action in $\mathbb{S}^\infty$, whose orbit space is denoted by $\mathbb{RP}^\infty$ and is called *infinite-dimensional real projective space*. In fact, the inclusion $\mathbb{S}^{n-1} \subset \mathbb{S}^n$ induces an inclusion $\mathbb{RP}^{n-1} \subset \mathbb{RP}^n$ and the union $\bigcup_{n=0}^\infty \mathbb{RP}^n$ coincides topologically with $\mathbb{RP}^\infty$.

On the other hand, the *circle group* $\mathbb{S}^1 = \{\zeta \in \mathbb{C} \mid \|\zeta\| = 1\}$ acts on $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ by multiplication on each complex coordinate, namely, $\zeta(z_1, \dots, z_{n+1}) = (\zeta z_1, \dots, \zeta z_{n+1})$. The orbit space of this action, which is the result of identifying $z \in \mathbb{S}^{2n+1}$ with $\zeta z \in \mathbb{S}^{2n+1}$, for all $\zeta \in \mathbb{S}^1$, is denoted by $\mathbb{CP}^n$ and is called *complex* *projective space* of dimension n (in fact, its *real dimension* is $2n$). The action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1}$ induces an action on $\mathbb{S}^\infty$, whose orbit space is denoted by $\mathbb{CP}^\infty$ and is called *infinite-dimensional complex projective space*. In analogy with the real case, the inclusion $\mathbb{S}^{2n-1} \subset \mathbb{S}^{2n+1}$, defined by the inclusion $\mathbb{C}^n \subset \mathbb{C}^{n+1}$, induces an inclusion $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ and the union $\bigcup_{n=0}^\infty \mathbb{CP}^n$ coincides topologically with $\mathbb{CP}^\infty$.

The *group of* $n\times n$ *invertible matrices* with real (complex) coefficients is denoted by $\mathrm{GL}_n(\mathbb{R})$ ($\mathrm{GL}_n(\mathbb{C})$) and consists of the matrices whose determinants are not zero. The subgroup $\mathrm{O}_n \subset \mathrm{GL}_n(\mathbb{R})$ ($\mathrm{U}_n \subset \mathrm{GL}_n(\mathbb{C})$) consisting of the *orthogonal matrices* (*unitary matrices*), that is, such that the matrix sends orthonormal bases to orthonormal bases with respect to the canonical scalar product in $\mathbb{R}^n$ (the canonical Hermitian product in $\mathbb{C}^n$) or, equivalently, such that its column vectors form an orthonormal basis, is called the *orthogonal group* (*unitary group*) of $n \times n$ matrices. In particular, $\mathrm{O}_1 = \mathbb{Z}_2$ and $\mathrm{U}_1 = \mathbb{S}^1$.

1.3 Categories

The spirit of algebraic topology consists in assigning to topological spaces certain algebraic objects, such as groups, modules, or algebras, and to continuous maps, corresponding algebraic homomorphisms in a sensitive way. This means, in more technical terms, that algebraic topology defines functors from topological categories to algebraic categories. This section explains briefly what categories and functors are, as well as some other related concepts.

1.3.1 Definition. A *category* $\mathfrak{C}$ consists of a class of objects, denoted by $\mathrm{ob}\,\mathfrak{C}$, and for any $A, B \in \mathrm{ob}\,\mathfrak{C}$, a set of *morphisms* $\mathfrak{C}(A,B)$. An element $f \in \mathfrak{C}(A,B)$ is usually denoted by $f : A \longrightarrow B$. They are such that the following hold:

(i) If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are morphisms, there is a *composite* $g \circ f : A \longrightarrow C$. In other words, there is a function $\circ : \mathfrak{C}(A,B) \times \mathfrak{C}(B,C) \longrightarrow \mathfrak{C}(A,C)$. It is associative in the sense that if $h : C \longrightarrow D$, then $(f \circ g) \circ h = f \circ (g \circ h)$

(ii) For all $C \in \mathrm{ob}\,\mathfrak{C}$, there is a unique morphism $\mathrm{id}_C \in \mathfrak{C}(C,C)$, called the *identity* of C, such that if $f : A \longrightarrow B$, then $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.

If a morphism $f : A \longrightarrow B$ is such that there is another morphism $g : B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$, then we say that f is an *isomorphism* with inverse g, which is also an isomorphism.

If $f : A \longrightarrow B$, $g : B \longrightarrow C$, $f' : A \longrightarrow B'$, and $g' : B' \longrightarrow C$ are morphisms such that $g \circ f = g' \circ f'$, we say that the diagram (or square)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f'"'] & B \arrow[d, "g"] \\ B' \arrow[r, "g'"'] & C \end{tikzcd}$$

commutes (or is *commutative*).

1.3.2 DEFINITION. Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. Under a *functor* $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ we understand an assignment $F : \operatorname{ob}\mathfrak{C} \longrightarrow \operatorname{ob}\mathfrak{D}$, $A \mapsto F(A)$, and a function $F : \mathfrak{C}(A,B) \longrightarrow \mathfrak{D}(F(A),F(B))$ such that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$.

A *subcategory* $\mathfrak{C}'$ of a category $\mathfrak{C}$ consists of a subclass $\operatorname{ob}\mathfrak{C}' \subseteq \operatorname{ob}\mathfrak{C}$ such that for each pair of objects $A,B \in \operatorname{ob}\mathfrak{C}'$ the set $\mathfrak{C}'(A,B)$ is a subset of $\mathfrak{C}(A,B)$. If these two sets are equal for all $A,B \in \operatorname{ob}\mathfrak{C}'$, then we say that the subcategory is *full.* If $\mathfrak{C}'$ is a subcategory of $\mathfrak{C}$, then one has an *inclusion functor*, denoted by $i : \mathfrak{C}' \longrightarrow \mathfrak{C}$, such that for any object $A \in \operatorname{ob}\mathfrak{C}'$, $i(A) = A$ and for any morphism $f : A \longrightarrow B$, $i(f) = f$.

A *natural transformation* τ between two functors $F, G : \mathfrak{C} \longrightarrow \mathfrak{D}$ consists of a morphism $\tau_A : F(A) \longrightarrow G(A)$ for each $A \in \operatorname{ob}\mathfrak{C}$ such that for every morphism $f : A \longrightarrow B$ in $\mathfrak{C}$, one has $G(f) \circ \tau_A = \tau_B \circ F(f)$. We write this fact stating that the square

$$\begin{tikzcd} A \arrow[d, "f"'] & F(A) \arrow[r, "\tau_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ B & F(B) \arrow[r, "\tau_B"'] & G(B) \end{tikzcd}$$

commutes. We say that τ is a *natural isomorphism* if for each $A \in \operatorname{ob}\mathfrak{C}$ the morphism τ_A is an isomorphism.

1.3.3 DEFINITION. Given functors $F : \mathfrak{C} \longrightarrow \mathfrak{D}$, $G : \mathfrak{D} \longrightarrow \mathfrak{C}$, we say that F is *left-adjoint* to G and that G is *right-adjoint* to F if there is a natural isomorphism

$$\Phi_{C,D} : \mathfrak{D}(F(C),D) \longrightarrow \mathfrak{C}(C,G(D))\,,$$

that is, a bijection such that for any objects $C, C' \in \operatorname{ob}\mathfrak{C}$ and $D, D' \in \mathfrak{D}$ and morphisms $f : C \longrightarrow C'$ and $g : D \longrightarrow D'$, one has commutative squares

$$\begin{tikzcd} \mathfrak{D}(F(C'),D) \arrow[r, "\Phi_{C',D}"] \arrow[d, "F(f)^*"'] & \mathfrak{C}(C',G(D)) \arrow[d, "f^*"] \\ \mathfrak{D}(F(C),D) \arrow[r, "\Phi_{C,D}"'] & \mathfrak{C}(C,G(D)) \end{tikzcd}\;, \qquad \begin{tikzcd} \mathfrak{D}(F(C),D) \arrow[r, "\Phi_{C,D}"] \arrow[d, "g_*"'] & \mathfrak{C}(C,G(D)) \arrow[d, "G(g)_*"] \\ \mathfrak{D}(F(C),D') \arrow[r, "\Phi_{C,D'}"'] & \mathfrak{C}(C,G(D')) \end{tikzcd}\;,$$

where $F(f)^*(\psi) = \psi \circ F(f) : F(C) \longrightarrow D$, $f^*(\varphi) = \varphi \circ f : C \longrightarrow G(D)$, $g_*(\xi) = g \circ \xi : F(C) \longrightarrow D'$, and $G(g)_*(\eta) = G(g) \circ \eta : C \longrightarrow G(D')$.

The following is a well-known result and will be used in the sequel.

1.3.4 **Proposition**. *Let $F : \mathfrak{C} \longrightarrow \mathfrak{D}$, $G : \mathfrak{D} \longrightarrow \mathfrak{C}$ be covariant functors. Then*

(a) *There is a one-to-one correspondence between natural transformations $\Phi_{C,D} : \mathfrak{D}(F(C), D) \longrightarrow \mathfrak{C}(C, G(D))$ and $\varphi : 1_{\mathfrak{C}} \longrightarrow G \circ F$.*

(b) *There is a one-to-one correspondence between natural transformations $\Psi_{C,D} : \mathfrak{C}(C, G(D)) \longrightarrow \mathfrak{D}(F(C), D)$ and $\psi : F \circ G \longrightarrow 1_{\mathfrak{D}}$.*

Proof: (a) Given Φ, define φ by $\varphi_C = \Phi_{C,F(C)}(\mathrm{id}_{F(C)}) : C \longrightarrow GF(C)$. Conversely, given φ, define Φ by $\Phi_{C,D}(g) = G(g) \circ \varphi_C : C \longrightarrow G(D)$.

(b) Given Ψ, define ψ by $\psi_D = \Psi_{G(D),D}(\mathrm{id}_{G(D)}) : FG(D) \longrightarrow D$. Conversely, given ψ, define Ψ by $\Psi_{C,D}(f) = \psi_D \circ F(f) : F(C) \longrightarrow D$. □

Given two functors $S, T : \mathcal{D} \longrightarrow \mathcal{C}$, we denote by $\mathcal{N}\mathrm{at}(S,T)$ the set of natural transformations from S to T. The following result is easy but very important.

1.3.5 **Lemma**. (Yoneda lemma) *Let $\mathcal{D}$ be a category and take a covariant functor $K : \mathcal{D} \longrightarrow \mathcal{S}et$. Then given any object X in $\mathcal{D}$, there is a bijection*

$$\varphi : \mathcal{N}\mathrm{at}(\mathcal{D}(X,-), K) \xrightarrow{\cong} K(X)\,,$$

which maps a natural transformation $\eta : \mathcal{D}(X,-) \longrightarrow K$ to $\eta_X(1_X)$.

Proof: Consider the following commutative diagram.

$$\begin{array}{ccc} X & \quad & \mathcal{D}(X,X) \xrightarrow{\eta_X} K(X) \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f_*}\big\downarrow \qquad\qquad \big\downarrow{\scriptstyle K(f)} \\ Y & & \mathcal{D}(X,Y) \xrightarrow[\eta_Y]{} K(Y)\,. \end{array}$$

Then, given an element $x \in K(X)$, define a natural transformation by $\eta_Y(f) = D(f)(x)$. This yields an inverse to φ,

$$\psi : K(X) \longrightarrow \mathcal{N}\mathrm{at}(\mathcal{D}(X,-), K)\,.$$

□

We shall use the following basic categories: Sets and functions, denoted by $\mathfrak{Set}$, topological spaces and continuous maps, denoted by $\mathfrak{Top}$, as well as several (full) subcategories of $\mathfrak{Top}$.

1.4 A convenient category of topological spaces

The category of all topological spaces $\mathfrak{Top}$ does not behave well with respect to identifications. For instance, if $p : X \longrightarrow X'$ is an identification and Y is any topological space, then the product $p \times \mathrm{id}_Y : X \times Y \longrightarrow X' \times Y$ need not be an identification. Kelley [30] solved this problem introducing the category of compactly generated Hausdorff spaces (see also [45] or [2]). This category, however, has another pathology. Namely, if X is a compactly generated Hausdorff space and $q : X \longrightarrow X'$ is an identification, then X' need not be Hausdorff and thus might be outside of the category. In this section we shall introduce another category, which was studied by R. Vogt [49] and has the desired properties, namely for spaces X and Y in the category and the product in the category we have:

- If $p : X \longrightarrow X'$ is an identification, then X' is in the category.
- If $p : X \longrightarrow X'$ is an identification, then the product $p \times \mathrm{id}_Y : X \times Y \longrightarrow X' \times Y$ is an identification.

1.4.1 Definition. Let X be any topological space. Define $k(X)$ as the space with the same underlying set as X and a finer topology given by

- $A \subseteq X$ is closed in $k(X)$ if and only if $\alpha^{-1}A \subseteq K$ is closed for any continuous map $\alpha : K \longrightarrow X$, where K is an arbitrary compact Hausdorff space. We then say that A is k-*closed*.

Clearly, if A is closed in X, then A is closed in $k(X)$. Thus the identity $\mathrm{id}_X^k : k(X) \longrightarrow X$ is always continuous. We say that X is a k-*space* if $X = k(X)$, i.e. if the closed sets in X are precisely the ones in $k(X)$. Denote by $\mathfrak{K}$-$\mathfrak{Top}$ the category of k-spaces and continuous map. We call the topology of $k(X)$ the k-*topology*.

1.4.2 Exercise. Show that indeed the k-closed sets in X constitute a topology.

1.4.3 **Proposition**. *The assignment $X \mapsto k(X)$ determines a covariant functor $k : \mathfrak{Top} \longrightarrow \mathfrak{K}$-$\mathfrak{Top}$.*

Proof: Let $f : X \longrightarrow Y$ be continuous and denote by $k(f) : k(X) \longrightarrow k(Y)$ the same function. To verify the continuity of $k(f)$, take a closed set $B \subset k(Y)$, namely such a set that $\beta^{-1}B \subset K$ is closed for any continuous map $\beta : K \longrightarrow Y$. Since the map $\beta = \mathrm{id}_Y^k \circ k(f) \circ \alpha = f \circ \alpha : K \longrightarrow Y$ is continuous, one has that $\alpha^{-1}(k(f)^{-1}B) = \beta^{-1}B \subseteq K$ is closed. Thus $k(f)^{-1}B \subseteq k(X)$ is closed. $\square$

1.4.4 Exercise. Show the following:

(a) If X is a k-space and $A \subseteq X$ is closed, then A as a subspace with the usual relative topology is again a k-space.

(b) If X is a k-space and $A \subseteq X$ is open, then A as a subspace with the usual relative topology is again a k-space.

(c) Conclude that if X is a k-space and $A \subseteq X$ is locally closed, i.e., the intersection of an open set and a closed set, then A as a subspace with the usual relative topology is again a k-space.

(d) Give an example of a k-space X and a subspace A which with the usual relative topology is not a k-space.

We have the following.

1.4.5 DEFINITION. If X is a k-space and $A \subseteq X$ is any subset, then we define the k-*relative topology* of A as the topology of $k(A)$. A with this topology is called k-*subspace* of X. For simplicity, whenever we talk about a subspace A of a k-space X, we mean the k-subspace, even though we shall not write $k(A)$.

1.4.6 EXERCISE. Let X be a k-space and $A \subseteq X$ be a k-subspace. Show that the map $i : A \hookrightarrow X$ has the following universal property:

(a) i is continuous.

(b) If Y is a k-space, then a map $f : Y \longrightarrow A$ is continuous if and only if the composite $i \circ f : Y \longrightarrow X$ is continuous.

The following result summarizes the main properties of the functor K.

1.4.7 **Proposition**.

(a) *The identity function* $\mathrm{id}_X^k : k(X) \longrightarrow X$ *is continuous.*

(b) *The topology in* $k(X)$ *is the finest such that any continuous map* $\alpha : K \longrightarrow X$, *$K$ compact and Hausdorff, factors through the identity* $\mathrm{id}_X^k : k(X) \longrightarrow X$.

(c) *For every compact Hausdorff space K there is a one-to-one correspondence between continuous maps* $K \longrightarrow X$ *and continuous maps* $K \longrightarrow k(X)$.

(d) *For every compact Hausdorff space* K, $k(K) = K$.

(e) *The functor k is idempotent, namely* $k(k(X)) = k(X)$ *for every topological space* X.

(f) *A map $f : X \longrightarrow Y$ is continuous if and only if $f \circ \alpha : K \longrightarrow Y$ is continuous for every continuous map $\alpha : K \longrightarrow X$, where K is an arbitrary compact Hausdorff space.*

Proof: Property (a) is obvious and property (b) follows by definition. Property (c) follows immediately from (b). To show property (d), take $B \subseteq k(K)$ closed. Hence, by definition, $\alpha^{-1}B \subset L$ is closed for any continuous map $\alpha : L \longrightarrow K$, L compact Hausdorff. Hence, if we take $L = K$ and $\alpha = \mathrm{id} : K$, then $B = \alpha^{-1}B$ is closed and so $\mathrm{id}_K^k : k(K) \longrightarrow K$ is a homeomorphism. Property (e) follows from the definition of $k(X)$. To prove property (f) assume first that f is continuous. Then $f \circ \alpha$ is continuous for every continuous map $\alpha : K \longrightarrow X$, where K is an arbitrary compact Hausdorff space. Conversely assume that $f \circ \alpha$ is continuous for every continuous map $\alpha : K \longrightarrow X$, where K is an arbitrary compact Hausdorff space. To see that f is continuous, let $B \subseteq Y$ be closed. Since $f \circ \alpha$ is continuous, then $\alpha^{-1}f^{-1}B \subseteq K$ is closed. Hence by definition $f^{-1}B \subseteq X$ is closed and thus f is continuous. □

We have the following categorical result about the category $\mathfrak{K}$-$\mathfrak{Top}$ and the functor k.

1.4.8 **Corollary**. *The inclusion functor $i : \mathfrak{K}\text{-}\mathfrak{Top} \longrightarrow \mathfrak{Top}$ is left-adjoint to the functor $k : \mathfrak{Top} \longrightarrow \mathfrak{K}\text{-}\mathfrak{Top}$, namely the following equality of sets holds:*

$$\mathfrak{K}\text{-}\mathfrak{Top}(X, k(Y)) = \mathfrak{Top}(i(X), Y)\,,$$

where X is a k*-space and Y is an arbitrary topological space.*

Proof: If $f : X \longrightarrow k(Y)$ is continuous, then $f = \mathrm{id}_Y^k \circ f : i(X) \longrightarrow Y$ is continuous. Furthermore, if $g : i(X) \longrightarrow Y$ is continuous, then by 1.4.7 (e) $k(g) : X = ki(X) \longrightarrow k(Y)$ is continuous. Thus both sets $\mathfrak{K}\text{-}\mathfrak{Top}(X, k(Y))$ and $\mathfrak{Top}(i(X), Y)$ have exactly the same functions. □

1.4.9 EXAMPLES. The following are examples of k-spaces.

1. Compact Hausdorff spaces. Namely assume that C is a compact Hausdorff space and $A \subset C$ is such that for any continuous map $\alpha : K \longrightarrow C$ the inverse image $\alpha^{-1}(A) \subseteq K$ is closed. Then in particular the identity map $\mathrm{id}_C : C \longrightarrow C$ is continuous and thus $A = \mathrm{id}_C^{-1}(A) \subseteq C$ must be closed.

2. More generally, locally compact Hausdorff spaces. Namely assume that X is a locally compact Hausdorff space and $B \subset X$ is nonclosed. Hence there is a point $x \in \overline{B}$ (the closure of B) which is not in B. Hence there is a compact (Hausdorff) neighborhood K of x in X such that $x \notin K \cap B$, although $x \in \overline{K \cap B}$. To see this, take another neighborhhod V of x in X. Hence $V \cap K$

is a neighborhhod of x in X and consequently $(V \cap K) \cap B = V \cap (K \cap B) \neq \emptyset$. Thus, if $i : K \hookrightarrow X$ is the (continuous) inclusion, then $i^{-1}(B) = K \cap B$ is nonclosed in K. Consequently B is nonclosed in the k-topology.

3. Or even more generally, compactly generated Hausdorff spaces in the sense of Kelley [30] and Steenrod [45]. Namely assume that X is a compactly generated Hausdorff space and $A \subseteq X$ is such that for any continuous map $\alpha : K \longrightarrow C$ the inverse image $\alpha^{-1}(A) \subseteq K$ is closed. Then in particular, if $C \subset X$ is compact, the inclusion map $i : C \hookrightarrow X$ is continuous. Thus $A \cap C = i^{-1}A \subset C$ is closed and hence C is closed in the compactly generated topology of X.

4. First-countable Hausdorff spaces. Namely assume that X is a first-countable Hausdorff space and $B \subset X$ is nonclosed. Hence there is a point $x \in \overline{B}$ (the closure of B) which is not in B. Take a nested countable neighborhood basis $\cdots U_{n-1} \subset U_n \subset U_{n-1} \subset \cdots$ around x and for each n take $x_n \in U_n \cap B$. Then $x_n \to x$ and so the set $K = \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ is compact (Hausdorff). Thus, if $i : K \hookrightarrow X$ is the (continuous) inclusion, then $i^{-1}(B) = B \cap K = \{x_n \mid n \in \mathbb{N}\}$ is nonclosed in K. Consequently B is nonclosed in the k-topology.

5. CW-complexes. Namely assume that X is a CW-complex and $A \subset X$ is such that for any continuous map $\alpha : K \longrightarrow X$ the inverse image $\alpha^{-1}(A) \subseteq K$ is closed. Then, in particular, for any characteristic map $\varphi_i : \mathbb{D}^n_i \longrightarrow X$, the inverse image $\varphi_i^{-1}(A) \subset C$ is closed. By definition of the topology of X, A must be closed in X (see 1.6.3 below).

The following results give the properties of the category $\mathfrak{K}$-$\mathfrak{Top}$ which will be of interest in what follows.

1.4.10 **Theorem**. *Let X be a* k*-space and assume that $q : X \longrightarrow Y$ is an identification. Then Y is a* k*-space. Consequently, the category $\mathfrak{K}$-$\mathfrak{Top}$ is closed under identifications.*

Proof: Assume that $B \subseteq Y$ is such that for any continuous map $\beta : K \longrightarrow Y$, with K a compact Hausdorff space, one has that the inverse image $\beta^{-1}B \subset K$ is closed. To prove that $B \subseteq Y$ is closed, we must show that $q^{-1}B \subseteq X$ is closed. Since X is a k-space, it is enough to see that for every continuous map $\alpha : K \longrightarrow X$, with K a compact Hausdorff space, the inverse image $\alpha^{-1}(q^{-1}B) \subseteq K$ is closed. This is indeed the fact, since $\alpha^{-1}(q^{-1}B) = (q \circ \alpha)^{-1}B = \beta^{-1}B$, for $\beta = q \circ \alpha : K \longrightarrow Y$. $\square$

1.4.11 **Proposition**. *Assume that $X = \bigcup_{n=0}^{\infty} X^n$ has the union topology (i.e. $B \subset X$ is closed if and only if $B \cap X^n$ is closed in X^n), where X^n is a* k*-space. Then X is a* k*-space.*

Proof: Assume that $B \subset X$ is such that for any continuous map $\alpha : K \longrightarrow X$ with K compact Hausdorff, the inverse image $\alpha^{-1}(B) \subset K$ is closed in K. We have to prove that B is closed in X, namely that $B \cap X^n$ is closed in X^n. Since X^n is by assumption a k-space, we must show that for any continuous map $\beta : K \longrightarrow X^n$ with K compact Hausdorff, the inverse image $\beta^{-1}(B \cap X^n)$ is closed in K. But $\beta^{-1}(B \cap X^n) = \alpha^{-1}(B)$ is closed, since $\alpha = i \circ \beta : K \longrightarrow X^n \hookrightarrow X$ is clearly continuous. □

1.4.12 EXERCISE. Give an alternative proof of 1.4.11 by showing the following:

(a) If $C \subset X$ is compact, then there exists n such that $C \subset X^n$. (*Hint:* Otherwise one can take a sequence of points $x_n \in C \cap (X^n - X^{n-1})$ for all n which has no cluster point.)

(b) If K is a compact Hausdorff space and $\alpha : K \longrightarrow X$ is continuous, then α factors through the inclusion $X^n \hookrightarrow X$ for some n.

Now use that X^n is a k-space.

Given two k-spaces X and Y, we consider their (usual) topological product $X \times_{\text{top}} Y$. There are examples that show that this product need not be a k-space (see [14]). We have the next.

1.4.13 DEFINITION. Given two k-spaces X and Y, we define their k-*product* by

$$X \times Y = k(X \times_{\text{top}} Y)\,.$$

Thus the product of k-spaces is again a k-space.

We must check that this is a product in the category $\mathfrak{K}$-$\mathfrak{Top}$, in other words, that it has the universal property. For infinite products, we do the same.

1.4.14 **Theorem**. *Let X, Y, and Z be* k-*spaces.*

(a) The projections $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are continuous.

(b) If $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ are continuous maps, then the induced map $(f, g) : Z \longrightarrow X \times Y$ is continuous.

Proof: (a) follows from the classical case applying functor k, since by 1.4.7 (e), $k(X) = X$ and $k(Y) = Y$. To prove (b) notice that by the universal property of the topological product, the map $(f, g) : Z \longrightarrow X \times_{\text{top}} Y$ is continuous. Applying functor k we have a continuous map $(f, g) : k(Z) \longrightarrow k(X \times_{\text{top}} Y)$. But again $k(Z) = Z$, hence the result. □

1.4.15 **Proposition**. *Let C be a compact Hausdorff space and Y an arbitrary space. Then the following hold:*

(a) $C \times Y = C \times_{\mathrm{top}} k(Y)$.

(b) *If Y is a* k-*space, then $C \times Y = C \times_{\mathrm{top}} Y$.*

Proof: Since (b) is clearly a consequence of (a), we just prove (a). To that end, let us take a k-closed set $B \subseteq C \times_{\mathrm{top}} Y$, namely such that for any continuous map $\alpha : K \longrightarrow C \times_{\mathrm{top}} k(Y)$, K a compact Hausdorff space, the inverse image $\alpha^{-1}(B) \subseteq K$ is closed. We must prove that B is closed in $C \times_{\mathrm{top}} k(Y)$.

Take $(x,y) \in C \times_{\mathrm{top}} k(Y) - B$. The inclusion $C \times \{y\} \hookrightarrow C \times_{\mathrm{top}} k(Y)$ is a continuous map from a compact Hausdorff space, hence $B \cap (C \times \{y\})$ is closed (compact) and thus there is a neighborhood U of x in C such that $(\overline{U} \times \{y\}) \cap B = \emptyset$. Let $A \subseteq Y$ be the image under the projection of $(\overline{U} \times_{\mathrm{top}} Y) \cap B \subseteq C \times_{\mathrm{top}} Y$ and let $\beta : K \longrightarrow Y$ be a continuous map from a compact Hausdorff space. The map $i \times \beta : \overline{U} \times_{\mathrm{top}} K \longrightarrow C \times_{\mathrm{top}} Y$, where i is the inclusion, is continuous with a compact Hausdorff domain. Hence by assumption, $(i \times \beta)^{-1}(B) \subset \overline{U} \times_{\mathrm{top}} K$ is closed and thus also $\beta^{-1}A \subset K$ is closed. Therefore, A is k-closed in Y. Since $y \notin A$, we have that $U \times_{\mathrm{top}} (Y - A)$ is a neighborhood of (x,y) in $C \times_{\mathrm{top}} Y$ such that $(U \times_{\mathrm{top}} (Y - A)) \cap B = \emptyset$. Consequently $B \subseteq C \times_{\mathrm{top}} k(Y)$ is closed. □

In order to prove the second property of the category $\mathfrak{K}$-$\mathfrak{Top}$, namely that the product of identifications is an identification, we have to develop some theory. First recall that given two topological spaces X and Y one can endow $\mathfrak{Top}(X,Y)$ with the *compact-open topology* defined as follows. Let $C \subseteq X$ be any compact set and let $U \subseteq Y$ be any open set. Then the topology has as subbasis the sets $(C,U) = \{f \in \mathfrak{Top}(X,Y) \mid f(C) \subset U\}$. Denote the resulting topological space by $\mathfrak{Top}_{\mathfrak{co}}(X,Y)$. This *map space* need not be a k-space, even if X and Y are k-spaces. Therefore we have the next.

1.4.16 Definition. Let X and Y be k-spaces. Define their k-*map space* by

$$M(X,Y) = k(\mathfrak{Top}_{\mathfrak{co}}(X,Y)) .$$

Thus the k-map space of k-spaces is again a k-space. If we are dealing with pointed spaces, denote by $M_*(X,Y)$ the k-subspace of $M(X,Y)$ of pointed maps.

We shall need the following result.

1.4.17 **Proposition**. *The mapping $f \mapsto \widehat{f}$, where $\widehat{f}(x)(y) = f(x,y)$, yields a natural bijection*

$$\Theta : \mathfrak{K}\text{-}\mathfrak{Top}(X \times Y, Z) \longrightarrow \mathfrak{K}\text{-}\mathfrak{Top}(X, M(Y,Z)) .$$

Before we proceed to prove Proposition 1.4.17, we prove the following lemma.

1.4.18 **Lemma**. *If K is a compact Hausdorff space and Y is any topological space, then the* evaluation map $e_{K,Y} : \mathfrak{Top}_{\mathfrak{co}}(K,Y) \times_{\mathrm{top}} K \longrightarrow Y$ *given by* $e_{K,Y}(f,x) = f(x)$ *is continuous.*

Proof: This is a special case of the more general result where K is a locally compact Hausdorff space (see Proposition 1.3.1 in [2]). □

The evaluation map has the following universal property which will be used below.

1.4.19 **Proposition**. *Assume that the evaluation map* $e_{Y,Z} : \mathfrak{Top}_{\mathfrak{co}}(Y,Z) \times Y \longrightarrow Z$ *is continuous and consider a commutative diagram*

$$\begin{tikzcd} X \times_{\mathrm{top}} Y \arrow[rr, "f"] \arrow[dr, "\widehat{f}\times \mathrm{id}_Y"'] & & Z \\ & \mathfrak{Top}_{\mathfrak{co}}(Y,Z) \times_{\mathrm{top}} Y . \arrow[ur, "e_{Y,Z}"'] & \end{tikzcd}$$

Then f is continuous if and only if $\widehat{f}$ is continuous. □

1.4.20 **Lemma**. *Given a continuous map $f : X \times Y \longrightarrow Z$, where X and Y are* k*-spaces, the adjoint map $\widehat{f} : X \longrightarrow M(Y,Z)$ given by $\widehat{f}(x)(y) = f(x,y)$ is also continuous.*

Proof: Since X is a k-space, it is enough to prove that the composite $K \xrightarrow{\alpha} X \xrightarrow{\widehat{f}} M(Y,Z)$ is continuous, where K is a compact Hausdorff space and α is continuous. First notice that the commutative diagram

$$\begin{tikzcd} K \times Y \arrow[r, "\alpha\times\mathrm{id}_Y"] & X \times Y \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & Z \\ K \times_{\mathrm{top}} Y \arrow[u, equal] \arrow[r, "\alpha\times\mathrm{id}_Y"'] & X \times_{\mathrm{top}} Y \arrow[ur, "f"'] & \end{tikzcd}$$

shows that the composite $f \circ (\alpha \times \mathrm{id}_Y) : K \times_{\mathrm{top}} Y \longrightarrow Z$ is continuous. Hence it has a continuous adjoint $f \circ \widehat{(\alpha \times \mathrm{id}_Y)} : K \longrightarrow \mathfrak{Top}_{\mathfrak{co}}(Y,Z)$. The following is a commutative diagram:

$$\begin{tikzcd} K \arrow[r, "f\circ\widehat{(\alpha\times\mathrm{id}_Y)}"] \arrow[d, "\alpha"'] & \mathfrak{Top}_{\mathfrak{co}}(Y,Z) \\ X \arrow[r, "\widehat{f}"'] & M(Y,Z) . \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

Applying Proposition 1.4.7, the continuity of $\widehat{f}$ follows. □

1.4.21 **Lemma**. *If Y is a* k*-space, then the evaluation map $e_{Y,Z} : M(Y,Z)\times Y \longrightarrow Z$ is continuous.*

Proof: It is enough to prove that for an arbitrary continuous map $\alpha : K \longrightarrow M(Y,Z)\times Y$, where K is compact Hausdorff, the composite $e_{Y,Z}\circ\alpha$ is continuous. Let $\alpha = (\alpha_1, \alpha_2)$ and consider the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "\alpha"] \arrow[d, "\delta"'] & M(Y,Z)\times Y \arrow[rr, "e_{Y,Z}"] & & Z \\
K\times K \arrow[r, "\alpha_1\times\mathrm{id}"'] & M(Y,Z)\times K \arrow[u, "\mathrm{id}\times\alpha_2"'] \arrow[r, "\alpha_2^*\times\mathrm{id}"'] & M(K,Z)\times K \arrow[r, "\mathrm{id}"'] & \mathfrak{Top}_{\mathfrak{co}}(Y,Z)\times_{\mathrm{top}} Y \arrow[u, "e_{Y,Z}"']
\end{tikzcd},$$

where δ is the diagonal map. Since the composite $e_{Y,Z}\circ\mathrm{id}\circ(\alpha_2^*\times\mathrm{id})\circ(\alpha_1\times\mathrm{id})\circ\delta$ is continuous, the map $e_{Y,Z}\circ\alpha$ on the top is continuous as desired. □

Since the evaluation map $e_{Y,Z} : M(Y,Z)\times Y \longrightarrow Z$ is continuous if Y and Z are k-spaces, we may now restate Proposition 1.4.19 to obtain the universal property of the evaluation maps for the category $\mathfrak{K}$-$\mathfrak{Top}$.

1.4.22 **Proposition**. *Consider a commutative diagram*

$$\begin{tikzcd}
X\times Y \arrow[rr, "f"] \arrow[dr, "\widehat{f}\times\mathrm{id}_Y"'] & & Z \\
& M(Y,Z)\times Y \arrow[ur, "e_{Y,Z}"'] &
\end{tikzcd}.$$

Then f is continuous if and only if $\widehat{f}$ is continuous. □

We are now ready for the *proof* of Proposition 1.4.17. Namely, take first $f \in \mathfrak{K}\text{-}\mathfrak{Top}(X\times Y, Z)$, that is, a continuous map $f : X\times Y \longrightarrow Z$. Then $\Theta(f) = \widehat{f} : X \longrightarrow M(Y,Z)$ is continuous by Lemma 1.4.20. Conversely, if $\widehat{f} : X \longrightarrow M(Y,Z)$ is continuous, then

$$f :\ X\times Y \xrightarrow{\widehat{f}\times\mathrm{id}_Y} M(Y,Z)\times Y \xrightarrow{e_{Y,Z}} Z$$

is continuous, since by Lemma 1.4.21, $e_{Y,Z}$ is continuous. □

Before passing to the proof, we state our desired result.

1.4.23 **Theorem**. *Assume that X and Z are* k*-spaces and that $p : X \longrightarrow Y$ is an identification (thus Y is a* k*-space too). Then $p\times\mathrm{id}_Z : X\times Z \longrightarrow Y\times Z$ is an identification.*

Proof: Since p is surjective, $p \times \mathrm{id}_Z$ is surjective too. Assume that $g : Y \times Z \longrightarrow W$ is such that the composite $g \circ (p \times \mathrm{id}_Z)$ is continuous. To prove that $p \times \mathrm{id}_Z$ is an identification, we must show that from the assumption it follows that g is continuous.

Consider the bijections, as given in Proposition 1.4.17,

$$\Theta : \mathfrak{K}\text{-}\mathfrak{Top}(X \times Z, W) \longrightarrow \mathfrak{K}\text{-}\mathfrak{Top}(X, M(Z, W)),$$

$$\Theta' : \mathfrak{K}\text{-}\mathfrak{Top}(Y \times Z, W) \longrightarrow \mathfrak{K}\text{-}\mathfrak{Top}(Y, M(Z, W)).$$

Since $g \circ (p \times \mathrm{id}_Z) : X \times Z \longrightarrow W$ is continuous, so is its image $\Theta(g \circ (p \times \mathrm{id}_Z)) : X \longrightarrow M(Z, W)$, which is given by

$$\Theta(g \circ (p \times \mathrm{id}_Z))(x)(z) = g(p \times \mathrm{id}_Z)(x, z) = g(p(x), z).$$

Furthermore we have the following commutative triangle

$$\begin{tikzcd} X \arrow[rr, "\Theta(g\circ(p\times\mathrm{id}_Z))"] \arrow[dr, "p"'] & & M(Z, W) \\ & Y . \arrow[ur, "\Theta'(g)"'] & \end{tikzcd}$$

Indeed, $\Theta'(g)(y)(z) = g(y, z)$, so that if $x \in X$, then $\Theta'(g)p(x)(z) = g(p(x), z) = \Theta(g \circ (p \times \mathrm{id}_Z))(x)(z)$.

But by hypothesis, p is an identification. Thus, since $\Theta'(g)p$ is continuous and p is an identification, so is $\Theta'(g)$ continuous. On the other hand, since Θ' is bijective, it follows that $g = \Theta'^{-1}\Theta'(g)$ is continuous. $\square$

Since the composite of identifications is an identification, we have the following.

1.4.24 **Corollary**. *Assume that X, Y, W, and Z are* K*-spaces and that $p : X \longrightarrow Y$ and $q : W \longrightarrow Z$ are identifications. Then $p \times q : X \times W \longrightarrow Y \times Z$ is an identification.*

Proof: Just observe that $p \times q = (\mathrm{id}_Y \times q) \circ (p \times \mathrm{id}_W)$ and that by the previous theorem, $\mathrm{id}_Y \times q$ and $p \times \mathrm{id}_W$ are identifications. $\square$

To finish this section, we prove that Θ induces isomorphisms in $\mathfrak{K}\text{-}\mathfrak{Top}$.

1.4.25 **Theorem**. *Let X and Y be* k*-spaces. Then*

$$\Theta : M(X, M(Y, Z)) \longrightarrow M(X \times Y, Z)$$

is a natural homeomorphism.

Proof: Consider the diagram

$$\begin{tikzcd}
M(X,M(Y,Z)) \times X \times Y \arrow[r, "e_1\times\mathrm{id}"] \arrow[d, "\Theta\times\mathrm{id}\times\mathrm{id}"'] & M(Y,Z) \times Y \arrow[d, "e_2"] \\
M(X\times Y,Z) \times X \times Y \arrow[ur, "\widehat{e_3}\times\mathrm{id}"] \arrow[r, "e_3"'] & Z\,,
\end{tikzcd}$$

where $e_1 = e_{X,M(Y,Z)}$, $e_2 = e_{Y,Z}$, and $e_3 = e_{X\times Y,Z}$. Since Θ makes the diagram commute, it is continuous by the universal property 1.4.22 of e_3. The triangle on the bottom commutes by definition of $\widehat{e_3}$. Thus $\widehat{e_3}\circ(\Theta\times\mathrm{id}) = e_1$ due to the universal property of e_2. By the universal property of e_1, there is a unique map

$$\Psi : M(X\times Y,Z) \longrightarrow M(X,M(Y,Z))$$

such that $e_1\circ(\Psi\times\mathrm{id}) = \widehat{e_3}$. On the other hand,

$$e_3\circ((\Theta\circ\Psi)\times\mathrm{id}\times\mathrm{id}) = e_2\circ(e_1\times\mathrm{id})\circ(\Psi\times\mathrm{id}\times\mathrm{id}) = e_2\circ(\widehat{e_3}\times\mathrm{id}) = e_3\,,$$

$$e_1\circ((\Psi\circ\Theta)\times\mathrm{id}) = \widehat{e_3}\circ(\Theta\times\mathrm{id}) = e_1\,.$$

Hence, by the universal properties of e_3 and e_1, we have $\Theta\circ\Psi = \mathrm{id}$ and $\Psi\circ\Theta = \mathrm{id}$. Thus Ψ and Θ are inverse homeomorphisms. □

1.4.1 The pointed category of k-spaces

Pointed spaces will play a central role in this book. Recall that a *pointed space* is a k-space X together with a *base point* $x_0 \in X$. There are several constructions we shall be interested in. Many of them rest upon the unit interval $I = \{t\in\mathbb{R} \mid 0 \le t \le 1\}$ considered as a pointed space with 0 as the base point. Whenever we take quotient spaces of the form X/A, we take as base point of them the point onto which A collapses. If $A = \emptyset$, then we put $X/A = X^+ = X \sqcup \{*\}$ taking the isolated point as base point. In what follows, given two pointed spaces X and Y, we shall denote by $M_*(X,Y)$ the map space of pointed maps, which is a subspace of the map space $M(X,Y)$ as defined above.

1.4.26 DEFINITION. Let X and Y be pointed spaces.

(i) We define the *wedge sum* or simply the *wedge* $X\vee Y$ of the spaces X and Y as the subspace $X\times\{y_0\}\cup\{x_0\}\times Y \subset X\times Y$ and we define their *smash product* $X\wedge Y$ as the quotient space $X\times Y/X\vee Y$. We denote the class of the pair (x,y) in $X\wedge Y$ by $x\wedge y$. Notice that the 0-sphere $\mathbb{S}^0 = \{0,1\}$ acts as a neutral for the smash product, namely $X\wedge\mathbb{S}^0 \approx X$.

(ii) A special role will be played by the *cone* of X, $CX = X\wedge I$, and the *suspension* of X, $\Sigma X = X\wedge\mathbb{S}^1$, where by definition $\mathbb{S}^1 = I/\mathbb{S}^0$. There is an embedding $X \hookrightarrow CX$ given by $x\mapsto x\wedge 1$. It is an easy exercise to show that $CX/X \approx \Sigma X$.

(iii) We shall also use the *path space* PX defined as the set of pointed maps $M_*(I,X) = \{\sigma \in M(I,X) \mid \sigma(0) = x_0\}$. We consider also the *loop space* ΩX defined as the set of pointed maps $M_*(\mathbb{S}^1, X) = \{\lambda \in M(I,X) \mid \lambda(0) = x_0 = \lambda(1)\}$.

1.4.27 Exercise. Show that for all $m,n \geq 0$ there is a homeomorphism $\mathbb{S}^m \wedge \mathbb{S}^n \approx \mathbb{S}^{m+n}$. (*Hint:* Use the facts that a sphere $\mathbb{S}^k$ is the one-point compactification of $\mathbb{R}^k$ and that the one-point compactification of a product of spaces is the smash product of the one-point compactification of each of the spaces.)

Recall the homeomorphism $M(X \times Y, Z) \approx M(X, M(Y,Z))$. If we consider the subspace of $M(X \times Y, Z)$ of those maps which send $X \vee Y$ to the base point $z_0 \in Z$, this space corresponds via the homeomorphism to the subspace $M_*(X, M_*(Y,Z)$ of $M(X, M(Y,Z)$. Since maps which send $X \vee Y$ into a point are in one-to-one correspondence with pointed maps of the smash product, we obtain the following pointed version of 1.4.25.

1.4.28 **Theorem**. *Let X, Y, and Z be pointed* k*-spaces. Then there is a natural homeomorphism*

$$\Theta_* : M_*(X, M_*(Y,Z)) \longrightarrow M_*(X \wedge Y, Z)\,.$$

□

This means in particular that the functor $\mathfrak{K}\text{-}\mathfrak{Top} \longrightarrow \mathfrak{K}\text{-}\mathfrak{Top}$ given by $X \mapsto X \wedge Y$ is left adjoint the functor $Z \mapsto M_*(Y,Z)$. A special case is obtained taking $Y = \mathbb{S}^1$. Then we obtain the next.

1.4.29 **Theorem**. *Let X and Z be pointed* k*-spaces. Then there is a natural homeomorphism*

$$\widetilde{\Theta} : M_*(X, \Omega Z) \longrightarrow M_*(\Sigma X, Z)\,.$$

□

We finish the section with a basic definition of a general character.

1.4.30 Definition. Under a *topological (abelian) group* we shall understand an abelian group F such that F is also a k-space and the map

$$F \times F \longrightarrow F\,, \quad (u,v) \longmapsto u - v$$

is continuous. Here the product is, as always, the product in $\mathfrak{K}\text{-}\mathfrak{Top}$.

1.4.31 Exercise. Show that the continuity condition imposed in the definition of a topological (abelian) group is equivalent to the continuity of the sum, namely of the map $F \times F \longrightarrow F$, $(u,v) \mapsto u+v$, and the continuity of the pass to inverse, namely of the map $F \longrightarrow F$, $u \mapsto -u$

1.5 Simplicial complexes

In this section we shall speak about simplicial complexes, also called triangulated spaces or polyhedra. There are two closely related concepts: that of an abstract simplicial set, which has combinatorial character, and that of a geometric simplicial complex which is already a topological space.

1.5.1 Definition. An *abstract simplicial complex* C consists of a set of *vertices* $V(C)$ and for each integer $k \geq 0$ a set C^k consisting of subsets of $V(C)$ of cardinality $k+1$, which satisfy the following conditions

(a) For each vertex $v \in V(C)$, the singleton $\{v\} \in C^0$.

(b) Each subset of a set in C^k with $j+1$ elements of them must lie in C^j.

In other words, $V(C)$ is an arbitrary set of points v called *vertices*. Each C^k consists of subsets $\sigma = \{v_0, v_1, \ldots, v_k\}$ of V which satisfy that if $\sigma' \subseteq \sigma$ has cardinality $j+1$, say $\sigma' = \{v_{i_0}, v_{i_1}, \ldots, v_{i_j}\}$, then σ' lies in C^j. Formally, $C = C^0 \cup C^1 \cup \cdots \cup C^k \cup \cdots$. A simplicial complex C is *finite dimensional* of dimension n if $C^k = \emptyset$ for all $k > n$. An element σ of C^k is called a *simplex of dimension* k, or k*-simplex* of C. Furthermore, a nonempty subset σ' of a k-simplex σ is called a *face* of σ. The set C^k is called the k*-skeleton* of C. A simplicial complex D such that for each k, $D^k \subset C^k$ is called a *simplicial subcomplex*.

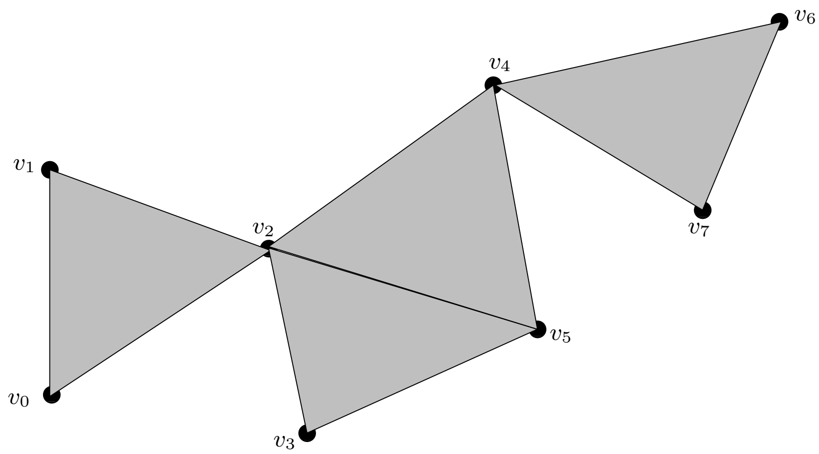

Figure 1.1 A simplicial complex

It is convenient to consider *ordered* simplicial complexes, namely simplicial complexes C such that the set of vertices C^0 is partially ordered, and each simplex σ with the induced order is totally ordered. In this case we shall write $[v_{i_0}, v_{i_1}, \ldots, v_{i_k}]$ for a k-simplex if and only if $v_{i_j} < v_{i_l}$ if and only if $j < l$. Notice that a simplex $\sigma \in C$ can be considered as a simplicial complex (a subcomplex of C).

Hence, given a partially ordered set V, then the set $C(V, \leq)^k$ of totally ordered subsets $\sigma \subset V$ with $k+1$ elements is the k-skeleton of an ordered simplicial complex $C(V, \leq)$.

1.5.2 Examples. The following are important examples of simplicial complexes.

1. For each $n \in \mathbb{N}$, define a simplicial complex D_n as follows. Consider $V(D_n) = \{0, 1, \ldots, n\}$, $n \geq 0$, with the obvious order. Then D_n^k consists of all sets of the form $\{i_0, i_1, \ldots, i_k\}$ such that $i_0 < i_1 < \cdots < i_k$.

2. Let $\{U_i \mid i \in \mathcal{I}\}$ be a family of subsets of a set X and consider the set
$$C(\mathcal{I})^k = \{\{i_1, \ldots, i_k\} \subset \mathcal{I} \mid i_m \neq i_n \text{ if } m \neq n \text{ and } U_{i_1} \cap \cdots \cap U_{i_k} \neq \emptyset,\ k \in \mathbb{N}\}\,.$$
Then these sets constitute a simplicial complex $C(\mathcal{I})$. If the family $\{U_i \mid i \in \mathcal{I}\}$ is a cover of a topological space X, then the simplicial complex $C(\mathcal{I})$ is called the *nerve* of the given cover.

3. Given a simplicial complex C there is an associated ordered simplicial complex $\operatorname{sd} C$, called the *barycentric subdivision* whose partially ordered set of vertices $V(\operatorname{sd} C)$ consists of all simplexes of C with the partial order relation given by
$$\sigma \leq \tau \quad \text{if and only if} \quad \sigma \subset \tau\,.$$

1.5.3 Definition. Given a simplicial complex C , we define its *geometric realization* as the set
$$|C| = \left\{ \alpha : V(C) \longrightarrow I \mid \alpha^{-1}(0,1] \in C \ \text{ and } \sum_{v \in V(C)} \alpha(v) = 1 \right\}.$$
Notice that $\alpha^{-1}(0,1]$ is finite. The metric topology of $|C|$ is given by the metric defined by $\mu_C(\alpha, \beta) = \sqrt{\sum_{v \in V(C)} (\alpha(v) - \beta(v))^2}$ (where the sum is finite). We denote this metric space by $|C|_{\mathrm{metric}}$. By restriction of the metric to the geometric realization $|\sigma| \subset |C|_{\mathrm{metric}}$, we furnish $|\sigma|$ with a topology. The topology that we take in $|C|$ is the coherent (weak) topology with respect to the closed subsets $|\sigma|$, that is, we declare a subset $A \subseteq |C|$ to be closed if and only if the intersection $A \cap |\sigma|$ is closed for all simplexes $\sigma \in C$.

1.5.4 Example. Let D be the simplicial complex of Example 1.5.2 1. Its geometric realization $|D_n|$ is homeomorphic to the *standard n-simplex*
$$\Delta^n = \left\{ \sum_{i=0}^n t_i e_i \mid \sum_{i=0}^n t_i = 1 \right\}$$
and the homeomorphism is given by $\alpha \mapsto \sum_{k=0}^n \alpha(k) e_k$, where $e_0 = (1, 0, \ldots, 0)$, $e_1 = (0, 1, 0, \ldots, 0)$,..., $e_n = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. See Figure 1.2.

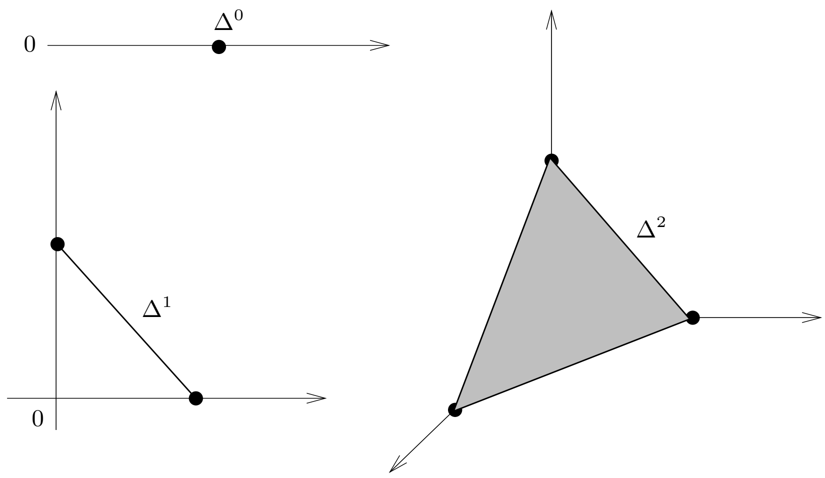

Figure 1.2 The standard 1-, 2-, and 3-simplexes

More generally than in the example, if we take any set of points $V = \{x_0, \ldots, x_k\}$ in $\mathbb{R}^n$ which are *affinely independent* (i.e. the k vectors $x_1 - x_0, \ldots, x_k - x_0$ are linearly independent), then its *convex hull*, namely the set

$$\xi = \left\{ \sum_{i=0}^{k} \lambda_i x_i \mid \lambda_i \geq 0,\ \sum_{i=0}^{k} \lambda_i = 1 \right\} \subset \mathbb{R}^n$$

is a *Euclidean simplex* in $\mathbb{R}^n$. The *interior* of ξ is given by

$$\overset{\circ}{\xi} = \left\{ \sum_{i=0}^{k} \lambda_i x_i \mid \lambda_i > 0,\ \sum_{i=0}^{k} \lambda_i = 1 \right\} \subset \xi\,.$$

The *faces* of ξ are the convex hulls of the subsets of V.

1.5.5 REMARK. $|C|$ is the union of its skeletons. More precisely, it is a filtered space

$$|C| \supset \cdots \supset |C^k| \supset |C^{k-1}| \supset \cdots \supset |C^0|\,,$$

where the 0-skeleton $|C^0|$ is discrete and the difference between the k-skeleton and the $(k-1)$-skeleton $|C^k| - |C^{k-1}|$ is a disjoint union of k-cells, namely the interior of geometric k-simplexes (see the previous example). Hence $|C|$ is a CW-complex. A topological space X together with a homeomorphism $\varphi : |C| \longrightarrow X$ is called a *polyhedron* and the map φ is called a *triangulation* of X.

1.5.6 EXAMPLE. The five regular polyhedra, namely the tetraheder, the cube, the octaheder, the dodecaheder, and the icosaheder are Euclidean simplicial complexes with 4, 8, 6, 20, and 12 vertices, respectively.

More generally than in the example above, we have the next.

1.5.7 **Lemma**. *Given a k-simplex σ in a simplicial complex C, the geometric realization $|\sigma|$ is homeomorphic to the standard k-simplex Δ^q.*

Proof: Given $\alpha \in |\sigma|$, where $\sigma = \{v_0, v_1, \dots, v_k\}$, define $\varphi : |\sigma| \longrightarrow \Delta^k$ by $\varphi(\alpha) = \sum_{i=0}^k \alpha(v_i)e_i$, where $e_i = (0, \dots, 1, \dots, 0)$, $i = 0, 1, \dots, k$, is the canonical generator of $\mathbb{R}^{k+1}$. Then φ is a homeomorphism with inverse $\psi : \Delta^n \longrightarrow |\sigma|$ given by $\psi(\sum_{i=0}^k t_i e_i)(v_j) = t_j$. □

Given two (ordered) simplicial complexes C and D, by a *simplicial map* we shall understand a (-n order-preserving) function $f : V(C) \longrightarrow V(D)$ such that if $\{v_{i_0}, v_{i_1}, \dots, v_{i_k}\}$ is a k-simplex of C, then $\{f(v_{i_0}), f(v_{i_1}), \dots, f(v_{i_k})\}$ is an l-simplex of D. Given such an f, it defines a map $|f| : |C| \longrightarrow |D|$ given by $|f|(\alpha)(w) = \sum_{v \in f^{-1}(w)} \alpha(v)$.

1.5.8 **Proposition**. *The map $|f| : |C| \longrightarrow |D|$ is continuous.*

Proof: It is enough to check the continuity of the restriction of $|f|$ to an arbitrary k-simplex $|\sigma|$. By 1.5.7 it is equivalent to check the continuity of the corresponding map between standard simplexes. Thus take the square

$$\begin{tikzcd}
|\sigma| \arrow[r, "\varphi", "\approx"'] \arrow[d, "|f|_{|\sigma|}"'] & \Delta^k \arrow[d, dashed, "f'"] \\
|\tau| \arrow[r, "\approx", "\psi"'] & \Delta^l ,
\end{tikzcd}$$

where φ and ψ are the homeomorphisms given in Lemma 1.5.7 and f' is such that the square commutes. Thus $|f|_{|\sigma|}$ is continuous if and only if f' is continuous. So it is enough to compute f'. Let σ be the simplex $\{v_0, v_1, \dots, v_k\}$ and take $\alpha \in |\sigma|$, i.e. $\alpha : \{v_0, v_1, \dots, v_k\} \longrightarrow I$ such that $\alpha(v_0) + \alpha(v_1) + \cdots + \alpha(v_k) = 1$. If $f(\sigma) = \{w_0, w_1, \dots, w_l\}$, then clearly $f' : \Delta^k \longrightarrow \Delta^l$ is given by $f'(e_i) = \left(\sum_{i \in f^{-1}(w_j)} \alpha(v_i)\right) e_j$ and then extending the map affinely. Thus clearly f' is continuous. □

1.5.9 Example. If $k < n$, we may define a simplicial map $D_k \longrightarrow D_n$ by sending i to i if $0 \leq i \leq k$. If $k \geq n$ we may define a simplicial map $D_k \longrightarrow D_n$ by sending i to i if $0 \leq i \leq n$, and sending i to n if $i \geq n$. Figure 1.3 shows $D_1 \longrightarrow D_2$ and $D_2 \longrightarrow D_1$.

It is an easy *exercise* to prove the next result.

1.5.10 **Proposition**.

(a) *Abstract simplicial complexes together with simplicial maps build a category* $\mathfrak{Simcom}$.

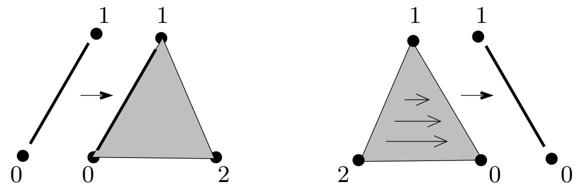

Figure 1.3 The simplicial maps $D_1 \longrightarrow D_2$ and $D_2 \longrightarrow D_1$

(b) *The geometric realization is a functor* $\mathfrak{Simcom} \longrightarrow \mathfrak{Top}$. □

In contrast to the concept of an abstract simplicial complex, we have the next.

1.5.11 Definition. A *Euclidean simplicial complex* X is a finite family of simplexes in $\mathbb{R}^n$ for some n, such that the following hold.

(a) If ξ is a simplex of X, then every face of ξ is a simplex of X.

(b) If ξ_1 and ξ_2 are two simplexes of X such that their interiors meet, i.e. $\overset{\circ}{\xi}_1 \cap \overset{\circ}{\xi}_2 \neq \emptyset$, then $\xi_1 = \xi_2$.

1.6 CW-complexes

A category which will be used along the text is that of the CW-complexes. Since we shall relate CW-complexes with simplicial complexes and even simplicial sets, it is convenient to replace unit balls with standard simplexes and spheres with the boundary of standard simplexes. Thus we start with the following.

1.6.1 Definition. Define the *nth standard simplex* $\Delta^n \subset \mathbb{R}^{n+1}$, $n \geq 0$, as the set

$$\Delta^n = \left\{ t_0e_0 + t_1e_1 + \cdots + t_ne_n \mid t_i \geq 0,\ \sum_{i=0}^{n} t_i = 1 \right\} ,$$

where $e_0, e_1, \ldots, e_n$ are the unit vectors $(1,0,\ldots,0)$, $(0,1,\ldots,0), \ldots, (0,0,\ldots,1) \in \mathbb{R}^{n+1}$, respectively. It is sometimes convenient to write the elements of Δ^n as $n+1$-tuples $(t_0, t_1, \ldots, t_n)$ such that $t_i \geq 0$ for all i and $\sum_{i=0}^{n} t_i = 1$. In the case of $n = 1$, one has that the elements of Δ^1 are pairs $(1-t, t)$ with $0 \leq t \leq 1$. Thus one can canonically identify the standard 1-simplex Δ^1 with the unit interval via $I \longrightarrow \Delta^1$ given by $t \mapsto (1-t, t)$ with inverse $\Delta^1 \longrightarrow I$ given by $(t_0, t_1) \mapsto t_1$ (this identifies $0 \in I$ with $e_0 \in \mathbb{R}^2$ and $1 \in I$ with $e_1 \in \mathbb{R}^2$. The *boundary* $\dot{\Delta}^n$ of Δ^n is defined as the set

$$\dot{\Delta}^n = \{t_0e_0 + t_1e_1 + \cdots + t_ne_n \in \Delta^n \mid t_i = 0 \text{ for at least one } i\} .$$

1.6.2 EXERCISE. Show that one has homeomorphisms

$$\varphi : \Delta^n \longrightarrow \mathbb{D}^n \qquad \text{and} \qquad \dot{\varphi} : \dot{\Delta}^n \longrightarrow \mathbb{S}^{n-1}\,.$$

(*Hint:* The orthogonal projection onto $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ sends Δ^n homeomorphically to a compact convex set in $\mathbb{R}^n$ with nonempty interior.)

1.6.3 DEFINITION. A CW-*complex* X is a filtered space $X = \bigcup_n X^n$ with the topology of the union, where the *0th skeleton* X^0 is a discrete subspace, and the *nth skeleton* X^n is defined from X^{n-1} as follows. Take an index set $\mathcal{I}_n$ and for each $i \in \mathcal{I}_n$ a continuous map $\psi_i : S_i^{n-1} \longrightarrow X^{n-1}$, where S_i^{n-1} is a copy of the boundary $\dot{\Delta}^n$ of the nth standard simplex Δ^n (or a copy of the unit $n-1$-sphere $\mathbb{S}^{n-1}$). Define X^n as the attaching space

$$X^{n-1} \cup_\psi \coprod_{i \in \mathcal{I} n} D_i^n\,,$$

where D_i^n is a copy of nth standard simplex Δ^n (or a copy of the unit n-ball $\mathbb{D}^n$) and $\psi : \coprod_{i\in\mathcal{I}n} S_i^{n-1} \longrightarrow X^{n-1}$ is given by $\psi|_{S_i^{n-1}} = \psi_i$. Clearly X^{n-1} is a closed subset of X^n, hence $A \subset X$ is closed if and only if $A \cap X^n$ is closed for all n. If $q_n : \coprod D_i^n \cup X^{n-1} \longrightarrow X^n$ is the identification map, $n > 0$, then for each $i \in \mathcal{I}_n$ the map $\varphi_i = q|_{D_i^n} : D_i^n \longrightarrow X^n$ is called the *characteristic map of the ith n-cell.* The image of φ_i is denoted by $\overline{e_i^n}$ and is called a *closed n-cell* of X. We say that the CW-complex X is *regular* if for every cell e_i^n, the characteristic map is an embedding. We denote the category of CW-complexes by $\mathfrak{CW}$ and the category of regular CW-complexes by $\mathfrak{RCW}$.

We have the next.

1.6.4 **Proposition**. *Let X be a* CW-*complex. A subset $A \subset X$ is closed if and only if $\varphi_i^{-1}A \subseteq D_i^n = \Delta^n$ is closed for every $i \in \mathcal{I}_n$ and every $n > 0$. Equivalently, A is closed if and only if $A \cap \overline{e_i^n}$ is closed for any closed cell $\overline{e_i^n}$ of X.* $\square$

1.6.5 EXAMPLES. In the following examples it is convenient to take $D_i^q = \mathbb{D}^q$ and $S_i^{q-1} = \mathbb{S}^{q-1}$.

1. The unit q-sphere $\mathbb{S}^q$ is a CW-complex such that $X^0 = X^1 = \cdots = X^{q-1} = \{x_0\}$ and $\mathbb{S}^q = X^q$ is obtained as an attaching space $X^0 \cup_\psi \mathbb{D}^q$, where $\psi : \mathbb{S}^{q-1} \longrightarrow X^0$ is the obvious map.

2. The q-sphere can be seen as a regular CW-complex. One may alternatively define $\mathbb{S}^q$ as the space filtered by the subspaces $X^n = \mathbb{S}^n \subset \mathbb{R}^{n+1}$, $n = 0, 1, \ldots, q$ with the canonical inclusions. There are two attaching maps $\psi_i^{n-1} : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$, $i = 1, 2$, each of which is the identity, $n = 1, 2 \ldots, q-1$.

3. The *real projective space* $\mathbb{RP}^q$ is a CW-complex filtered by the subspaces $X^n = \mathbb{RP}^n$, $n = 0, 1, \ldots, q$ with the canonical inclusions. The attaching map $\psi^{n-1} : \mathbb{S}^{n-1} \longrightarrow X^{n-1} = \mathbb{RP}^{n-1}$ is the antipodal identification.

4. The *complex projective space* $\mathbb{CP}^q$ is a CW-complex filtered by the subspaces $X^{2n-1} = X^{2n} = \mathbb{CP}^n$, $n = 0, 1, \ldots, q$ with the canonical inclusions. The attaching map $\psi^{n-1} : \mathbb{S}^{2n-1} \longrightarrow X^{n-1} = \mathbb{CP}^{n-1}$ is the usual identification given by the equivalence relation $z \sim z'$ if there is $\zeta \in \mathbb{S}^1$ such that $z' = \zeta z$, where $z, z' \in \mathbb{S}^{2n-1} \subset \mathbb{R}^{2n} = \mathbb{C}^n$.

5. The geometric realization $|K|$ of an abstract simplicial complex is a regular CW-complex with one n-cell for each nondegenerate n-simplex of K.

1.6.6 Exercise. Describe the examples 1.-5. above in terms of the standard simplexes $D_i^q = \Delta^q$ and their boundaries $S_i^{q-1} = \dot{\Delta}^q$.

Chapter 2 Elements of simplicial sets

In this chapter we shall study the notions on simplicial sets that will be needed in the book. Simplicial sets are an alternative to the study of (nice) topological spaces from a combinatorial viewpoint. This approach to algebraic topology dates back to the early fifties, when Eilenberg and Zilber studied the singular homology theory. Simplicial sets lie halfways between topology and algebra, since their nearness to algebra makes the computation of homotopy and homology groups easier, while one can define for simplicial sets many topological concepts, such as connectedness, homotopy, fibrations, et cetera.

2.1 Simplicial sets

In this section we define the concept of simplicial set. We start considering the category Δ whose objects are the ordered sets $\mathbf{n} = \{0, 1, 2, \ldots, n\}$, $n \in \mathbb{N}$, and whose morphisms are monotonic (order-preserving) functions $\mu : \mathbf{m} \longrightarrow \mathbf{n}$, namely, if $i \le j \in \mathbf{m}$, then $\mu(i) \le \mu(j) \in \mathbf{n}$.

2.1.1 Definition. Let $\mathfrak{C}$ be any category. Then a *simplicial object* of $\mathfrak{C}$ is a contravariant functor $K : \Delta \longrightarrow \mathcal{C}$. We denote by K_n the value of K at $\mathbf{n}$ and we call its elements *n-simplexes*. Furthermore, if $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in Δ then we denote by $\mu^K : K_n \longrightarrow K_m$ the morphism $K(\mu)$. Given two simplicial objects K and K' of $\mathfrak{C}$, we define a *morphism* $\varphi : K \longrightarrow K'$ to be a natural transformation of functors. Namely, for each $n \in \mathbb{N}$, one has a morphism $\varphi_n : K_n \longrightarrow K'_n$ such that for any morphism $\mu : \mathbf{m} \longrightarrow \mathbf{n}$, the following diagram commutes:

$$\begin{tikzcd} K_n \arrow[r, "\varphi_n"] \arrow[d, "\mu^K"'] & K'_n \arrow[d, "\mu^{K'}"] \\ K_m \arrow[r, "\varphi_m"'] & K'_m . \end{tikzcd}$$

We have a category simp-$\mathfrak{C}$ of simplicial objects of $\mathfrak{C}$.

2.1.2 Definition. Assume that $\mathfrak{C}$ is a category that admits subobjects of its objects. Given a simplicial object K of $\mathfrak{C}$, we shall understand by a *simplicial subobject* a simplicial object L of $\mathfrak{C}$ such that for each n, L_n is a subobject of K_n and the inclusions $L_n \hookrightarrow K_n$ induce a morphism $L \hookrightarrow K$ of simplicial objects.

2.1.3 EXAMPLES. The next will be useful examples in what follows.

1. If $\mathfrak{C} = \mathfrak{Set}$ is the category of sets, then the simplicial objects are called *simplicial sets*. We denote the category of simplicial sets simply by $\mathfrak{SSet}$. In this case, given a simplicial set K, we have the concept of a *simplicial subset*, which is a simplicial set L such that for each n, $L_n \subseteq K_n$. This allows to define the concept of a *quotient* of simplicial sets K/L, which is a simplicial set which for each n is given by $(K/L)_n = K_n/L_n$. If L is a simplicial subset of K, it is an easy *exercise* to show that K/L is a simplicial set too.

2. If $\mathfrak{C} = \mathfrak{Grp}$ (or $\mathfrak{Ab}$) is the category of (abelian) groups, then the simplicial objects are called *simplicial (abelian) groups*. We denote the category of simplicial groups by $\mathfrak{SGrp}$ (or $\mathfrak{SAb}$ in the abelian case). As above, given a simplicial group Γ, we have the concept of a (*normal*) *simplicial subgroup* Λ, such that for each n, $\Lambda_n \subseteq \Gamma_n$ is a (normal) subgroup. Defining $(\Gamma/\Lambda)_n = \Gamma_n/\Lambda_n$ (group quotient), it is an easy *exercise* to show that we obtain a simplicial set (or group).

A source of new simplicial objects is given by the following.

2.1.4 **Proposition**. *Let K be a simplicial object of a category $\mathfrak{C}$ and take a covariant functor $F : \mathfrak{C} \longrightarrow \mathfrak{D}$. Then the composite functor $F \circ K$ is a simplicial object of $\mathfrak{D}$. For each $n \in \mathbb{N}$, the object $(F \circ K)_n$ is given by $F(K_n)$ and if $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in Δ, then $\mu^{F\circ K} = F(\mu^K)$.* $\square$

Given a simplicial set K, for each $n \in \mathbb{N}$ there are two types of *operators*, namely the *degeneracy operators* $s_i : K_n \longrightarrow K_{n+1}$ and the *face operators* $d_i : K_n \longrightarrow K_{n-1}$, $0 \leq i \leq n$. These operators are determined by $s_i = K(\sigma_i)$ and $d_i = K(\delta_i)$, where $\sigma_i : \mathbf{n+1} \longrightarrow \mathbf{n}$ and $\delta_i : \mathbf{n-1} \longrightarrow \mathbf{n}$ are given by

$$\sigma_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases} \qquad \delta_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i\,. \end{cases}$$

2.1.5 EXERCISE. Show that any morphism $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ of Δ is a composite of face and degeneracy morphisms.

2.1.6 EXERCISE.

(a) Show that the morphisms $\delta_i : \mathbf{n} \longrightarrow \mathbf{n+1}$ and $\sigma_i : \mathbf{n+1} \longrightarrow \mathbf{n}$ satisfy the following relations:

$$\begin{aligned}
\delta_j\delta_i &= \delta_i\delta_{j-1} && \text{if } i<j\,,\\
\sigma_j\delta_i &= \delta_i\sigma_{j-1} && \text{if } i<j\,,\\
\sigma_j\delta_j &= \sigma_j\delta_{j+1} = \mathrm{id}_{\mathbf{n}}\,, &&\\
\sigma_j\delta_i &= \delta_{i-1}\sigma_j && \text{if } i>j+1\,,\\
\sigma_j\sigma_i &= \sigma_i\sigma_{j+1} && \text{if } i\le j\,.
\end{aligned}$$

(b) Conclude that the following *simplicial identities* hold:

$$\begin{aligned}
d_id_j &= d_{j-1}d_i && \text{if } i<j\,,\\
d_is_j &= s_{j-1}d_i && \text{if } i<j\,,\\
d_js_j &= d_{j+1}s_j = \mathrm{id}_{K_n}\,, &&\\
d_is_j &= s_jd_{i-1} && \text{if } i>j+1\,,\\
s_is_j &= s_{j+1}s_i && \text{if } i\le j\,.
\end{aligned}$$

Recall the *nth standard simplex* $\Delta^n \subset \mathbb{R}^{n+1}$, $n \geq 0$, defined as the set

$$\Delta^n = \left\{ t_0e_0 + t_1e_1 + \cdots + t_ne_n \mid t_i \geq 0,\ \sum_{i=0}^{n} t_i = 1 \right\},$$

where $e_0, e_1, \ldots, e_n$ are the unit vectors $(1,0,\ldots,0)$, $(0,1,\ldots,0),\ldots,(0,0,\ldots,1) \in \mathbb{R}^{n+1}$. The face morphisms $\delta_i : \mathbf{n-1} \longrightarrow \mathbf{n}$ and the degeneracy morphisms $\sigma_i : \mathbf{n+1} \longrightarrow \mathbf{n}$ induce maps $\delta_{i\#} : \Delta^{n-1} \longrightarrow \Delta^n$ by sending each vertex e_j to $e_{\delta_i(j)}$ and $\sigma_{i\#} : \Delta^{n+1} \longrightarrow \Delta^n$ by sending each vertex e_j to $e_{\sigma_i(j)}$, respectively, and then extending them affinely.

2.1.7 Examples. The following are two manners of associating simplicial sets to topological spaces.

1. Given a topological space X, we can associate to it the *singular* simplicial set $\mathcal{S}(X) : \Delta \longrightarrow \mathfrak{Set}$ as follows

   $$\mathcal{S}(X)(\mathbf{n}) = \mathcal{S}_n(X) = \{\alpha : \Delta^n \longrightarrow X \mid \alpha \text{ is continuous}\}\,.$$

   Its face and degeneracy maps are given by

   $$d_i(\alpha) = \alpha \circ \delta_{i\#} : \Delta^{n-1} \longrightarrow X\,, \qquad s_i(\alpha) = \alpha \circ \sigma_{i\#} : \Delta^{n+1} \longrightarrow X\,.$$

   As a functor, $\mathcal{S}(X) : \Delta \longrightarrow \mathfrak{Set}$ is defined as follows. If $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism of Δ, then μ determines a map $\mu_\# : \Delta^m \longrightarrow \Delta^n$ by mapping each vertex $e_i \in \Delta^m$ to $e_{\mu(i)} \in \Delta^n$ and then extending affinely. Then the function $\mu^{\mathcal{S}}(X) : \mathcal{S}_n(X) \longrightarrow \mathcal{S}_m(X)$ is given by $\mu^{\mathcal{S}}(\alpha) = \alpha \circ \mu_\#$. An element $\sigma \in \mathcal{S}_n(X)$ shall be called a *singular n-simplex* of X.

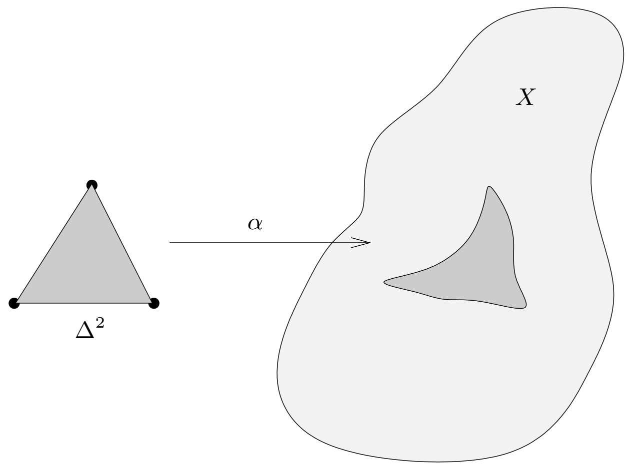

Figure 2.1 A singular 2-simplex in a space X

2. If C is an ordered abstract simplicial complex, then we can associate to it a simplicial set $K(C)$ such that

$$K(C)_m = \{(v_0, v_1, \ldots, v_m) \mid \{v_0, v_1, \ldots, v_m\} \in C, v_0 \leq v_1 \leq \cdots \leq v_m\},$$

and the face operator $d_i^{K(C)} : K(C)_m \longrightarrow K(C)_{m-1}$ is given by

$$d_i^{K(C)}(v_0, \ldots, v_m) = (v_0, \ldots, \widehat{v_i}, \ldots, v_m),$$

where the hat means that the vertex v_i is removed, and $s_i^{K(C)} : K(C)_m \longrightarrow K(C)_{m+1}$ is given by

$$s_i^{K(C)}(v_0, \ldots, v_m) = (v_0, \ldots, v_i, v_i, \ldots, v_m)$$

It is an easy exercise to show that these operators satisfy the simplicial identities 2.1.6 (b).

This construction is a functor, namely if $f : C \longrightarrow D$ is a simplicial map, then we define $K(f) : K(C) \longrightarrow K(D)$ as follows. Give $K(f)_n : K(C)_n \longrightarrow K(D)_n$ by $K(f)(v_0, v_1, \ldots, v_n) = (f(v_0), f(v_1), \ldots, f(v_n))$. Thus we have a functor $\mathfrak{Simcom} \longrightarrow \mathfrak{SSet}$.

2.2 Geometric realization

The last example in the previous section shows how to produce from a topological space simplicial sets. The following definition links the combinatorics of the simplicial sets with the topology. Concretely it assigns to a simplicial set a topological space.

2.2.1 Definition. Let K be a simplicial set. Endow each set K_n with the discrete topology and take the standard simplex Δ^n with its relative topology as a subspace of $\mathbb{R}^{n+1}$. The *geometric realization* of K is given by

$$|K| = \coprod_{n=0}^{\infty} K_n \times \Delta^n/\!\sim\,,$$

with the quotient topology, where the equivalence relation $\sim$ is generated by the relation $(x, \mu_\#(t)) \sim (\mu^K(x), t)$ for $\mu : \mathbf{m} \longrightarrow \mathbf{n}$, where $x \in K_n$ and $t \in \Delta^m$. We shall denote by $[x, s]$ the equivalence class of $(x, s) \in K_n \times \Delta^n$ and call it *geometric simplex.*

According to 2.1.5, it should be enough to consider the equivalence relation generated by the relations $(x, \delta_{i\#}(s)) \sim (d_i(x), s)$ and $(x, \sigma_{i\#}(t)) \sim (s_i(x), t)$, where $x \in K_n$, $s \in \Delta^{n-1}$, and $t \in \Delta^{n+1}$.

Given a morphism of simplicial sets $\varphi : K \longrightarrow K'$, we may define a continuous map $|\varphi| : |K| \longrightarrow |K'|$ as follows. Consider the diagram

$$\begin{tikzcd}
\coprod_n K_n \times \Delta^n \arrow[r, "\coprod \varphi_n \times \mathrm{id}"] \arrow[d, "q"'] & \coprod_n K'_n \times \Delta^n \arrow[d, "q'"] \\
|K| \arrow[r, dashed, "|\varphi|"'] & |K'|\,.
\end{tikzcd}$$

where q and q' are the respective quotient maps. The map $|\varphi|$ is well defined and thus continuous, since it is compatible with the quotient maps, namely if $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in Δ, then $(\varphi_m \mu^K(x), s) = (\mu^{K'} \varphi_n(x), s) \sim (\varphi_n(x), \mu_\#(s))$, where the equality holds because φ is natural. Thus equivalent elements are mapped by $\coprod \varphi_n \times \mathrm{id}$ to equivalent elements.

It is a routine *exercise* to prove the following.

2.2.2 **Proposition**. *The realization is a functor, namely the equalities* $|\mathrm{id}_K| = \mathrm{id}_{|K|}$ *and* $|\psi \circ \varphi| = |\psi| \circ |\varphi|$ *hold if* $\varphi : K \longrightarrow K'$ *and* $\psi : K' \longrightarrow K''$ *are morphisms of simplicial sets.* □

2.2.3 **Proposition**. *Let C be a simplicial complex and let $K(C)$ be its associated simplicial set. There is a natural homeomorphism $\varphi : |C| \longrightarrow |K(C)|$*

Proof: Take $\alpha \in |C|$, $\alpha : V(C) \longrightarrow I$, and define $\varphi(\alpha) = [x, s] \in |K(C)|$, where $x = (v_0, v_1, \ldots, v_n) \in K(C)_n$ if $\alpha^{-1}(0, 1] = \{v_0, v_1, \ldots, v_n\}$, $v_0 < v_1 < \cdots < v_n$, and $s = \alpha(v_0)e_0 + \alpha(v_1)e_1 + \cdots + \alpha(v_n)e_n \in \Delta^n$. Conversely, define $\psi : |K(C)| \longrightarrow |C|$ as follows. Let $(x, s) \in K(C)_n \times \Delta^n$ be a representative of an element in $|K(C)|$ such that $x = (v_0, v_1, \ldots, v_n)$ and $s = s_0e_0 + s_1e_1 + \cdots + s_ne_n$, $s_i \neq 0$, $i = 0, 1, \ldots, n$, and define

$$\psi'(x, s) : V(C) \longrightarrow I \quad \text{by } \psi'(x, s)(v) = \begin{cases} s_i & \text{if } v = v_i, i = 0, 1, \ldots, n\,, \\ 0 & \text{otherwise.} \end{cases}$$

It is an easy *exercise* to show that ψ' is compatible with the equivalence relation which defines $|K(C)|$ and thus it determines ψ. It is also easy to verify that both φ and ψ are continuous and inverse to each other. $\square$

The next family of simplicial sets will be very important in what follows.

2.2.4 Definition. The simplicial set $\Delta[q] : \Delta \longrightarrow \mathfrak{Set}$ given by $\Delta[q]_n = \Delta(\mathbf{n}, \mathbf{q})$ is called the *q-model* and the simplicial subset $\dot{\Delta}[q] : \Delta \longrightarrow \mathfrak{Set}$ given by $\dot{\Delta}[q]_n = \{\sigma : \mathbf{n} \longrightarrow \mathbf{q} \mid \sigma \text{ is not surjective}\}$ is the *boundary* the q-model.

2.2.5 Exercise. Show that the geometric realization of the q-model is the standard q-simplex, namely $|\Delta[q]| \approx \Delta^q$. Furthermore show that $|\dot{\Delta}[q]| \approx \dot{\Delta}^q$.

2.2.6 Exercise.

(a) Verify that $\Delta[q]$ is indeed a simplicial set and show that its geometric realization $|\Delta[q]|$ is canonically homeomorphic to the standard q-simplex Δ^q.

(b) Verify that $\dot{\Delta}[q]$ is also a simplicial set and show that its geometric realization $|\dot{\Delta}[q]|$ is canonically homeomorphic to the boundary $\dot{\Delta}^q$ of the standard q-simplex.

If one takes a monotonic function $\alpha : \mathbf{q} \longrightarrow \mathbf{n}$, one can associate to it the simplex $\alpha^*(\mathbf{n}) \in \Delta[n]_q$, which clearly determines a one-to-one relation between monotonic functions $\mathbf{q} \longrightarrow \mathbf{n}$ and the q-simplexes of $\Delta[n]$. Indeed we have the next.

2.2.7 **Proposition**. *There is a bijection between the set of monotonic functions $\Delta(\mathbf{q}, \mathbf{n})$ and the set of morphisms of simplicial sets $\mathfrak{SSet}(\Delta[q], \Delta[n])$.*

Proof: Map the function $\alpha : \mathbf{q} \longrightarrow \mathbf{n}$ to the morphism given by

$$[q] \longmapsto (\alpha(0), \alpha(1), \dots, \alpha(q)),$$

namely $\alpha : \Delta[q] \longrightarrow \Delta[n]$. Then $\alpha([q]) = \alpha^*([n])$. $\square$

In other words, the last result states that we have a full subcategory of the category of models which is isomorphic to Δ.

2.2.8 Exercise. Consider the simplicial set $\Sigma^{q-1} : \Delta \longrightarrow \mathfrak{Set}$ given by $\Sigma^{q-1}_n = \{\sigma : \mathbf{n} \longrightarrow \mathbf{q-1} \mid \sigma(i) = 0,\ i = 0, \dots, q-2\}$, $\Sigma^{q-1}_{q-1} = \{\mathrm{id}_{\mathbf{q-1}}\}$, and $\Sigma^{q-1}_n = \{\sigma : \mathbf{n} \longrightarrow \mathbf{q-1} \mid \sigma \text{ is surjective}\}$. Verify that Σ^{q-1} is indeed a simplicial set and show that its geometric realization is homeomorphic to the $q-1$-sphere $\mathbb{S}^{q-1}$.

2.2.9 Example. If $\mathfrak{C} = \mathfrak{Ab}$ is the category of abelian groups and $G : \Delta \longrightarrow \mathfrak{Ab}$ is a simplicial abelian group, then its geometric realization $|G|$ is an abelian topological group (see below). We postpone the proof of this nontrivial fact.

2.2.10 **Theorem**. *Let K be a simplicial set and Y be a* k*-space. Then there is a natural bijection*

$$\mathfrak{K}\text{-}\mathfrak{Top}(|K|, X) \cong \mathfrak{SSet}(K, \mathcal{S}(X))\,.$$

In other words, the functor $|\cdot|$ is left-adjoint to the functor $\mathcal{S}$.

Proof: According to Lemma 1.3.5, for the proof we shall need the following canonical morphisms which relate the geometric realization and the singular simplicial set construction. First consider a simplicial set K and let

(2.2.18) $\qquad \alpha_K : K \longrightarrow \mathcal{S}(|K|)$ be given by $\alpha_n(x)(s) = [x, s]$

for $x \in K_n$ and $s \in \Delta^n$. We have to show that α_K is natural, i.e., that given $\mu : \mathbf{m} \longrightarrow \mathbf{n}$, the following is a commutative diagram:

$$\begin{tikzcd} K_n \arrow[r, "\alpha_n"] \arrow[d, "\mu^K"'] & \mathcal{S}_n(|K|) \arrow[d, "\mu^{\mathcal{S}}"] \\ K_m \arrow[r, "\alpha_m"'] & \mathcal{S}_m(|K|)\,, \end{tikzcd}$$

where we shorten the notation $(\alpha_K)_n$ by writing simply α_n, and similarly for m. To see this, we chase an element $x \in K_n$ along the diagram. Going right and down and evaluating in $t \in \Delta^m$, we obtain $\mu^{\mathcal{S}}\alpha_n(x)(t) = \alpha_n(x)(\mu_\#(t)) = [x, \mu_\#(t)]$, while going down and right, we obtain $\alpha_m(\mu^K(x))(t) = [\mu^K(x), t]$. Thus both are equal.

If X is a topological space, let also

(2.2.19) $$\rho_X : |\mathcal{S}(X)| \longrightarrow X$$

be given by

$$\rho_X([\sigma, t]) = \sigma(t) \text{ where } \sigma \in \mathcal{S}_n(X),\ t \in \Delta^n\,.$$

To see that ρ_X is well defined consider a morphism $\mu : \mathbf{m} \longrightarrow \mathbf{n}$. Then by definition $(\mu^{\mathcal{S}}(\sigma))(s) = \sigma(\mu_\#(s))$. Therefore, both $\rho_X([\mu^{\mathcal{S}}(\sigma), s]) = (\mu^{\mathcal{S}}(\sigma))(s)$ and $\rho_X([\sigma, \mu_\#(s)] = \sigma(\mu_\#(s))$ are equal.

We pass to the proof of the theorem. Consider the function

$$R_{K,X} : \mathfrak{SSet}(K, \mathcal{S}(X)) \longrightarrow \mathfrak{Top}(|K|, X)$$

given by $R_{K,X}(\varphi) = \rho_X \circ |\varphi|$. Explicitly, if $\varphi : K \longrightarrow \mathcal{S}(X)$ is a morphism of simplicial sets, then φ maps each simplex $x \in K_n$ to a singular simplex $\varphi(x) : \Delta^n \longrightarrow X$. So we have $R_{K,X}(\varphi)([x, s]) = \varphi_n(x)(s)$.

On the other hand, consider the function

$$A_{K,X} : \mathfrak{Top}(|K|, X) \longrightarrow \mathfrak{SSet}(K, \mathcal{S}(X))$$

given by $A_{K,X}(f) = \mathcal{S}(f) \circ \alpha_K$. Explicitly, if we have a continuous map $f : |K| \longrightarrow X$, $x \in K_n$, and $s \in \Delta^n$, then $A_{K,X}(f)_n(x)(s) = f([x,s])$.

To see that A and R are inverse of each other, take $x \in K_n$ and $s \in \Delta^n$ and consider

$$A_{K,X}(R_{K,X}(\varphi))_n(x)(s) = R_{K,X}(\varphi)([x,s]) = \varphi_n(x)(s)\,,$$

thus $A_{K,X}R_{K,X}(\varphi) = \varphi$, and

$$R_{K,X}A_{K,X}(f)([x,s]) = A_{K,X}(f)_n(x)(s) = f([x,s])\,,$$

thus $R_{K,X}A_{K,X}(f) = f$. □

2.2.20 Definition. Let K be a simplicial set. An *n-simplex* x, i.e. an element x of K_n is said to be *degenerate* if x is of the form $s_i(y)$ for some $y \in K_{n-1}$ and any i. Otherwise, we say that x is *nondegenerate*. Furthermore we say that a representative $(x,s) \in K_n \times \Delta^n$ of a geometric simplex $[x,s] \in |K|$ is *nondegenerate* if x is nondegenerate and $s \in \overset{\circ}{\Delta}{}^n$ (i.e. s is an interior point).

2.2.21 Exercise. Show that a n-simplex $x \in K_n$ is degenerate if and only if there is a surjective morphism $\mu : \mathbf{n} \longrightarrow \mathbf{m}$ and an m-simplex $y \in K_m$ such that $x = \mu^K(y)$.

2.2.22 **Proposition**. *If x is a degenerate simplex, then there is a unique nondegenerate simplex x_0 such that $x = s_{i_0}s_{i_1}\cdots s_{i_k}(x_0)$, for some finite sequence of degeneracy maps $s_{i_0}, s_{i_1}, \ldots, s_{i_k}$.*

Proof: First notice that if $x = \mu_{i_0}\mu_{i_1}\cdots\mu_{i_k}(x_0)$ for some nondegenerate simplex x_0, where each μ_{i_j} is either a face or a degeneracy map, then one can switch the face maps to the right using the simplicial identities 2.1.6 (b) and then replace x_0 with one of its proper faces. If this face turns out to be degenerate, we apply the process once more. Eventually this process stops. Hence a degenerate simplex can always be written in the form $s_{i_0}s_{i_1}\cdots s_{i_k}(x_0)$.

To show the uniqueness, assume that x_0 and y_0 are nondegenerate simplexes, possibly in different sets K_m and K_n, such that $\mathbf{s}(x_0) = \mathbf{t}(y_0)$, where $\mathbf{s}$ and $\mathbf{t}$ are composites of degeneracy operators. Assume $\mathbf{s} = s_{i_0} \circ s_{i_1} \circ \cdots \circ s_{i_k}$ and take $\mathbf{d} = d_{i_k} \circ d_{i_{k-1}} \circ \cdots \circ d_{i_0}$. Then by the third simplicial identity $x_0 = \mathbf{ds}(x_0) = \mathbf{dt}(y_0)$ and using again the simplicial identities for the first equality, and switching the face maps to the right, we obtain $x_0 = \mathbf{t}'\mathbf{d}'(y_0)$ for some composite of face operators $\mathbf{d}'$ and some composite of degeneracy operators $\mathbf{t}'$. But since by assumption x_0 is nondegenerate, $\mathbf{t}'$ must be the identity and so $x_0 = \mathbf{d}'(y_0)$. Thus x_0 is a face of y_0. But repeating this argument reversing x_0 and y_0, we may prove that y_0 is a face of x_0. This can only happen if $x_0 = y_0$. □

2.2.23 **Proposition**. *Every geometric simplex $[x, s] \in |K|$ has a unique nondegenerate representative.*

Proof: If $x \in K_n$ is already nondegenerate but s lies in the boundary of Δ^n, then $s = \delta(t)$, where $t \in \overset{\circ}{\Delta}{}^m$, $m < n$, and δ is the composite of finitely many face operators $\delta_{i\#}$. Thus $(x, s) \sim (d(x), t)$, where $\mathbf{d} : K_n \longrightarrow K_m$ is the function corresponding to δ.

If $x \in K_n$ is degenerate, then $x = \mathbf{s}(x_0)$, where $\mathbf{s} : K_m \longrightarrow K_n$, $m < n$, is the composite of finitely many degeneracy operators s_i and $x_0 \in K_m$ is nondegenerate. In this case, $(x, s) \sim (x_0, \sigma(s))$, where σ corresponds to $\mathbf{s}$. If $\sigma(s)$ is not an interior point of Δ^m, we may proceed as in the first part of the proof. □

2.2.24 Example. Given a topological space X, a singular n-simplex $\sigma : \Delta^n \longrightarrow X$ is nondegenerate if it cannot be written as a composite $\Delta^n \xrightarrow{\pi} \Delta^k \xrightarrow{\tau} X$, where π is a simplicial *collapse* with $k < n$ and τ is a singular k-simplex.

Notice that a nondegenerate simplex might have a degenerate face. Also a degenerate simplex might have a nondegenerate face (for instance, we know that $d_j s_j(x) = x$ for any x, degenerate or not).

2.3 Product and quotients of simplicial sets

We shall need products of simplicial sets.

2.3.1 Definition. Let K and K' be simplicial sets. Define their *product* $K \times K'$ by

(a) $(K \times K')_n = K_n \times K'_n$,

(b) If $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in Δ and $(x, x') \in (K \times K')_n$, then

$$\mu^{K \times K'}(x, x') = (\mu^K(x), \mu^{K'}(x')) .$$

This is obviously a simplicial set. Observe that the projections $\pi_n : K_n \times K'_n \longrightarrow K_n$ and $\pi'_n : K_n \times K'_n \longrightarrow K'_n$ are simplicial morphisms.

2.3.2 Examples.

1. Given any (abstract) group G, one can define a simplicial group Γ as follows. Let $\Gamma_n = G$ for all n and let $\mu^\Gamma = 1_G$ for all morphisms $\mu : \mathbf{n} \longrightarrow \mathbf{m}$.

2. Let F be a topological (abelian) group. The singular simplicial set $\mathcal{S}(F)$ is a simplicial (abelian) group, if one defines the product of two singular n-simplexes σ and τ by elementwise multiplication, namely

$$\sigma \cdot \tau : \Delta^n \longrightarrow F \quad \text{is given by} \quad (\sigma \cdot \tau)(s) = \sigma(s)\tau(s) \in F\,.$$

If $f : F \longrightarrow G$ is a continuous homomorphism of topological groups, so is $\mathcal{S}(f) : \mathcal{S}(F) \longrightarrow \mathcal{S}(G)$ a simplicial homomorphism.

2.3.3 **Theorem**. *Let K and K' be simplicial sets. Then there is an isomorphism $\eta : |K \times K'| \xrightarrow{\approx} |K| \times |K'|$.*

Proof: Consider the maps $|\pi| : |K \times K'| \longrightarrow |K|$ and $|\pi'| : |K \times K'| \longrightarrow |K'|$. By the universal property of the product, there is a map

$$\eta : |K \times K'| \longrightarrow |K| \times |K'|\,.$$

We prove first that this continuous map is bijective.

Consider $z \in |K \times K'|$ and take a nondegenerate representative $((x, x'), t) \in (K_n \times K'_n) \times \Delta^n$. Then, according to 2.2.23 $|\pi|(z) = [x, t]$ has a nondegenerate representative (x_0, t_0) and $|\pi'|(z) = [x', t]$ has a nondegenerate representative (x'_0, t'_0).

We define an inverse function $\xi : |K| \times |K'| \longrightarrow |K \times K'|$ as follows. Take an element $(a, a') \in |K| \times |K'|$ and nondegenerate representatives (x_a, t_a) and $(x_{a'}, t_{a'})$ of a and a'. If $t_a = (t_0, t_1, \ldots, t_k)$ and $t_{a'} = (t'_0, t'_1, \ldots, t'_{k'})$, then for each m, n, define

$$t^m = \sum_{i=0}^{m} t_i \quad \text{and} \quad t'^n = \sum_{j=0}^{n} t'_j\,.$$

Let $r_0 < r_1 < \cdots < r_b = 1$ be the sequence obtained by ordering the different elements of $\{t^m\} \cup \{t'^n\}$ by size, and define $t''_i = r_i - r_{i-1}$, $0 \le i \le b$, $r_{-1} = 0$. Clearly $0 < t''_i < 1$ and

$$\sum_{i=0}^{k} t''_i = r_b = 1\,.$$

Therefore the point $u_b = (t''_0, t''_1, \ldots, t''_b)$ lies in the interior of the standard simplex Δ^b. Let $i_1 < \cdots < i_{b-a}$ be those integers i such that $r_i \notin \{t^m\}$ and let $j_1 < \cdots < j_{b-a'}$ be those integers j such that $r_j \notin \{t'^n\}$. Hence $\{i_\alpha\}$ and $\{j_\beta\}$ are disjoint, $u_a = \sigma_{i_1} \cdots \sigma_{i_{b-a}}(u_b)$ y $u_{a'} = \sigma_{j_1} \cdots \sigma_{j_{b-a'}}(u_b)$. Now put

$$\xi(a, a') = [(s_{i_{b-a}} \cdots s_{i_1}(x_a), s_{j_{b-a'}} \cdots s_{j_1}(x_{a'})), u_b]\,.$$

Then we have

$$|\pi|\xi(a, a') = [s_{i_{b-a}} \cdots s_{i_1}(x_a), u_b] = [x_a, t_a] = a$$

and

$$|\pi'|\xi(a, a') = [s_{j_{b-a'}} \cdots s_{j_1}(x_{a'}), u_b] = [x_{a'}, t_{a'}] = a'\,.$$

Hence $\eta \circ \xi = \mathrm{id}_{|K|\times|K'|}$. On the other hand, taking $z \in |K \times K'|$ as above,

$$\xi\eta(z) = \xi([x_a, t_a], [x_{a'}, t_{a'}]) = [(x, x'), t] = z\,.$$

To finish, notice that ξ is continuous in each product cell of $|K| \times |K'|$ and since we are working in the category of k-spaces this implies that ξ is continuous. Therefore η is a homeomorphism with inverse ξ. □

2.3.4 **Theorem**. *Let G be a simplicial (abelian) group. Then the geometric realization $|G|$ is a topological (abelian) group (in $\mathfrak{K}$-$\mathfrak{Top}$).*

Proof: The group structure of G is given by a family of set functions $\mu_n : G_n \times G_n \longrightarrow G_n$ (multiplication) and $\iota_n : G_n \longrightarrow G_n$ (pass to the inverse) that make each G_n into a group. Define the maps

$$m : |G| \times |G| \longrightarrow |G| \quad \text{ and } \quad i : |G| \longrightarrow |G|$$

by $m = |\mu| \circ \xi$ and $i = |\iota|$, where μ and ι are the simplicial morphisms defined by the families μ_n and ι_n, respectively. The conditions on the functions μ_n and ι_n that make G_n into a group as well as the naturality of ξ imply that m and i convert $|G|$ in a topological group. □

The following will have much interest in Chapter 6 below.

For the next, recall Example 2.1.3 2.

2.3.5 **Proposition**. *Let K be a simplicial set and $Q \subset K$ be a simplicial subset. Then $|K/Q| \approx |K|/|Q|$.*

Proof: The sequence of simplicial sets $Q \overset{i}{\hookrightarrow} K \overset{p}{\twoheadrightarrow} K/Q$ determines a sequence of spaces $|Q| \overset{|i|}{\hookrightarrow} |K| \overset{|p|}{\twoheadrightarrow} |K/Q|$, where $|i|$ is an embedding and $|p|$ is an identification. Consider, on the other hand, the quotient map $q : |K| \twoheadrightarrow |K|/|Q|$. We shall see that both $|p|$ and q identify exactly the same points.

Assume first that $q([\sigma, t]) = q([\sigma', t'])$. If $[\sigma, t] \neq [\sigma', t']$, then this means that $[\sigma, t], [\sigma', t'] \in |Q|$. Therefore $(\sigma, t) \in Q_m \times \Delta^m$ and $(\sigma', t') \in Q_n \times \Delta^n$ for some m and n. Hence $|p|([\sigma, t]) = [p_m(\sigma), t] = [\overline{\tau}, t]$ and $[|p|([\sigma', t']) = [p_n(\sigma'), t'] = [\overline{\tau}', t']$, where the bar means the class in the quotient. Both representatives $(p_m(\sigma), t)$ and $(p_n(\sigma'), t')$ are degenerate if $m > 0$ and $n > 0$ and the common nondegenerate representative of both is $(\overline{\tau}_0, 1) \in (K_0/Q_0) \times \Delta^0$. Consequently, $|p|([\sigma, t]) = |p|([\sigma', t'])$.

Conversely, assume that $|p|([\sigma, t]) = |p|([\sigma', t'])$, where (σ, t) and (σ', t') are nondegenerate. We claim that if $\sigma \notin Q_m$, then $p_m(\sigma)$ is also nondegenerate. This is true, because if $p_m(\sigma)$ is degenerate, then the classes $\overline{\sigma} = \overline{s}^K(\overline{\sigma}_r)$ with $r < m$. But in this case $\overline{\sigma} = \{\sigma\} = \overline{s}^K(\overline{\sigma}_r)$, and so $s^K(\sigma_r) = \sigma$, which is a contradiction.

Therefore, the assumption means that $[p_m(\sigma), t] = [p_n(\sigma'), t']$, and we have two possible cases:

(i) $\sigma \in Q_m$ and $\sigma' \in Q_n$, hence $[\sigma, t], [\sigma', t'] \in |Q|$, so that $q([\sigma, t]) = q([\sigma', t'])$.

(ii) $\sigma \in Q_m$ and $\sigma' \notin Q_n$, hence $[p_m(\sigma), t] = [\overline{\tau}_0, 1] = [p_n(\sigma'), t']$. But by the claim above, $p_n(\sigma')$ is nondegenerate. Therefore, $p_n(\sigma') = \overline{\tau}_0$ and thus $\sigma' \in Q_0$, contradicting the assumption.

Hence, only case (i) can hold. □

2.4 Kan sets and Kan fibrations

In what follows we shall define a special subclass of simplicial sets, which is quite useful. One should call them *Kan simplicial sets* but we stick to one of the usual terms, namely simply *Kan sets*. Before stating the definition, we need some previous concepts.

2.4.1 Definition. The *simplicial kth horn* $\Lambda[n, k]$ is the union of all faces of the simplicial set $\Delta[n]$ except for the kth face. Equivalently it can be defined as the simplicial subset of $\Delta[n]$ generated by the set $\{d_0, d_1, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n\}$, $d_i : \Delta[n-1] \longrightarrow \Delta[n]$. It has the property that $\Lambda[n, k]_j = \Delta[n]_j$ for $j < n-1$, and $\Delta[n, k]_{n-1} = \Delta[n]_{n-1} - \{d_k\}$. Given a simplicial set K, a sequence of $n-1$-simplexes of K $(x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n-1}, x_n)$ is said to be a *horn* in K with a *hole* in the kth place if

$$d_i(x_j) = d_{j-1}(x_i)\,, \quad i < j \quad \text{and} \quad i \neq k \neq j$$

holds.

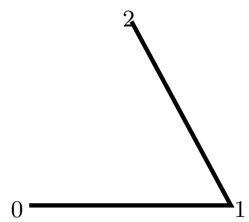

Figure 2.2 The simplicial horn $\Lambda[2, 1]$

Notice that the geometric realization $|\Lambda[n, k]|$ of the simplicial kth horn is the subcomplex of the geometric realization $|\Delta[n]|$ of the n-simplex obtained by removing the interior of $|\Delta[n]|$ and the interior of the kth face of it (see Figure 2.2).

2.4.2 Definition. A simplicial set K is said to satisfy the *extension condition* or *Kan condition* if for all $n = 1, 2, \ldots,$ and all $0 \leq k \leq n$, any morphism of simplicial sets

$$\Lambda[n, k] \longrightarrow K$$

admits an extension to a morphism of simplicial sets

$$\Delta[n] \longrightarrow K\,.$$

Such a simplicial set K is called a *Kan set* or a *Kan complex*. In a diagram

$$\begin{tikzcd} \Lambda[n,k] \arrow[r] \arrow[d, hook] & K \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

Nowadays a Kan set is also called a *fibrant* simplicial set. We shall stick to the term *Kan set*.

2.4.3 Exercise. Show that a simplicial set K is a Kan set if and only if for all $n = 1, 2, \dots,$ and all $0 \leq k \leq n$ the following holds: For any horn $T = (x_0, x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n-1}, x_n)$ in K there is an $x \in K_n$ such that $d_i(x) = x_i$ for all $i \neq k$. Such an x is called the *filling* of the horn. This is an alternate description of the Kan condition.

2.4.4 Definition. Given a simplicial set K and a simplex $x \in K_n$, we define its *boundary* by

$$D(x) = (d_0(x), d_1(x), \dots, d_n(x))\,.$$

2.4.5 Exercise. Show that if $x, y \in K_n$ are degenerate and $D(x) = D(y)$, then $x = y$.

The following result yields a very important example of a Kan set.

2.4.6 **Proposition**. *For any space X, the singular simplicial set $\mathcal{S}(X)$ is a Kan set.*

Proof: By the adjunction 2.2.10 there is a correspondence between the following diagram of simplicial sets and the one of spaces:

$$\begin{tikzcd} \Lambda[n,k] \arrow[r] \arrow[d, hook] & \mathcal{S}(X) \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} |\Lambda[n,k]| \arrow[r] \arrow[d, hook] & X\,. \\ |\Delta[n]| \arrow[ur, dashed] & \end{tikzcd}$$

Since the topological horn $|\Lambda[n,k]|$ is a deformation retract of the topological n-simplex $|\Delta[n]|$, the extension problem on the right has a solution and the dashed arrow exists. Passing this back along the adjunction 2.2.10 once more, we obtain a solution for the extension problem on the left as desired. □

There are other nice examples as follows.

2.4.7 Examples.

1. It is easy to verify that the 0-model $\Delta[0]$ satisfies the Kan condition, namely it is a Kan set.

2. The n-model $\Delta[n]$, $n > 0$ does not satisfy the Kan condition. For instance, take the 1-model $\Delta[1]$ and consider the horn $\Lambda[2,0]$ which consists of the edges $[0,2]$ and $[0,1]$ of $\Delta[2]$. Now take the simplicial morphism that maps $[0,1]$ and $[0,2] \in \Lambda[2,0]$ to $[0,1]$ and $[0,0] \in \Delta[1]$, respectively. There is a unique such simplicial map, since we specified what happens in the nondegenerate simplexes of $\Lambda[2,0]$. Observe that these functions are order preserving. Notice that this simplicial morphism cannot be extended to a map $\Delta[2] \longrightarrow \Delta[1]$, since the given prescriptions
$$0 \longmapsto 0\,, \quad 1 \longmapsto 1\,, \quad 2 \longmapsto 0$$
are order preserving as morphisms $\Lambda[2,0] \longrightarrow \Delta[1]$, but they are not so as morphisms $\Delta[2] \longrightarrow \Delta[1]$.

3. Any simplicial group (ignoring its group structure) satisfies the Kan condition. For a proof of this fact see [39, Theorem 2.2].

Recall the morphism of simplicial sets $\alpha_K : K \longrightarrow \mathcal{S}(|K|)$ given by $\alpha_{K\,n}(\sigma)(s) = [\sigma, s]$ and the continuous map $\rho_X : |\mathcal{S}(X)| \longrightarrow X$ given by $\rho_X([\sigma, s]) = \sigma(s)$. It is an immediate result that
$$\mathcal{S}(\rho_X) \circ \alpha_{\mathcal{S}(X)} = \mathrm{id}_{\mathcal{S}(X)}\,.$$
The following is an interesting result for the other composite.

2.4.8 **Proposition**. *For any space X, the composite*
$$\alpha_{\mathcal{S}(X)} \circ \mathcal{S}(\rho_X) : \mathcal{S}(|\mathcal{S}(X)|) \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{S}(|\mathcal{S}(X)|)$$
is homotopic to $\mathrm{id}_{\mathcal{S}(|\mathcal{S}(X)|)}$ relative to $\mathcal{S}(X)$.

Proof: By Proposition 2.4.6, the singular simplicial set $\mathcal{S}(X)$ is a Kan set. Thus there is a simplicial homotopy
$$H : \mathcal{S}(|\mathcal{S}(X)|) \times \Delta[1] \longrightarrow \mathcal{S}(|\mathcal{S}(X)|) \text{ rel } \mathcal{S}(X)$$
which starts with $\mathrm{id}_{\mathcal{S}(|\mathcal{S}(X)|)}$ and ends with the composite $\alpha_{\mathcal{S}(X)} \circ r$, where $r : \mathcal{S}(|\mathcal{S}(X)|) \longrightarrow \mathcal{S}(X)$ is some simplicial retraction. Hence the composite $\alpha_{\mathcal{S}(X)} \circ \mathcal{S}(\rho_X) \circ H$ is a simplicial homotopy relative to $\mathcal{S}(X)$, which starts with $\alpha_{\mathcal{S}(X)} \circ \mathcal{S}(\rho_X)$ and ends with $\alpha_{\mathcal{S}(X)} \circ \mathcal{S}(\rho_X) \circ \alpha_{\mathcal{S}(X)} \circ r = \alpha_{\mathcal{S}(X)} \circ r$, because $\mathcal{S}(\rho_X) \circ \alpha_{\mathcal{S}(X)} = \mathrm{id}_{\mathcal{S}(X)}$. Since $\mathcal{S}(|\mathcal{S}(X)|)$ is a Kan set too, the homotopy relation relative to $\mathcal{S}(X)$ is an equivalence relation on the set of all simplicial morphisms $\mathcal{S}(|\mathcal{S}(X)|) \longrightarrow \mathcal{S}(|\mathcal{S}(X)|)$. $\square$

In what follows we define the concept of a Kan fibration, which is a generalization of the notion of Kan set, and show that given a simplicial group Γ and a subgroup Λ, then the simplicial quotient map $\Gamma \twoheadrightarrow \Gamma/\Lambda$ is a Kan fibration with fiber Λ. In the definiton of a Kan fibration we shall not use the (explicit) concept of homotopy; later on we shall give characterizations that depend on homotopy.

2.4.9 Definition. Let $\pi : K \longrightarrow Q$ be a morphism of simplicial sets. We shall say that π is a *Kan fibration* if for every horn $T = (x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1})$ in K with a hole in the kth place (see 2.4.1), and for any n-simplex y of Q such that $d_i(y) = \pi(x_i)$, for $i \neq k$, there is an $n+1$-simplex x of K such that $d_i(x) = x_i$, $i \neq k$, and $\pi(x) = y$. We say that the horn T in K can be *filled* in K over y. The simplicial set K will be called the *total simplicial set*, Q the *base simplicial set*, and π the *projection*. If we take a 0-simplex y_0 seen as a simplicial subset of Q, then $K' = \pi^{-1}(y_0)$ is a simplicial subset of E which will be called the *fiber* of π.

If x_0 is a 0-simplex of the fiber F', we shall frequently refer to the sequence

$$(K', x_0) \overset{i}{\hookrightarrow} (K, x_0) \overset{\pi}{\twoheadrightarrow} (Q, y_0)$$

as *simplicial fiber sequence*. If there is no danger of confusion, we shall also write $K' \hookrightarrow K \twoheadrightarrow Q$ for this sequence.

The following is a convenient characterization of a Kan fibration.

2.4.10 **Proposition**. *A morphism of simplicial sets $\pi : K \longrightarrow Q$ is a Kan fibration if and only if given morphisms of simplicial sets $g : \Lambda[n,k] \longrightarrow K$ and $f : \Delta[n] \longrightarrow Q$ such that $f|_{\Lambda[n,k]} = \pi \circ g$, where $\Lambda[n,k]$ is the simplicial kth horn in $\Delta[n]$, there exists a morphism of simplicial sets $h : \Delta[n] \longrightarrow K$ which extends g and lifts f. In a diagram, we have:*

$$\begin{tikzcd} \Lambda[n,k] \arrow[r, "g"] \arrow[d, hook] & K \arrow[d, "\pi"] \\ \Delta[n] \arrow[r, "f"'] \arrow[ur, dashed, "h"] & Q . \end{tikzcd} \tag{2.4.3}$$

Proof: Clearly to have a morphism $g : \Lambda[n,k] \longrightarrow K$ is equivalent to having a horn $T = (x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1})$ in K; furthermore a morphism $f : \Delta[n] \longrightarrow Q$ is an n-simplex y in Q, and the commutativity of (2.4.3) means that $d_i(y) = \pi(x_i)$, for $i \neq k$. The extension $h : \Delta[n] \longrightarrow K$ determines an n-simplex x in K such that $d_i(x) = x_i$, $i \neq k$, and $\pi(x) = y$. Hence the horn T can be filled by x over y. $\square$

The following concept provides a system to construct Kan fibrations.

2.4.4 DEFINITION. Let Γ be a simplicial group and let K be a simplicial set. We say that Γ *acts* on K from the left, if there is a morphism of simplicial sets

$$\Gamma \times K \longrightarrow K\,, \qquad (g, x) \longmapsto gx\,,$$

such that

$$1x = x\,, \qquad (gh)x = g(hx) \quad \text{for all} \quad x \in K\,,\ \ g, h \in \Gamma\,.$$

Here 1 means the unit element in each group Γ_n. We thus have a *left action* of Γ on K. One can correspondingly define a *right action* $K \times \Gamma \longrightarrow K$.

Given a left (right) action of Γ on K, we say that two n-simplexes $x, x' \in K_n$ lie in the same *orbit* if $x' = gx$ ($x' = xg$) for some $g \in \Gamma_n$. This relation is clearly an equivalence relation and the quotient sets, denoted by $K_n \backslash \Gamma_n$ (K_n/Γ_n) build up a simplicial set $K\backslash\Gamma$ (K/Γ) called the *simplicial orbit set.* Each equivalence class is called *orbit.*

The action of Γ on K is said to be *free* if $gx = x \;\Rightarrow\; g = 1$, equivalently, the action is free if for each $x \in K_n$ the mapping $\Gamma_n \longrightarrow K_n$ given by $g \mapsto gx$ is injective for all n.

2.4.5 EXAMPLE. If Λ is a simplicial subgroup of Γ, then Λ acts on Γ by left (right) multiplication. The simplicial orbit set in this case is the homogeneous simplicial set $\Gamma\backslash\Lambda$ (Γ/Λ). See Example 2.1.3 2.

2.4.6 NOTE. If we take $\pi : K \longrightarrow *$, where $*$ means the one-point simplicial set, then π is a Kan fibration if and only if K is a Kan set.

2.4.7 **Proposition**. *Let $\pi : K \longrightarrow Q$ be a Kan fibration. Then its fiber K' is Kan set.*

2.4.8 EXERCISE. The collection

$$T = (x_0, \ldots, x_{i_1-1}, x_{i_1+1}, \ldots, x_{i_2-1}, x_{i_2+1}, \ldots, x_{i_r-1}, x_{i_r+1}, \ldots, x_n)$$

is a so-called *horn* with r holes in K, if it satisfies a compatibility condition similar to that of a horn with a hole. Let $\pi : K \longrightarrow Q$ be a Kan fibration. Show that if $y \in Q_n$ is such that T lies over the faces of y, then T may be filled by some $x \in K_n$ over y.

2.4.9 EXERCISE. Given a Kan fibration $\pi : K \longrightarrow Q$, show that K is a Kan set if and only if Q is a Kan set.

2.4.10 **Theorem**. *If a simplicial group Γ acts freely on a simplicial set K, then the orbit map $\pi : K \longrightarrow K\backslash G$ is a Kan fibration.*

Proof: Consider a horn $T = (x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$ in K and an n-simplex y of $K\backslash G$ such that $d_i(y) = \pi(x_i)$, for $i \neq k$.

First step. For $r = 0, 1, \dots, k-1$ we construct consecutively simplexes $t_r \in K_n$ in such a way that $\pi(t_r) = y$ and $d_i(t_r) = x_i$ for $0 \leq i \leq r$. Since π is surjective, there is an simplex $t_{-1} \in K_n$ such that $\pi(t_{-1}) = y$. If we assume that we already have constructed t_r for $r \leq k-2$, since $\pi d_{r+1}(t_r) = d_{r+1}(y) = \pi(x_{r+1})$ and the action is free, there is a unique $g_r \in \Gamma_n$ such that

$$g_r d_{r+1}(t_r) = x_{r+1}\,. \tag{2.4.11}$$

If we apply d_i with $i \leq r$ to this equation, then we obtain

$$d_i(g_r d_{r+1}(t_r)) = d_i(x_{r+1}) = d_r(x_i)\,. \tag{2.4.12}$$

Once more, since the action is free, we must have

$$d_i(g_r) = 1 \quad \text{ for } 0 \leq i \leq r\,. \tag{2.4.13}$$

We define $t_{r+1} = s_{r+1}(g_r t_r)$. Hence $\pi(t_{r+1}) = \pi(t_r) = y$ and from equations (2.4.11) and (2.4.12) it follows that $d_i(t_{r+1}) = x_i$ for $0 \leq i \leq r+1$.

Second step. For $q = 0, 1, \dots, n-k$ we construct consecutively simplexes $v_q \in K_n$ in such a way that $\pi(v_q) = y$ and $d_i(v_q) = x_i$ for $0 \leq i < k$ and $n - q < i \leq n$. We start setting $v_0 = t_{k-1}$. By induction, assume that v_q has been constructed already and we construct v_{q+1} as in the first step.

Third step. We then have that v_{n-q} is a filling of the given horn over y. In case that $k = 0$, we skip the first step and start with the second with a v_0 such that $\pi(v_0) = y$. □

By 2.4.5, an immediate consequence is the next.+

2.4.14 **Corollary**. *Given a simplicial group* Γ *and a subgroup* Λ*, then the simplicial quotient map* $\Gamma \twoheadrightarrow \Gamma\backslash\Lambda$ *is a Kan fibration with fiber* Λ. □

2.5 Simplicial abelian groups

In this section we shall study in certain detail the structure of simplicial abelian groups, namely the objects of the category $\mathfrak{SAb}$ introduced in 2.1.3, Example 2, above. We shall see two interesting properties. One is the fact that any simplicial abelian group can essentially be seen as a chain complex, while the other property is the fact that any surjective simplicial homomorphism is a Kan fibration, whose fiber is the kernel of the homomorphism.

2.5.1 DEFINITION. Let A be a simplicial abelian group, namely an object of $\mathfrak{SAb}$, and consider the face operators $d_i : A_n \longrightarrow A_{n-1}$, $0 \leq i \leq n$. Define the + *boundary operator*

$$\partial_n : A_n \longrightarrow A_{n-1} \quad \text{by} \quad \partial_n(a) = \sum_{i=0}^{n}(-1)^i d_i(a)\,.$$

2.5.2 **Proposition**. *The composite* $A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1}$ *is equal to zero.*

Proof: Take an element $a \in A_{n+1}$ and consider

$$\begin{aligned}
\partial_n\partial_{n+1}(a) &= \sum_{i=0}^{n}(-1)^i d_i \left(\sum_{j=0}^{n+1}(-1)^j d_j(a)\right)\\
&= \sum_{(i,j)=(0,0)}^{(n,n+1)} (-1)^{i+j} d_i d_j(a)\\
&= \sum_{i<j}(-1)^{i+j} d_i d_j(a) + \sum_{i\geq j}(-1)^{i+j} d_i d_j(a)\\
&= \sum_{i<j}(-1)^{i+j} d_{j-1} d_i(a) + \sum_{i\geq j}(-1)^{i+j} d_i d_j(a)\\
&= \sum_{k\geq l}(-1)^{k+l+1} d_k d_l(a) + \sum_{i\geq j}(-1)^{i+j} d_i d_j(a)\\
&= 0\,,
\end{aligned}$$

where we put $k = j - 1$ and $l = i$ and the fourth equality follows from the first simplicial identity 2.1.6 (b). □

Thus we have proved that $A_* = (A_n; \partial_n \mid n \in \mathbb{N} \cup \{0\})$ is a *chain complex* (see Chapter 5).

Assume that F is a simplicial abelian group and that $G \subseteq F$ is a simplicial subgroup, namely G is a simplicial group and for each n, the group G_n is a subgroup of F_n. Moreover, the inclusion homomorphisms $i : G_n \hookrightarrow F_n$ determine a homomorphism of simplicial groups $i : G \hookrightarrow F$. In this case, one can also define a simplicial quotient group F/G by $(F/G)_n = F_n/G_n$, i.e. the quotient group of cosets of G_n in F_n for each n. If $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in Δ then we take the homomorphism $\mu^{F/G} : (F/G)_n \longrightarrow (F/G)_m$ induced by the homomorphism $\mu^F : F_n \longrightarrow F_m$. The fact that $G \subset F$ is a simplicial subgroup guarantees that $\mu^{F/G}$ is well defined. The other property of the simplicial abelian groups which we want to exhibit was already proven in the last section. We put here the case of simplicial abelian groups, which follows immediately from 2.4.14.

2.5.3 **Proposition**. *Let F be a simplicial abelian group and let $G \subset F$ be a simplicial subgroup. Then the quotient morphism $\pi : F \twoheadrightarrow F/G$ is a Kan fibration.*
□

Chapter 3 Homotopy theory of simplicial sets

In many ways, simplicial sets behave very much like spaces. This is also the case with homotopy concepts.

3.1 Simplicial homotopy

The role usually played by the interval will be played here by the 1-model $\Delta[1]$, which has the nondegenerate 1-simplex $[0,1]$ and two nondegenerate 0-simplexes $[0]$ and $[1]$, while all other simplexes are degenerate. Each of these other simplexes has the form $[0,\ldots,0,1,\ldots,1]$ with possibly no 0s or no 1s. We shall abuse notation and write $[0]$ and $[1]$ for the degenerate simplexes $[0,0,\ldots,0]$ and $[1,1,\ldots,1]$, respectively.

3.1.1 Definition. Under a *path* in a simplicial set K we shall understand a simplicial morphism

$$\lambda : \Delta[1] \longrightarrow K\,.$$

Equivalently, a path in K is a 1-simplex λ of K. Given a path λ in K, $d_1 \circ \lambda = \lambda[0]$ is the *origin* of the path, and $d_0 \circ \lambda = \lambda[1]$ is its *end*.

Two 0-simplexes a and b of K are said to belong to *the same path component* of K if there is a path λ whose origin is a and whose end is b. We write this fact by $\lambda : a \simeq b$ or simply by $a \simeq b$, and say that λ is a path from a to b.

3.1.2 **Theorem**. *In a Kan set, the path connectedness relation $\simeq$ is an equivalence relation.*

Proof: We shall verify each of the axioms of an equivalence relation.

Reflexivity. For any vertex $[a]$, $s_0 \circ [a]$ is a path from a to a.

Transitivity. Given $\lambda_1 : a \simeq b$ and $\lambda_2 : b \simeq c$, define $\mu : \Delta[2,1] \longrightarrow K$ by letting μ take $[0,1]$ to λ_1 and $[1,2]$ to λ_2. The Kan condition allows to extend μ to $\mu' : \Delta[2] \longrightarrow K$. The composite $\lambda = \mu' \circ [0,2]$ is a path from a to c. See Figure 3.1

Symmetry. Take a path $\lambda : a \simeq b$. We need another path in the reversed direction. We consider the path as the edge $[0,1]$ of $\Delta[2]$. If the edge $[0,2]$ of $\Delta[2]$

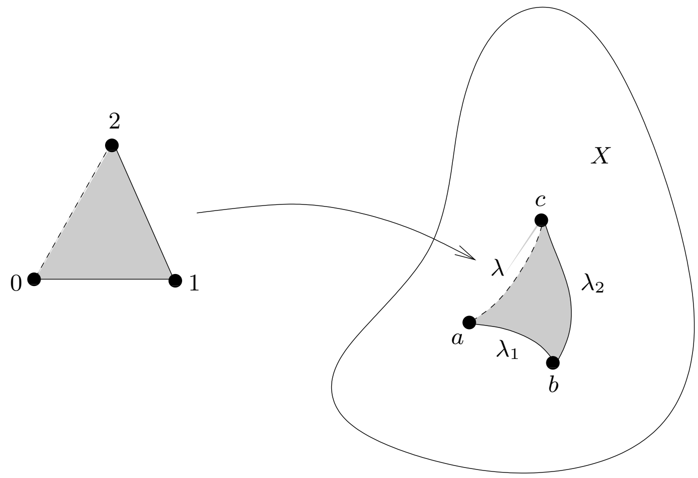

Figure 3.1 Transitivity

represents $s_0 \circ [a]$, which exists since K is a simplicial set. Notice that $d_0 \circ s_0 \circ [a] = d_1 \circ s_0 \circ [a] = [a]$. From here on, we label the three vertices of $[0, 1, 2]$ of $\Delta[2]$ as $[a, b, a]$. Hence we have a simplicial morphism defined on $\Lambda[2, 0]$ that takes $[0, 1]$ to λ and $[0, 2]$ to $s_0 \circ [a]$. Once more, by the Kan condition, we extend this morphism to all of $\Delta[2]$, and then $[1, 2]$ gets taken to a path μ from b to a. See Figure 3.2.□

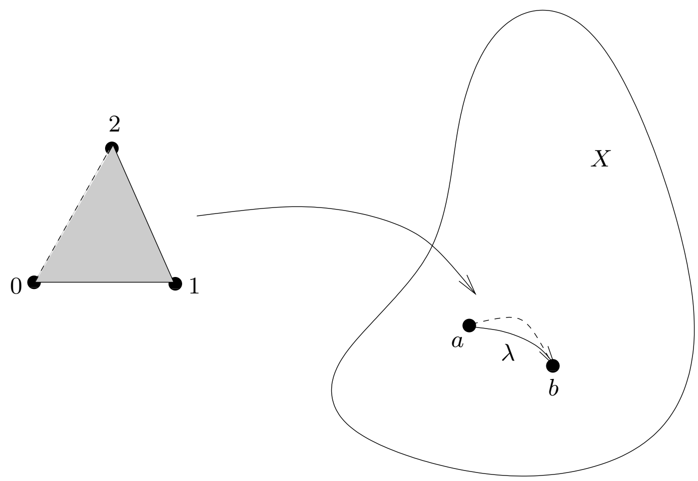

Figure 3.2 Symmetry

3.1.1 Homotopy of morphisms of simplicial sets

A special role in homotopy is played by the cylinders $K \times \Delta[1]$, since a homotopy in the simplicial case is a morphism $\eta : K \times \Delta[1] \longrightarrow Q$. Particularly important is the next.

3.1.3 DEFINITION. The product $\Delta[n] \times \Delta[1]$ is called *prism*. A q-simplex in the prism is of the form

$$((a_0, a_1, \dots, a_q), (\underbrace{0 \dots 0}_{i} 1 \dots 1))\,,$$

where $0 \le a_0 \le a_1 \le \dots \le a_q \le n$ are integers. We abbreviate by

$$(a_0, a_1, \dots, a_i, a'_{i+1}, \dots, a'_q) = ((a_0, a_1, \dots, a_q), (0 \cdots \overset{i}{0}\, 1 \dots 1)$$

(see Figure 3.3).

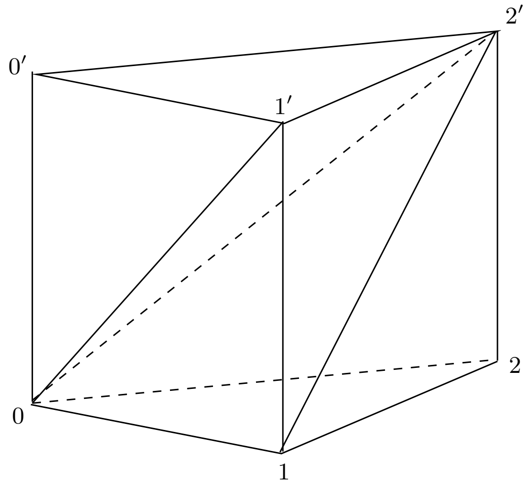

Figure 3.3 Prism $\Delta[2] \times \Delta[1]$

The prism $\Delta[n] \times \Delta[1]$ is generated by the nondegenerate $n+1$-simplexes

$$c_i = (0, 1, \dots, i, (i+1)', \dots, n')\,, \quad i = 0, 1, \dots, n\,.$$

One may easily verify the following

$$\begin{aligned} d_0(c_0) &= ([n], 1)\,, \quad d_{n+1}(c_n) = ([n], 0)\,, \\ d_{i+1}(c_i) &= d_{i+1}(c_{i+1})\,, \quad i = 0, 1, \dots, n-1\,, \\ d_j(c_i) &\in \dot{\Delta}[n] \times \Delta[1]\,, \quad i \ne j \ne i+1\,. \end{aligned}$$

3.1.4 NOTE. Since the prism $\Delta[n] \times \Delta[1]$ is generated by the simplexes c_i, each morphism of simplicial sets (simplicial homotopy) $\eta : \Delta[n] \times \Delta[1] \longrightarrow K$ is determined by its values $\eta(c_i)$. They can be arbitrarily given, provided that the equations

$$d_{i+1}\eta(c_i) = d_{i+1}\eta(c_{i+1})\,, \quad i = 0, 1, \dots, n-1$$

hold. If L and L' are simplicial subsets of $\Delta[n]$ or $\Delta[n] \times \Delta[1]$, then each morphism $\alpha : L \longrightarrow L'$ is uniquely determined by its values $\alpha(x)$ for all $x \in L_0$.

3.1.5 DEFINITION. Let K and K' be simplicial sets and $L \subset K$, $L' \subset K'$ be simplicial subsets. If $\varphi, \psi : K \longrightarrow K'$ are morphisms of simplicial sets such that

$\varphi(L), \psi(L) \subset L' \subset K'$, then we say that φ and ψ are *morphisms of pairs*. We say that φ and ψ are *homotopic* if there is a *simplicial homotopy* $\eta : \varphi \simeq \psi$, namely a morphism of simplicial sets $\eta : K \times \Delta[1] \longrightarrow K'$ such that $\eta(L \times \Delta[1]) \subset L'$, and its restrictions $\eta_0 : K = K \times [0] \longrightarrow K'$ and $\eta_1 : K = K \times [1]$ coincide with φ and ψ respectively.

3.1.6 Examples.

1. If $f, g : X \longrightarrow Y$ are homotopic maps of topological spaces, then the induced morphisms of the singular simplicial sets $\mathcal{S}(f), \mathcal{S}(g) : \mathcal{S}(X) \longrightarrow \mathcal{S}(Y)$ are simplicially homotopic. Namely if $H : X \times I \longrightarrow Y$ is a homotopy, then the composite
$$\mathcal{S}(X) \times \Delta[1] \xrightarrow{\mathrm{id}\times i} \mathcal{S}(X) \times \mathcal{S}(I) = \mathcal{S}(X \times I) \xrightarrow{\mathcal{S}(H)} \mathcal{S}(Y)$$
is a homotopy from $\mathcal{S}(f)$ to $\mathcal{S}(g)$. Notice that the morphism $i : \Delta[1] \longrightarrow \mathcal{S}(I)$ is given by
$$i_0(0) = \kappa_0\,; \quad i_0(1) = \kappa_1\,; \quad i_1([0,1]) = \alpha : \Delta^1 \longrightarrow I\,,\ \alpha((1-t)e_0 + te_1) = t\,.$$

2. The constant morphism $\kappa : \Delta[n] \longrightarrow \Delta[n]$ given by $(01\ldots n) \mapsto (00\ldots 0)$ and the identity morphism $\mathrm{id} : \Delta[n] \longrightarrow \Delta[n]$ are homotopic, i.e., $c \simeq \mathrm{id}$ via the homotopy
$$\begin{aligned} \eta : \Delta[n] \times \Delta[1] &\longrightarrow \Delta[n]\,, \\ (a_0, a_1, \ldots, a_i, a'_{i+1}, \ldots, a'_q) &\mapsto (0\ldots 0a_{i+1}\ldots a_q)\,. \end{aligned} \tag{3.1.7}$$
However $\mathrm{id} \not\simeq c$, since if there were a homotopy, it should be a morphism $\Delta[n] \times \Delta[1] \longrightarrow \Delta[n]$ such that $(1, 1') \mapsto (1, 0)$, where 0 and 1. But $(1, 0)$ is not a simplex of $\Delta[n]$. Thus the homotopy relation is not reflexive and thus it is not an equivalence relation.

3.1.8 **Theorem**. *Let $L \subset K$, $L' \subset K'$ be pairs of simplicial sets. Then we have the following:*

(a) *If K' is a Kan set, then the homotopy relation for morphisms $\varphi, \psi : K \longrightarrow K'$ is an equivalence relation.*

(b) *If L' and K' are Kan sets, then the homotopy relation for morphisms $\varphi, \psi : (K, L) \longrightarrow (K', L')$ is an equivalence relation.*

(c) *If A is a Kan set, then the homotopy relation for morphisms $\varphi, \psi : K \longrightarrow K'$ relative to L, i.e. the restriction of the homotopy to L is stationary, is an equivalence relation.*

3.2 Homotopy extension and lifting properties

We now come back to Kan fibrations under the light of the homotopy concept. We state a new concept.

3.2.1 Definition. Let $\pi : K \longrightarrow Q$ be a morphism of simplicial sets. We shall say that π has the *homotopy lifting property* (the HLP for short) for the pair of simplicial sets (L, L') if given the commutative diagram

$$\begin{tikzcd} L \times e \cup L' \times \Delta[1] \arrow[r, "\gamma"] \arrow[d, hook] & K \arrow[d, "\pi"] \\ L \times \Delta[1] \arrow[r, "\eta"'] \arrow[ur, dashed, "\widetilde{\eta}"] & Q \end{tikzcd} \tag{3.2.10}$$

of morphisms of simplicial sets (shown in solid arrows), where $e = [0]$ or $[1]$, there is a simplicial homotopy $\widetilde{\eta}$ (shown in dashed arrow) such that both triangles commute. Equivalently we may say that the pair (L, L') has the *homotopy extension property* (the HEP for short) for the morphism of simplicial sets $\pi : K \longrightarrow Q$.

The following provides a characterization of Kan fibrations.

3.2.11 **Theorem**. *The following assertions are equivalent:*

(a) $\pi : K \longrightarrow Q$ *is a Kan fibration.*

(b) $\pi : K \longrightarrow Q$ *is surjective and has the HLP for the pairs* $(\Delta[n], \dot{\Delta}[n])$, $n = 0, 1, \ldots$.

(c) $\pi : K \longrightarrow Q$ *is surjective and has the HLP for all pairs* (L, L') *of simplicial sets.*

Proof: Clearly (a) $\Rightarrow$ (b). Namely assume first that $e = [0]$ and consider the generating simplexes c_i of $\Delta[n] \times \Delta[1]$ (see 3.1.3). According to 3.1.3, $G(d_j(c_i))$ for $i \neq j \neq i+1$ and to 2.2.7, $G(d_{n+1}(c_n))$ are defined. One may choose $\widetilde{\eta}(c_n)$ so that it is a filling of the horn

$$(G(d_0(c_n)), G(d_1(c_n)), \ldots, G(d_{n-1}(c_n)), G(d_{n+1}(c_n))) \quad \text{over} \quad \eta(c_n)\,,$$

and then we may choose $\widetilde{\eta}(c_{n-1})$ to be a filling of

$$(G(d_0(c_{n-1})), G(d_1(c_{n-1})), \ldots, G(d_{n-2}(c_{n-1})), G(d_{n+1}(c_{n-1})))\,,$$

et cetera. Thus $d_i\widetilde{\eta}(c_i) = d_i\widetilde{\eta}(c_{i-1})$ for all i. By 3.1.4, $\widetilde{\eta}$ is a morphism of simplicial sets which obviously makes diagram (3.2.10) commute. In case that $e = [1]$ one begins with c_0, then c_1 and so on and proceeds analogously as above.

To see that (b) $\Rightarrow$ (c), one should proceed skeleton by skeleton inductively under the assumption that $\widetilde{\eta}^{n-1}$ has been already defined in such a way that the next diagram commutes:

(3.2.12)
$$\begin{tikzcd}
L \times e \cup L' \times \Delta[1] \arrow[r, "\gamma"] \arrow[d, hook] & K \arrow[dd, "\pi"] \\
L \times e \cup (L' \cup L^{n-1}) \times \Delta[1] \arrow[ur, dashed, "\widetilde{\eta}^{n-1}"] \arrow[d, hook] & \\
L \times \Delta[1] \arrow[r, "\eta"'] & Q
\end{tikzcd}$$

Next diagram (with the solid arrows) commutes:

(3.2.13)
$$\begin{tikzcd}
\Delta[n] \times e \cup \dot{\Delta}[n] \times \Delta[1] \arrow[r, "x\times\mathrm{id}"] \arrow[d, hook] & L \times e \cup (L' \cup L^{n-1}) \times \Delta[1] \arrow[r, "\widetilde{\eta}^{n-1}"] \arrow[d, hook] & K \arrow[d, "\pi"] \\
\Delta[n] \times \Delta[1] \arrow[r, "x\times\mathrm{id}"'] \arrow[urr, dashed, "\widetilde{\eta}_x"] & L \times \Delta[1] \arrow[r, "\eta"'] & Q
\end{tikzcd}$$

By (b) we can complete it with $\widetilde{\eta}_x$ (dashed arrow) making the triangles commute. Then we define

$$\widetilde{\eta}^n : L \times e \cup (L' \cup L^n) \times \Delta[1] \longrightarrow K$$

by

$$\widetilde{\eta}^n(x,t) = \begin{cases} \widetilde{\eta}^{n-1}(x,t) & \text{if } (x,t) \in L \times e \cup (L' \cup L^{n-1}) \times \Delta[1]\,, \\ \widetilde{\eta}_x([n],t)\,. & \end{cases}$$

for the nondegenerate $x \in L_n$. $\widetilde{\eta}^n$ is a morphism of simplicial sets for which diagram (3.2.12) with n instead of $n-1$ commutes.

Now we prove (c) $\Rightarrow$ (a). In the diagram

(3.2.14)
$$\begin{tikzcd}
\Delta[n] \times e \cup \Lambda[n,i] \times \Delta[1] \arrow[r, "\eta|"] \arrow[d, hook] & \Lambda[n,i] \arrow[r] \arrow[d, hook] & K \arrow[d, "\pi"] \\
\Delta[n] \times \Delta[1] \arrow[r, "\eta"'] \arrow[urr, dashed, "\widetilde{\eta}"] & \Lambda[n] \arrow[r] \arrow[ur, dashed, "\zeta"'] & Q
\end{tikzcd}$$

the right subdiagram is given (solid arrows) and we wish to find the dashed arrow ζ so that the two triangles commute. Take first $i > 0$ and we choose $e = [1]$ and define η according to 3.1.4 by $\eta(j) = j$ for all j, $\eta(j') = j$ for all $j \neq i-1$ and $\eta((i-1)') = i$. Then η is a morphism of simplicial sets for which $\eta(\Delta[n] \times [1] \cup \Lambda[n,i] \times \Delta[1]) \subset \Lambda[n,i]$. If $i = 0$, we choose $e = [0]$ and $\eta = \omega$ (see 3.1.7). In both cases, using (b), one can find a morphism of simplicial sets $\widetilde{\eta}$ which makes diagram (3.2.14) commutative. One defines ζ by $\zeta([n]) = \widetilde{\eta}([n], 1-e)$. $\square$

We finish this section stating a result which is the simplicial version of the topological fact that that the inclusion $A \hookrightarrow X$ of a subcomplex A of a CW-complex X has the homotopy extension property (see Definition 4.3.1). It follows from the previous result if we take $Q = \Delta[0]$.

3.2.15 **Theorem**. *A simplicial set K is a Kan set if and only if it has the following homotopy extension property. For $e = [0]$ or $[1]$ and every pair of simplicial sets (L, L'), given a morphism $\alpha : L \longrightarrow K$ and a simplicial homotopy $\eta : L' \times \Delta[1] \longrightarrow K$ such that $\eta|_{L'\times[0]} = \alpha|_{L'}$, there is a homotopy $\widetilde{\eta} : L \times \Delta[1] \longrightarrow K$ such that $\widetilde{\eta}|_{L'\times\Delta[1]} = \eta$ and $\widetilde{\eta}|_{L\times[0]} = \alpha$. In diagrams*

(3.2.16)
$$\begin{tikzcd} L' \arrow[r, "j_0"] \arrow[d, hook, "j"'] & L' \times \Delta[1] \arrow[d, "j\times\mathrm{id}"] \arrow[ddr, bend left, "\eta"] & \\ L \arrow[r, "j_0"'] \arrow[drr, bend right, "\alpha"'] & L \times \Delta[1] \arrow[dr, dashed, "\widetilde{\eta}"] & \\ & & K . \end{tikzcd} \qquad or \qquad \begin{tikzcd} L \times [0] \cup L' \times \Delta[1] \arrow[r, "(\alpha,\eta)"] \arrow[d, hook] & K . \\ L \times \Delta[1] \arrow[ur, dashed, "\widetilde{\eta}"'] & \end{tikzcd}$$

□

3.3 Simplicial homotopy groups

In this section we define the simplicial homotopy groups of a simplicial set and we construct the long exact sequence of simplicial homotopy groups determined by a Kan fibration. Given a pointed Kan set K, i.e., each set K_n has a distinguished base point $*$, there are two different definitions of $\pi_n(K)$. The first is the following. We wish to acknowledge here that for many of the proofs we were inspired by [16], [31], and [34].

3.3.1 Definition. Let K be a pointed Kan set. We consider the set

$$\pi_n(K) = [\dot{\Delta}[n+1], *; K, *] \quad n = 1, 2, \ldots$$

of homotopy classes of morphisms of pairs of simplicial sets, where $\dot{\Delta}[n+1]$ denotes the *boundary* of $\Delta[n+1]$, .

Notice that we require that K is a Kan set in order for homotopy to be an equivalence relation. For the other definition, we need some preparation.

3.3.2 Definition. Two n-simplexes $x, x' \in K_n$ are said to be *homotopic* if the following hold:

(a) $d_i(x) = d_i(x')$ for $0 \le i \le n$.

(b) There is a simplex $y \in X_{n+1}$ such that

(i) $d_n(y) = x$,

(ii) $d_{n+1}(y) = x'$, and

(iii) $d_i(y) = s_{n-1}d_i(x) = s_{n-1}d_i(x')$, $0 \le i \le n-1$.

In other words, such that

$$D(y) = (s_{n-1}d_0(x), \dots, s_{n-1}d_{n-1}(x), x, x')\,.$$

The previous definition means that the two homotopic simplexes x and x' have the same boundaries and that y acts as a homotopy between them relative to the boundary, in such a way that x and x' become two of the faces of y. The rest of the faces of y degenerate. See Figure 3.4.

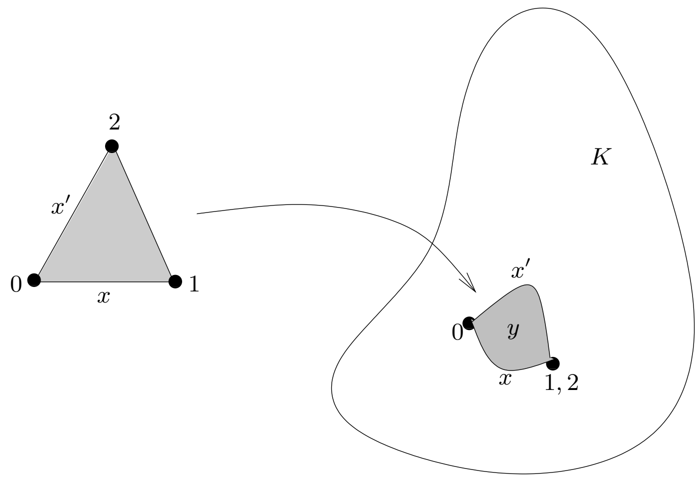

Figure 3.4

3.3.3 Exercise. Show that homotopy of simplexes is an equivalence relation for Kan sets. (*Hint:* Arrange a simplex in such a way that the known parts fall on certain faces of horns and the parts whose existence we want to show fall on the missing faces. The Kan extension condition allows these relations to exist.)

Now we can give the second definition of $\pi_n(K)$.

3.3.4 Definition. Let K be a pointed Kan set. We define $\pi_n(K)$ to be the set of equivalence classes of n-simplexes x of K such that $d_i(x) \in *$, $0 \leq i \leq n$, up to homotopy of simplexes.

There are two things to be done. One is to show that both definitions are equivalent. The second is how to endow $\pi_n(K)$ with a group structure. First we need some preparation.

3.3.5 **Lemma**. *Let K be a Kan set and assume that $d_i(x) = d_i(x')$ for all i. Then we obtain the same equivalence relation of Definition 3.3.2 if we instead require that $d_r(y) = x$, $d_{r+1}(y) = x'$ for some $0 \leq r \leq n$, and $d_i(y) = d_i s_r(x) = d_i s_r(x')$ for $i \neq r, r+1$. In other words, if*

$$D(y) = (s_{n-1}d_0(x), \dots, s_{n-1}d_{r-1}(x), x, x', s_{n-1}d_{r+2}(x), \dots, s_{n-1}d_n(x))\,.$$

Proof: A full proof is available in [34, Lemma 5.5]. We sketch here some ideas of it. It is a sort of induction, since one proves that for relevant values of r there is a y as in Definition 3.3.2, namely so that $d_r(y) = x$ and $d_{r+1}(y) = x'$, then there is some other y' such that $d_{r+1}(y') = x$ and $d_{r+2}(y') = x'$. One does this showing that there is an $n+2$-simplex z that contains the $n+1$-simplexes y and y' and then one uses the Kan extension property of K. Figure 3.5 shows this in small dimension.

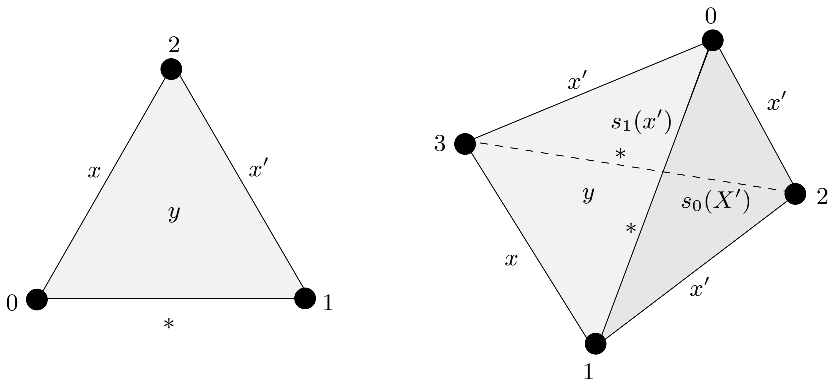

Figure 3.5

□

Given an n-simplex $x \in K_n$ of a simplicial set, there is a unique morphism of simplicial sets $\xi : \Delta[n] \longrightarrow K$ which maps the only nondegenerate n-simplex e_n of $\Delta[n]$ exactly to x. We say that ξ *represents* x. We have the next.

3.3.6 **Lemma**. *Let K be a Kan set. Then two n-simplexes x and x' in K are homotopic in the sense of 3.3.2 if and only if the morphisms ξ and ξ' which represent them are homotopic in the sense of 3.1.5.*

Proof: We assume first that x and x' are homotopic n-simplexes of K which are represented by morphisms $\xi, \xi' : \Delta[n] \longrightarrow K$. Hence there is an $n+1$-simplex $y \in K_{n+1}$ that connects x and x', for instance $d_n(y) = x$, $d_{n+1}(y) = x'$, and $d_i(y) \in *$ for all other values of i. We must define a morphism $\eta : \Delta[n] \times \Delta[1] \longrightarrow K$ which yields the desired homotopy. We already know that in the bottom and top η must be given by ξ and ξ', respectively. Now η must be such that the image of one of the $n+1$-simplexes yields y, let it be the last, and put $s_i(x)$ in the others (see Figure 3.6).

Conversely, assume that ξ and ξ' are homotopic relative to $*$ via $\eta : \Delta[n] \times \Delta[1] \longrightarrow K$. The image of η is a simplicial subset such that each of the nondegenerate $n+1$-simplexes of the prism has two n-faces which are not in $\dot{\Delta}[n] \times \Delta[1]$ and the rest are in $\dot{\Delta}[n] \times \Delta[0]$ which goes totally to $*$ in K. It is not difficult to verify that the two n-faces not in $\dot{\Delta}[n] \times \Delta[1]$ are consecutive faces. These faces

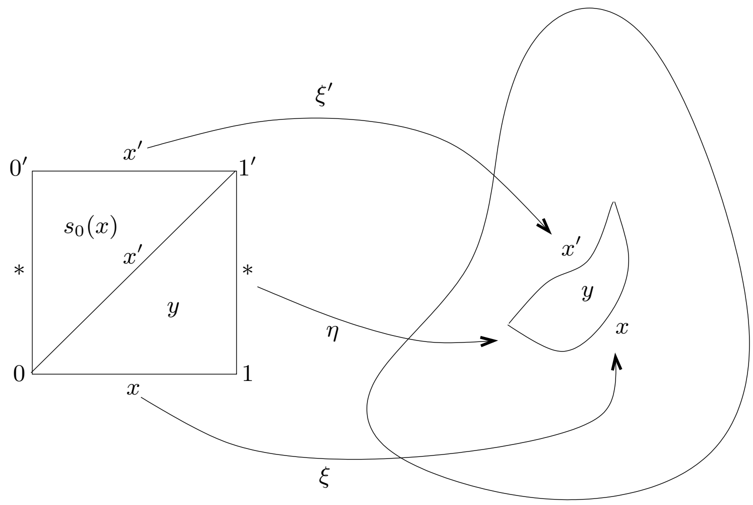

Figure 3.6

are consecutive homotopies which by the transitivity property of the homotopy relation imply the homotopy between x and x'. $\square$

In order to prove the equivalence between the two definitions of the homotopy groups, we need one more previous result, which is very familiar in the topological context.

3.3.7 **Lemma**. *Given a pointed Kan set K, there is a bijection*

$$[\Delta[n], \dot{\Delta}[n]; K, *] \longrightarrow [\dot{\Delta}[n+1], *; K, *] .$$

Proof: Take $\beta : (\Delta[n], \dot{\Delta}[n]) \longrightarrow (K, *)$, identify $\Delta[n]$ with the face $d_0\Delta[n+1]$, and define $\alpha : (\dot{\Delta}[n+1], *) \longrightarrow (K, *)$ by defining it by β on $d_0\Delta[n+1]$ and by the only morphism to $*$ on $d_i\Delta[n+1]$ for $i > 0$. Following the same procedure one can prove that a homotopy of β relative to $\dot{\Delta}[n]$ determines a homotopy of α relative to $*$. Thus we have a well-defined function

$$\Phi : [\Delta[n], \dot{\Delta}[n]; K, *] \longrightarrow [\dot{\Delta}[n+1], *; K, *] , \quad \Phi([\beta]) = [\alpha] .$$

Conversely, assume given a morphism $\alpha : (\dot{\Delta}[n+1], *) \longrightarrow (K, *)$. We shall show that there is a homotopy from α to a morphism α' which maps the horn $\Lambda[n+1, 0]$ to $*$. Once we have done this, we define β as $\alpha'|_{d_0\Delta[n+1]}$, modulo the obvious isomorphism.

As we did in 3.3.6, in order to construct a homotopy between two inclusions of k-simplexes x and x' in K, it is enough to find a a simplex y in K such that $d_k(y) = x$ and $d_{k+1}(y) = x'$, which can be taken as one of the blocks of a prism. The rest of it can be filled up with degeneracies of either x or x'.

The construction is by induction. The induction step is as follows. Assume that $\alpha_{k-1} : \dot{\Delta}[n+1] \longrightarrow K$ is such that $\alpha([0]) \in *$ and $\alpha(z) \in *$ for all simplexes $z \in \dot{\Delta}[n+1]$ of dimension $\leq k-1$ and such that $[0]$ is a vertex of z. Then there exists a homotopy from α_{k-1} to an α_k which takes all simplexes up to dimension k that have $[0]$ as a vertex to $*$. This homotopy can be assumed to be relative to the faces of dimension up to $k-1$ which have $[0]$ as a simplex.

Of course we can take $\alpha_0 = \alpha$, and assume that we have constructed α_{k-1} for $k \geq 1$. We have to define only the desired homotopy on the k-simplexes of $\Delta[n+1]$ which have $[0]$ as a vertex. Then we may apply the homotopy extension theorem 3.2.15. Hence we let z be a k-simplex of $\Delta[n+1]$ having 0 as a vertex. We know already that $\alpha_{k-1}d_i(z) \in *$ for all $i \neq 0$. Take the horn $\Lambda[k+1, 0]$ and observe that we can map it into K in such a way that the k-face corresponding to $d_{k+1}\Delta[k+1]$ is $\alpha_{k-1}(z)$ and that all other k-faces are mapped into $*$. This can be done since $\alpha_{k-1}d_i(z) \in *$ for all $i > 0$. Since K is a Kan set, we can extend the horn to a $k+1$-simplex y in K with the property that $d_{k+1}(y) = \alpha_{k-1}(z)$ and $d_k(y) \in *$. This is all we need to define a homotopy on z starting with $\alpha_{k-1}(z)$ and ending with with the only map from z into $*$. Moreover, this homotopy is relative to all faces of z for which $[0]$ is a simplex. These homotopies can be found for all such simplexes z independently and compatibly. This way, we obtain a homotopy on all k-simplexes of $\Delta[n+1]$ having $[0]$ as a vertex, which starts at α_{k-1} and ends with the map to $*$. The desired homotopy to α_k is now obtained extending this one using Theorem 3.2.15.

Inductively, we obtain a morphism $\alpha_{n+1} : \dot{\Delta}[n+1] \longrightarrow K$ which is homotopic to the given α and is such that the horn $\Lambda[n+1, 0]$ is mapped into $*$. We now define β as the restriction $\alpha_{n+1}|_{d_0\Delta[n+1]}$.

Now suppose that $\alpha, \alpha' : \dot{\Delta}[n+1] \longrightarrow K$ are homotopic relative to $*$. In order to show that the corresponding β and β' constructed as above, are homotopic, we must define a homotopy which starts with the homotopy $H : \dot{\Delta}[n+1] \times \Delta[1] \longrightarrow K$, $H : \alpha \simeq \alpha'$, and ends with a homotopy $H_{k+1} : \dot{\Delta}[n+1] \times \Delta[1] \longrightarrow K$ such that $H_{k+1}(\Lambda[n+1, 0] \times \Delta[1]) \in *$ and extends the homotopies constructed for α and α', as we did above. Then the restriction $H_{k+1}|_{d_0\Delta[n+1]\times\Delta[1]}$ will be the desired homotopy from β to β'. Thus the class $[\beta]$ depends only on the class $[\alpha]$ and so we have a well defined function

$$\Psi : [\dot{\Delta}[n+1], *; K, *] \longrightarrow [\Delta[n], \dot{\Delta}[n]; K, *]$$

which can be shown to be the inverse of Φ defined above. □

Now we can pass to the group structure of $\pi_n(K)$. Notice first that an element $[\alpha] \in \pi_n(K)$, which is given by a morphism of simplicial sets $\alpha : \Delta[n] \longrightarrow K$ such that $\alpha(\dot{\Delta}[n]) = *$ determines a unique n-simplex $a \in K_n$ such that $d_i(a) = *$ for all i.

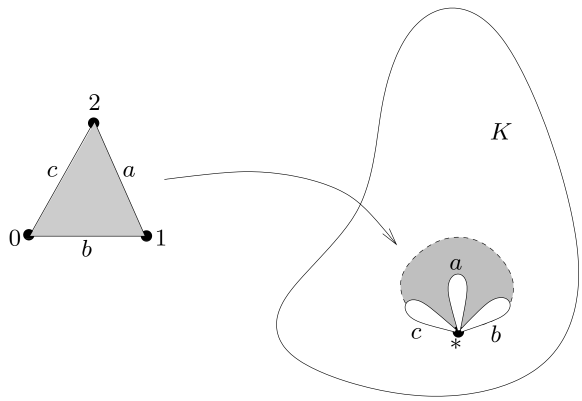

Figure 3.7

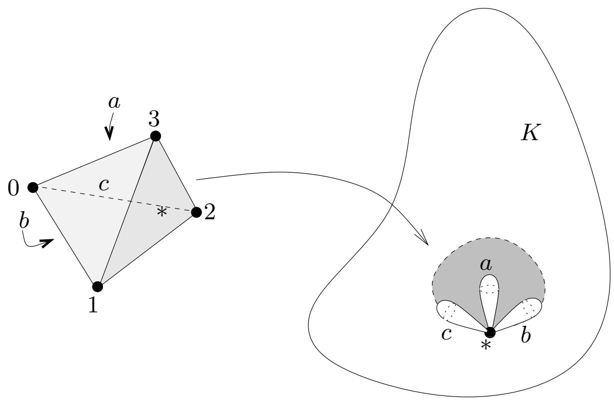

Figure 3.8

3.3.8 Definition. Let K be a Kan set. Given two simplexes $a, b \in K_n$ such that $d_i(a) = d_i(b) = *$ for all i, one may consider the horn $(*, \dots, *, a, -, b)$. Fill the horn with $v \in K_{n+1}$ and put $c = d_n(v)$, then c is such that $d_i(c) = *$ for all i. We can define the product of the elements of $\pi_n(K)$ represented by a and b by

$$[a] \cdot [b] = [c] \,.$$

Figures 3.7 and 3.8 illustrate this construction for $n = 1$ and $n = 2$ respectively.

3.3.9 **Proposition**. *The homotopy class of c depends only on the homotopy classes of a and b.*

Proof: Assume that $v' \in K_{n+1}$ also satisfies $d_{n-1}(v') = a$, $d_{n+1}(v') = b$, and otherwise $d_i(v') = *$. By the extension condition, there is a $w \in K_{n+2}$ such that $d_i(w) = *$ for $0 \leq i \leq n-2$, $d_{n-1}(w) = s_n d_{n-1}(v)$, $d_{n+1}(w) = v$ and $d_{n+2}(w) = v'$. Hence $d_{n-1}(w)$ is a homotopy from $d_n(v)$ and $d_n(v')$.

Assume now that b is homotopic to b', namely there is some $w \in K_{n+1}$ such that $d_i(w) = *$, $0 \leq i \leq n-1$, $d_n(w) = b'$ and $d_{n+1}(w) = b$. We can find $w' \in K_{n+1}$ such that $d_{n-1}(w') = a$, $d_{n+1}(w') = b'$, and $d_i(w') = *$, $0 \leq i \leq n-1$. It follows from the extension condition that there is a $u \in K_{n+2}$ such that $d_i(u) = *$, $0 \leq i \leq n-2$, $d_{n-1}(u) = s_{n-1}(a)$, $d_n(u) = w'$, and $d_{n+2}(u) = w$.

Hence $d_n d_{n+1}(u) = d_n(v')$, $d_{n+1}d_{n+1}(u) = b$, and $d_{n-1}(u) = a$. This means that we may take the same v for either b or b'. Analogously one may prove that the homotopy class of c is independent on how we choose a. □

3.3.10 **Proposition**. *With the operation defined above, $\pi_n(K)$ is a group if $n \geq 1$.*

Proof: Let $a, b \in K_n$ represent two elements of $\pi_n(K)$. By the extension property, there is $v \in K_{n+1}$ such that $d_i(z) = *$, $0 \leq i \leq n-1$, $d_n(v) = b$, and $d_{n+1}(v) = a$. Hence $[d_{n-1}(v)] \cdot [a] = [b]$. This proves divisibility on the left. Divisibility on the right is proven similarly. Taking $b = *$ one obtains left and right inverses of $[a]$.

If $a, b, c \in K_n$ represent three elements of $\pi_n(K)$, if we use the extension property, then we may choose $v_{n-1}, v_{n+1}, v_{n+2} \in K_{n+1}$ such that $d_i(v_\nu) = *$, $0 \leq i \leq n-2$ and $\nu = n-1, n+1, n+2$, $d_{n-1}(v_{n-1}) = a$, $d_{n+1}(v_{n-1}) = b$, $d_{n-1}(v_{n+1}) = d_n(v_{n-1})$, $d_{n+1}(v_{n+1}) = c$, $d_{n-1}(v_{n+2}) = b$, and $d_{n+1}(v_{n+2}) = c$. From these equalities, we obtain

$$\begin{aligned}([a] \cdot [b]) \cdot [c] &= [d_n(v_{n-1})] \cdot [c] = [d_{n-1}(v_{n+1})] \cdot [c] \\ &= [d_n(v_{n+1})] = [a] \cdot [d_n(w_{n+2})] = [a] \cdot ([b] \cdot [c])\,.\end{aligned}$$

Hence the operation is associative. □

The assignment $K \mapsto \pi_n(K)$ is a functor $\mathfrak{SSet} \longrightarrow \mathfrak{Grp}$. We have namely the next. Let $f : K \longrightarrow L$ be a morphism of pointed simplicial sets between Kan sets, and take $a \in K_n$ such that $d_i(a) = *$ for all i. Since f is a morphism of simplicial sets, $d_i f(a) = f d_i(a) = *$. If $a \simeq a'$, then by definition, there is an element $u \in K_{n+1}$ such that $d_n(u) = a$, $d_{n+1}(u) = a'$, and $d_i(u) = *$ for $0 \leq i < n$. Hence $d_n f(u) = f d_n(u) = f(a)$, $d_{n+1} f(u) = f d_{n+1}(u) = f(a')$, and $d_i f(u) = f d_i(u) = *$ for $0 \leq i < n$. Consequently $f(a) \simeq f(a')$ and so f defines a function

$$f_* : \pi_n(K) \longrightarrow \pi_n(L)\,.$$

3.3.11 **Proposition**. *The following hold:*

(a) *If $f_0, f_1 : K \longrightarrow L$ are homotopic pointed morphisms of simplicial sets, then $f_{0*} = f_{1*} : \pi_n(K) \longrightarrow \pi_n(L)$.*

(b) $(\mathrm{id}_K)_* = 1_{\pi_n(K)}$ *and if $f : K \longrightarrow L$ and $g : L \longrightarrow M$ are morphisms of simplicial sets, then $(g \circ f)_* = g_* \circ f_*$.*

(c) *If $f : K \longrightarrow L$ is a morphism of simplicial sets and $n \geq 1$, then $f_* : \pi_n(K) \longrightarrow \pi_n(L)$ is a homomorphism.*

Proof: We take the definition $\pi_n(K) = [\Delta[n], \dot{\Delta}[n]; K, *]$ for this. Therefore let $a : (\Delta[n], \dot{\Delta}[n]) \longrightarrow (K, *)$ be a morphism of simplicial sets. Then $f_0 \circ a, f_1 \circ a : (\Delta[n], \dot{\Delta}[n]) \longrightarrow (K, *)$ are homotopic. Hence

$$f_{0*}([a]) = [f_0 \circ a] = [f_1 \circ a] = f_{1*}([a])$$

and this proves (a).

Obviously $(\mathrm{id}_K)_* = 1_{\pi_n(K)}$ and on the other hand if $a : (\Delta[n], \dot{\Delta}[n]) \longrightarrow (K, *)$ is a morphism of simplicial sets, then

$$(gf)_*([a]) = [(g \circ f) \circ a] = [g \circ (f \circ a)] = g_* f_*([a])$$

and this proves (b).

Take $f : K \longrightarrow L$ and $a, b, c \in K_n$ are such that $d_i(a) = d_i(b) = d_i(c) = *$ for all i and $[c] = [a][b]$. Namely there is $u \in K_{n+1}$ such that

$$D(u) = (*, \ldots, *, a, c, b)\,.$$

Then $f(u) \in L_{n+1}$ is such that

$$D(f(u)) = (*, \ldots, *, f(a), f(c), f(b))\,.$$

Hence $[f(c)] = [f(a)][f(b)]$. In other words

$$f_*([a][b]) = f_*([a]) f_*([b])$$

and this proves (c). □

3.3.12 **Proposition**. *If $n \geq 2$, then the group $\pi_n(K)$ is abelian.*

Proof: Take elements $a, b, c, e \in K_n$. We shall give the proof in four steps.

First step. Assume that $v_{n+1} \in K_{n+1}$ is such that

$$D(v_{n+1}) = (*, \ldots, *, a, c, b, *)\,.$$

We show that $[a][b] = [c]$.

Choose $v_{n-1} \in K_{n+1}$ such that

$$d_n(v_{n-1}) = c\,, \quad d_{n+1}(v_{n-1}) = b\,, \quad d_i(v_{n-1}) = *\,, \quad 0 \leq i \leq n-2\,.$$

Define $c' = d_{n-1}(v_{n-1})$ and $v_i = *$ if $0 \leq i < n-2$, $v_{n-2} = s_n(b)$, and $v_{n+2} = s_{n-2}(b)$. Then the v_is satisfy Kan's extension condition, namely there is an $r \in K_{n+2}$ such that $d_i(r) = v_i$ for all $i \neq n$ and we may put $v_n = d_n(r)$. Then

$$D(v_n) = (*, \ldots, *, c', a, *)\,.$$

Hence $[c'][*] = [a]$. On the other hand, the choice of v_{n-1} implies $[c'][b] = [c]$, thus $[a][b] = [c]$.

Second step. Now assume that $v_n \in K_{n+1}$ satisfies

$$D(v_n) = (*, \dots, *, b, *, a, e)\,.$$

We show that $[b][a] = [e]$.

Choose $v_{n-1} \in K_{n+1}$ such that

$$d_i(v_{n-1}) = *\,, \quad 0 \le i < n-2\,, \quad d_{n-2}(v_{n-1}) = b\,, \quad d_{n-1}(v_{n-1}) = *\,, \quad d_{n+1}(v_{n-1}) = *\,.$$

Put $c' = d_n(v_{n-1})$ and let $v_i = *$ if $0 \le i < n-2$ and let $v_{n-2} = s_{n-2}(b)$ and $v_{n+2} = s_n(e)$. Then the v_is satisfy Kan's extension condition, namely there is an $r \in K_{n+2}$ such that $d_i(r) = v_i$ for all $i \neq n+1$ and we may put $v_{n+1} = d_{n+1}(r)$.

$$D(v_{n+1}) = (*, \dots, *, c', a, e)$$

hence $[c'][e] = [a]$. On the other hand, the choice of v_{n-1} and the first step imply $[c'][b] = [*]$, so that $[b] = [c']^{-1}$ and so $[b][a] = [c']^{-1}[a] = [e]$.

Third step. Suppose that $v_{n+2} \in K_{n+1}$ satisfies

$$D(v_{n+2}) = (*, \dots, *, b, c, a, e)\,.$$

We show that $[b]^{-1}[c][e] = [a]$.

Choose $v_{n-2} \in K_{n+1}$ with faces $d_i(v_{n-2}) = *$, $i \neq n-2,\, n+1$, $d_{n+1}v_{n+2}) = b$ and $d_{n-2}(v_{n-2}) = c'$. Take another element $v_{n-1} \in K_{n+1}$ such that $d_i(v_{n-1}) = *$, $0 \le i < n-2$ or $i = n-1$, and $d_{n-2}(v_{n-1}) = c'$, $d_{n+1}(v_{n-1}) = c$, and set $b' = (v_{n-1})$. Let $v_i = *$, $0 \le i < n-2$, $v_n = s_n(a)$. Then the v_i satisfy the extension connection, i.e. $d_i(r) = v_i$, $i \neq n+1$. Set $v_{n+1} = d_{n+1}(r)$. By the second step, $[c'] = [b]$ and $[c'][b'] = [c]$. On the other hand, $D(v_{n+1}) = (*, \dots, *, b', a, e)$. Therefore $[b'][e] = [a]$. Combining we obtain $[b]^{-1}[c][e] = [a]$.

Fourth step. In the third step put $e = *$. Hence $[b]^{-1}[c] = [a]$ and applying the first step to v_{n+2} of the third step in this case, it results that $[a] = [c][b]^{-1}$. Thus for every $[b]$ and $[c]$, $[b]^{-1}[c] = [c][b]^{-1}$, and this implies the desired result. □

3.4 Long exact sequence of a Kan fibration

In this section we shall derive the long exact sequence of a Kan fibration $p : K \longrightarrow Q$ with fiber M. We start by defining the connecting homomorphism as follows.

3.4.1 Definition. Take $a \in Q_n$ such that $d_i(a) = *$ for all i. By the Kan condition for a fibration, there is an $a' \in K_n$ such that $p(a') = a$ and $d_i(a') = *$ if $1 \le i \le n$. Hence $d_0(a') \in M_{n-1}$ and $d_i d_0(a') = *$ for all i. The homotopy class

$[d_0(a')] \in \pi_{n-1}(M)$ depends only on the homotopy class $[a] \in \pi_n(Q)$, namely if $[b] = [a]$, we take $b' \in K_n$ such that $p(b') = b$ and $d_i(b') = *$ if $1 \leq i \leq n$. Since a and b are homotopic, there is a $u \in Q_{n+1}$ such that $D(u) = (*, a, b, *, \ldots, *)$. By the Kan-fibration condition, there is a $u' \in K_{n+1}$ such that $p(u') = u$ and $D(u') = (d_0(u'), a', b', *, \ldots, *)$. Then $d_0(u') \in M_n$ and

$$D(d_0(u')) = (d_0(a'), d_0(b'), *, \ldots, *)\,.$$

Hence $[d_0(a')] = [d_0(b')]$. Therefore we have a well-defined function

$$\partial : \pi_n(Q) \longrightarrow \pi_{n-1}(M) \quad \text{given by} \quad [a] \mapsto [d_0(a')]\,.$$

3.4.2 **Proposition**. *The function $\partial : \pi_n(Q) \longrightarrow \pi_{n-1}(M)$ is a natural homomorphism.*

Proof: Recall that in the case of $n = 1$, the set $\pi_0(M)$ regularly has no group structure. In this case what one can expect is that $\partial([*]) = [*]$, which is indeed clearly true. Thus we assume that $n \geq 2$. Take $a, b, c \in Q_n$ such that $D(a) = D(b) = D(c) = *$ and $[a] + [b] = [c]$. This means that there is $u \in Q_{n+1}$ such that $D(u) = (*, b, c, a, *, \ldots, *)$. By the Kan-fibration condition, there is an element $u' \in K_{n+1}$ such that $p(u') = u$ and $D(u') = (d_0(u'), b', c', a', *, \ldots, *)$, where the elements a', b', c' are to a, b, c as in Definition 3.4.1 a' is to a. Then $d_0(u') \in M_{n-1}$ and

$$D(d_0(u')) = (d_0(b'), d_0(c'), d_0(a'), *, \ldots, *)\,.$$

Hence, by Definition 3.4.1,

$$\partial([a]) + \partial([b]) = [d_0(a')] + [d_0(b')] = [d_0(c')] = \partial([c])$$

and thus ∂ is a homomorphism.

3.4.3 **Theorem**. *Given a Kan fibration $p : K \longrightarrow Q$ with fiber M, there is a natural long exact sequence*

$$\cdots \longrightarrow \pi_n(M) \longrightarrow \pi_n(K) \longrightarrow \pi_n(Q) \xrightarrow{\partial} \pi_{n-1}(M) \longrightarrow \cdots$$
$$\longrightarrow \pi_1(Q) \xrightarrow{\partial} \pi_0(M) \longrightarrow \pi_0(K) \longrightarrow \pi_0(Q) \longrightarrow 0\,,$$

where exactness in the nongroup part simply means that the image of a function coincides with the inverse image of the base point under the next function.

Proof: Consider elements $a \in M_n$, $b \in K_n$, and $c \in Q_n$ such that $D(a) = D(b) = *$ and $D(c) = *$ and let $i : M \hookrightarrow K$ be the inclusion.

(a) We clearly have that $p_* \circ i_* = 0$ since $M = p^{-1}(*)$ and thus $p \circ i = *$.

(b) We have that $\partial \circ p_* = 0$ since $\partial p_*([b]) = [d_0(b)] = [*]$.

(c) One has $i_* \circ \partial = 0$. Namely, let $c' \in K_n$ be to $c \in Q_n$ as a' is to a in 3.4.1, namely $p(c') = c$ and $D(c') = (d_0(c'), *, \dots, *)$. Then $i_*\partial([c]) = [d_0(c')] \in \pi_{n-1}(K)$. But by the form of $D(c')$ it follows that $d_0(c)$ and $*$ are homotopic in K. Thus $i_*\partial([c]) = 0$.

(d) Now suppose that $i_*([a]) = 0$. This means that $a \in M_n$ is homotopic to $*$ in K. Hence there exists $u \in K_{n+1}$ such that $D(u) = (a, *, \dots, *)$, and so $D(p(u)) = *$ and $\partial([p(u)]) = [a]$, therefore $[a] \in \operatorname{im}(\partial)$. This together with (c) shows the exactness at $\pi_n(M)$.

(e) If $p_*([b]) = 0$, then there is $u \in Q_{n+1}$ such that $D(u) = (*, p(b), *, \dots, *)$. To such a u there is $u' \in K_{n+1}$ such that $p(u') = u$ and $D(u') = (d_0(u'), b, *, \dots, *)$. Hence $d_0(u')$ lies in M_n, $D(d_0(u')) = *$, and $d_0(u')$ is homotopic to b in K. Thus $i_*([d_0(u')]) = [b]$, and so $[b] \in \operatorname{im}(i_*)$. This together with (a) shows the exactness at $\pi_n(K)$.

(f) Assume $\partial([c]) = 0$. Let again $c' \in K_n$ be to $c \in Q_n$ as a' is to a in 3.4.1, namely such that $p(c') = c$ and $D(c') = (d_0(c'), *, \dots, *)$, but also such that $[d_0(c')] = 0$. Hence there is a $u \in M_n$ such that $D(u) = (*, d_0(c'), *, \dots, *)$. By the Kan-fibration condition, there is a $v \in K_{n+1}$ such that $p(v) = s_1(c)$ and $D(v) = (u, d_1(v), c', *, \dots, *)$. Therefore $D(d_1(v)) = *$ and $p(d_1(v)) = c$. Hence we have the equality $p_*([d_1(v)]) = [c]$, which together with (b) yields exactness at $\pi_n(Q)$. $\square$

Chapter 4 Fibrations, cofibrations, and homotopy groups

In this chapter we shall study fibrations and cofibrations in the topological case, and the exact sequences of homotopy groups determined by them.

For the purposes of this chapter, we shall understand by the *n-ball* $\mathbb{B}^n$ the unit n-cube I^n and the *$n-1$-sphere* $\mathbb{S}^{n-1}$ the boundary ∂I^n of the unit n-cube and via a canonical homeomorphism, also the quotient $I^{n-1}/\partial I^{n-1}$.

4.1 Topological fibrations

In this section we study the homotopy lifting property for different families of spaces and we analyze the special case of Serre and Hurewicz fibrations.

4.1.1 Definition. Let $\mathcal{C}$ be some class of k-spaces. We say that a map $p : E \longrightarrow X$ has the *homotopy lifting property* for the class $\mathcal{C}$ (the $\mathcal{C}$-HLP for short) or that it is a *$\mathcal{C}$-fibration* if given a homotopy $H : Y \times I \longrightarrow X$ and a map $h : Y \longrightarrow E$ such that $pf(y) = H(y, 0)$, for any space Y in $\mathcal{C}$, there is a lifting $\widetilde{H} : Y \times I \longrightarrow E$ such that $p\widetilde{H}(y, t) = H(y, t)$ and $\widetilde{H}(y, 0) = h(y)$. In a diagram

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "j_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & X\,. \end{tikzcd}$$

If $\mathcal{C}$ is the class of all unit balls $\mathbb{B}^n$, then we say that p is a *Serre fibration*. If $\mathcal{C}$ is the class of all k-spaces, then we say that p is a *Hurewicz fibration*

4.1.2 Exercise. Show that a Serre fibration has the homotopy lifting property for the class of all CW-complexes.

Assume that $p : E \longrightarrow X$ is a Hurewicz fibration and define $E \times_X M(I, X) = \{(e, \sigma) \in E \times M(I, X) \mid p(e) = \sigma(0)\}$. Consider the following commutative diagram:

$$\begin{tikzcd} E \times_X M(I, X) \arrow[r, "h"] \arrow[d, "j_0"'] & E \arrow[d, "p"] \\ E \times_X M(I, X) \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & X\,, \end{tikzcd}$$

where $h(e,\sigma) = e$ and $H(e,\sigma,t) = \sigma(t)$. Since all spaces involved are k-spaces, there is a solution $\widetilde{H} : E \times_X M(I,X) \times I \longrightarrow E$. If we take the adjoint $\Gamma_p : E \times_X M(I,X) \longrightarrow M(I,E)$ this map has the following properties:

$$\Gamma_p(e,\sigma)(0) = e \qquad \text{and} \qquad p \circ \Gamma_p(e,\sigma) = \sigma\,.$$

Namely, the map Γ_p at the point (e,σ) lifts the path σ starting at e,

4.1.3 Definition. The map $\Gamma_p : E \times_X M(I,X) \longrightarrow M(I,E)$ is called the *path-lifting map* for the Hurewicz fibration $p : E \longrightarrow X$.

The existence of a path-lifting map characterizes Hurewicz fibrations.

4.1.4 **Theorem**. *A map $p : E \longrightarrow X$ is a Hurewicz fibration if and only if it has a path-lifting map $\Gamma_p : E \times_X M(I,X) \longrightarrow M(I,E)$.*

Proof: We already saw that a Hurewicz fibration p has a path-lifting map Γ_p.

Assume conversely that p has a path lifting map Γ_p. Given a homotopy $H : Y \times I \longrightarrow X$ and a map $h : Y \longrightarrow E$ such that $pf(y) = H(y,0)$, consider the adjoint map to H, $\widehat{H} : Y \longrightarrow M(I,X)$. Define $K : Y \longrightarrow M(I,E)$ by $K(y) = \Gamma_p(h(y), \widehat{H}(y))$. Since $p(h(y)) = H(y,0) = \widehat{H}(y)(0)$, the map K is well defined. Defining $\widetilde{H} : Y \times I \longrightarrow Y$ to be the adjoint map to K, $\widetilde{H}$ is the desired lifting. Thus p is a Hurewicz fibration.

4.1.5 Examples.

1. A *trivial bundle*, defined as the projection on the first factor $p : X \times F \longrightarrow X$, is a Hurewicz fibration. Namely define a PLM by $\Gamma((x,y),\sigma) = (\sigma, \kappa_y)$, where κ_y is the constant path with value y. In other words, if a point $(x,y) \in X \times F$ and a path σ in X are such that $x = \sigma(0)$, then the path $t \mapsto (\sigma(t), y) \in X \times F$ defines a continuous PLM.

2. Let X be a pointed k-space and put $PX = \{\sigma : I \longrightarrow X \mid \sigma(0) = x_0\} \subset M(I,X)$ and let $\pi : PX \longrightarrow X$ be given by $\pi(\sigma) = \sigma(1)$. Define $\Gamma : PX \times_X M(I,X) \longrightarrow M(I,E)$ by

$$\Gamma(\omega,\sigma)(t)(s) = \begin{cases} x_0 & \text{if } 4s \leq t\,, \\ \omega(\frac{4s-t}{4-2t}) & \text{if } t \leq 4s \leq 4-t\,, \\ \sigma(\frac{4s+t-4}{2s-1}) & \text{if } 4-t \leq 4s\,. \end{cases}$$

One easily shows that the map is well defined and continuous as a function of ω and σ. Furthermore, $\Gamma(\omega,\sigma)(0)(s) = \omega(s)$, so that the path $\Gamma(\omega,\sigma)$ starts at ω, and $\pi\Gamma(\omega,\sigma)(t) = \Gamma(\omega,\sigma)(t)(1) = \sigma(t)$, so that $\pi \circ \Gamma(\omega,\sigma) = \sigma$. Hence Γ is a PLM for π and so π is a Hurewicz fibration. It is the so-called *path fibration* of X. Its total space PX is called the *path space* and its fiber $\Omega X = \pi^{-1}(b_0)$ is called the *loop space* of X. Notice that PX is contractible, namely the homotopy $H : PX \times I \longrightarrow PX$ given by $H(\sigma,t) = \sigma_{1-t}$, where $\sigma_{1-t}(s) = \sigma((1-t)s)$ is a contraction of PX to the constant path with value x_0.

3. Let X be a pointed k-space and let now $\pi : M(I,X) \longrightarrow X$ be given by $\pi(\sigma) = \sigma(0)$. Similarly as in Example 2, it is an *exercise* to construct a PLM for π and thus show that it is also a Hurewicz fibration.

4. Given any continuous map $f : X \longrightarrow Y$, the *mapping path space* of f is defined as
$$E_f = \{(x,\alpha) \in X \times M(I,Y) \mid \alpha(1) = f(x)\}\,.$$
The map $\pi : E_f \longrightarrow Y$ given by $p(x,\alpha) = \alpha(0)$ is a Hurewicz fibration, whose fiber $\pi^{-1}(y_0) = P_f = \{(x,\alpha) \in X \times M(I,Y) \mid \alpha(0) = y_0,\ \alpha(1) = f(x)\}$, is the so-called *homotopy fiber* of the pointed map f. The map $i : X \longrightarrow E_f$ given by $i(x) = (x, \kappa_{f(x)})$ is a homotopy equivalence with the nice property that the following is a commutative diagram:
$$\begin{tikzcd} & E_f \arrow[d, "\pi"] \\ X \arrow[ur, "i"] \arrow[ur, phantom, "\simeq"', near end, shift right=2] \arrow[r, "f"'] & Y \end{tikzcd}$$
which means that one can replace any continuous map, up to a homotopy equivalence, by a Hurewicz fibration.

4.1.6 DEFINITION. Let $p : E \longrightarrow X$ be a Hurewicz fibration with fiber $F = p^{-1}(x_0)$, and let $\Gamma_p : E \times_X M(I,X) \longrightarrow M(I,E)$ be a path-lifting map for p. Let $\Omega X \subset M(I,X)$ be the *loop space* of X, namely the subspace of *loops* $\lambda : I \longrightarrow X$ such that $\lambda(0) = \lambda(1) = x_0$ (see above). Given $\lambda \in \Omega X$, the path $\Gamma_p(e_0,\lambda)$ is well defined and has the property that $p\Gamma(e_0,\lambda)(1) = \lambda(1) = x_0$, i. e., $\Gamma(e_0,\lambda)(1) \in F$. The map $\theta : \Omega X \longrightarrow F$ given by $\theta(\lambda) = \Gamma(e_0,\lambda)(1)$ is called the *holonomy* of p.

4.2 LOCALLY TRIVIAL BUNDLES AND COVERING MAPS

A very important special case of fibrations is the following.

4.2.1 DEFINITION. A map $p : E \longrightarrow X$ is called a *locally trivial bundle* with *fiber* F if every point $x \in X$ has a neighborhood $U \subset X$ such that there is a homeomorphism $\varphi_U : U \times F \longrightarrow p^{-1}U$ making the triangle
$$\begin{tikzcd} U \times F \arrow[rr, "\varphi_U"] \arrow[dr, "\pi"'] & & p^{-1}U \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$
commute, where $p_U = p|p^{-1}U : p^{-1}U \longrightarrow U$ and where π is the projection onto U. From this commutative diagram we get that φ_U can be restricted to a homeomorphism of $\pi^{-1}(x) = \{x\} \times F \approx F$ onto $p^{-1}(x)$ for all $x \in U$. Because of this we say that the fiber is F. The open cover of such sets U is called a *trivializing cover* of the bundle, and the maps φ_U *trivializing maps*.

4.2.2 EXAMPLE. If we can take $U = X$, that is, if $E \approx X \times F$, then we have a *trivial bundle*. In particular, if $E = X \times F$, then $p = \mathrm{proj}_X$ is a trivial bundle.

A very important special case of a locally trivial bundle is given as follows.

4.2.3 DEFINITION. A locally trivial bundle $p : E \longrightarrow X$ whose fiber F is a discrete space is called a *covering map*. In particular, a covering map always is a local homeomorphism. Figure 4.1 shows what a covering map looks like locally.

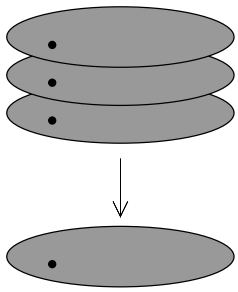

Figure 4.1

4.2.4 **Proposition**. *Assume that X is connected. Then a map $p : E \longrightarrow X$ is a covering map if and only if the following condition holds:*

> *There is an open cover $\mathcal{U}$ of X such that every $U \in \mathcal{U}$ is* evenly covered *by p, i.e. the inverse image*
> $$p^{-1}U = \coprod_{i \in \mathfrak{I}} \widetilde{U}_i \,,$$
> *and for each $i \in \mathfrak{I}$, $p|_{\widetilde{U}_i} : \widetilde{U}_i \longrightarrow U$ is a homeomorphism.*

The *proof* is an easy exercise. Noteworthy is the fact that the connectedness of X implies that all fibers $p^{-1}(x)$ are equivalent sets. Otherwise, given a fixed fiber $p^{-1}(x_0)$, the sets $A = \{x \in X \mid p^{-1}(x) \equiv p^{-1}(x_0)\}$ and $B = \{x \in X \mid p^{-1}(x) \not\equiv p^{-1}(x_0)\}$ are open disjoint nonempty sets that cover X. $\square$

4.2.5 **Theorem**. *Every locally trivial bundle is a Serre fibration.*

Proof: Let $p : E \longrightarrow X$ be a locally trivial bundle. We have to prove that for every commutative square

$$\begin{tikzcd} I^q \arrow[r, "f"] \arrow[d, "j_0"'] & E \arrow[d, "p"] \\ I^q \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & X \end{tikzcd}$$

there exists $\widetilde{H} : I^q \times I \longrightarrow E$ such that $p \circ \widetilde{H} = H$ and $\widetilde{H} \circ j_0 = f$. For each point $x \in H(I^q \times I)$ there exists a neighborhood $U(x)$ of x such that $p_{U(x)}$ is trivial, and so there exists a homeomorphism $\varphi_{U(x)} : F \times U(x) \longrightarrow E_{U(x)} = p^{-1}U(x)$. Since $H(I^q \times I)$ is compact, we can cover it with a finite number of such neighborhoods $U(x)$, say $U_1, \dots, U_k$. Since $I^q \times I$ is a compact metric space, there exists a number $\varepsilon > 0$, called the *Lebesgue number* of the cover $\{H^{-1}(U_i)\}$, such that every subset of diameter less than ε is contained in some $H^{-1}(U_i)$. Therefore, we can subdivide I^q into subcubes and take numbers $0 = t_0 < t_1 < \cdots < t_m = 1$ in such a way that if c is an n-face, then the image of $c \times [t_j, t_{j+1}]$ under H lies in some U_i. (Note that the 0-faces are vertices, the 1-faces are edges, etc.) Suppose that we have constructed $\widetilde{H}$ on $I^q \times [t_0, t_j]$. Then we shall construct $\widetilde{H}$ on $I^q \times [t_j, t_{j+1}]$ by defining it on each n-subface, using induction on n.

If c is a 0-face, then we pick some U_i such that $H(c \times [t_j, t_{j+1}]) \subset U_i$. Since $p\widetilde{H}(c, t_j) = H(c, t_j)$, we then have $\widetilde{H}(c, t_j) \subset E_{U_i}$. We define

$$\widetilde{H}(c,t) = \varphi_{U_i}(H(c,t), \mathrm{proj}_F \varphi_{U_i}^{-1}(\widetilde{H}(c,t_j))) \quad \text{for} \quad t \in [t_j, t_{j+1}]\,.$$

This is well defined and continuous.

Assume that we have already constructed $\widetilde{H}$ on $\widetilde{c} \times [t_j, t_{j+1}]$ for every face $\widetilde{c}$ of dimension less than n and let c be an n-face. Let us then pick some U_i such that $H(c \times [t_j, t_{j+1}]) \subset U_i$. By hypothesis $\widetilde{H}$ is defined on $c \times \{t_j\} \cup \partial c \times [t_j, t_{j+1}]$. Clearly, there exists a homeomorphism of $c \times [t_j, t_{j+1}]$ to itself that sends $c \times \{t_j\} \cup \partial c \times [t_j, t_{j+1}]$ onto $c \times \{t_j\}$, and so using Example 1 of 4.1.5 we can complete the diagram

$$\begin{tikzcd}
c \times \{t_j\} \cup \partial c \times [t_j, t_{j+1}] \arrow[r, "\varphi_i^{-1} \circ \widetilde{H}|"] \arrow[d, hook] & U_i \times F \arrow[d, "\mathrm{proj}"] \\
c \times [t_j, t_{j+1}] \arrow[r, "H|"'] \arrow[ur, dashed, "\widetilde{K}"] & U_i.
\end{tikzcd}$$

Composing this lifting $\widetilde{K}$ with φ_i, we define $\widetilde{H}$ on $c \times [t_j, t_{j+1}]$. In this way we complete the induction step and obtain $\widetilde{H}|I^q \times [0, t_{j+1}]$. Finally, by induction on j, we define $\widetilde{H}$ on $I^q \times I$. □

4.2.6 Exercise. Using the same method of proof as in 4.2.5, prove the following statement, which means that the concept of being a Serre fibration is a local concept:

4.2.7 **Proposition**. *Suppose that $p : E \longrightarrow X$ is continuous and that there exists an open cover $\{U\}$ of X such that for each open set U in the cover the restriction p_U is a Serre fibration. Then p is a Serre fibration.* □

4.2.8 Exercise. Assume that $p : E \longrightarrow X$ is a covering map. Prove that p has the *unique path-lifting property*; that is, p is such that for any given path $\alpha : I \longrightarrow X$

and any given point $y \in p^{-1}(\alpha(0))$ there exists a unique path $\widetilde{\alpha} : I \longrightarrow E$ satisfying $\widetilde{\alpha}(0) = y$ and $p \circ \widetilde{\alpha} = \alpha$. (*Hint:* Since p is a Serre fibration, the lifting always exists. To prove that it is unique, show that any two liftings with the same initial point y have to be homotopic fiber by fiber, using again the fact that p is a Serre fibration, and notice that this is possible only if both coincide, since the fiber is discrete.)

The following is a very important example.

4.2.9 EXAMPLE. Let $\mathbb{S}^3 \subset \mathbb{C} \times \mathbb{C}$ be defined as

$$\mathbb{S}^3 = \{(z, z') \in \mathbb{C} \times \mathbb{C} \mid z\overline{z} + z'\overline{z}' = 1\}\,.$$

Also let us identify the *Riemann sphere*, defined by $\mathbb{C} \cup \{\infty\}$, with $\mathbb{S}^2$ by means of the stereographic projection $e : \mathbb{S}^2 \longrightarrow \mathbb{C} \cup \{\infty\}$ defined by $e(\zeta) = (1/1 - z)(x + \mathrm{i}y)$ for $\zeta = (x, y, z)$ and $z < 1$ and by $e(0, 0, 1) = \infty$. This is shown in Figure 4.2.

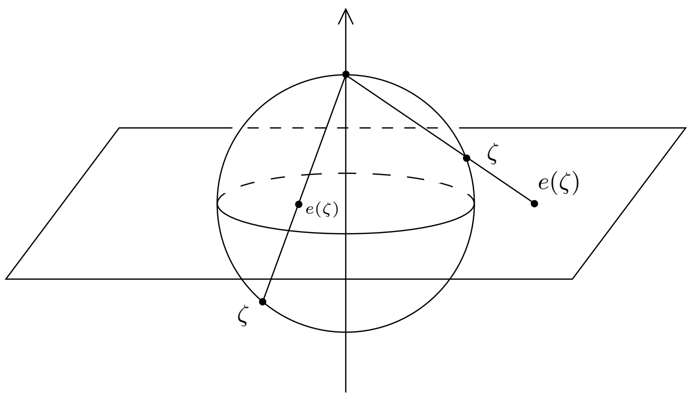

Figure 4.2

We have a map

$$p : \mathbb{S}^3 \longrightarrow \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$$

defined by

$$p(z, z') = \begin{cases} \frac{z}{z'} & \text{if } z' \neq 0\,, \\ \infty & \text{if } z' = 0\,. \end{cases}$$

Then p is a locally trivial bundle with fiber $\mathbb{S}^1 = \{\zeta \in \mathbb{C} \mid \zeta\overline{\zeta} = 1\}$, as we shall soon see.

Put $U = \mathbb{S}^2 - \{\infty\} (= \mathbb{C})$ and $V = \mathbb{S}^2 - \{0\}$. We define a homeomorphism

$$\varphi_U : U \times \mathbb{S}^1 \longrightarrow p^{-1}U$$

by $\varphi_U(z, \zeta) = \big(\zeta z/\sqrt{z\overline{z} + 1}, \zeta/\sqrt{z\overline{z} + 1}\big)$. It then has an inverse

$$\psi_U : p^{-1}U \longrightarrow U \times \mathbb{S}^1$$

given by $\psi_U(z,z') = (\frac{z}{z'}, \frac{z'}{|z'|})$.

We define another homeomorphism

$$\varphi_V : V \times \mathbb{S}^1 \longrightarrow p^{-1}V$$

by

$$\varphi_V(z,\zeta) = \left(\frac{|z|\zeta}{\sqrt{z\overline{z}+1}}, \frac{|z|\zeta}{z\sqrt{z\overline{z}+1}} \right)$$

if $z \in \mathbb{C} - \{0\}$ and by $\varphi_V(\infty,\zeta) = (\zeta, 0)$. Then its inverse

$$\psi_V : p^{-1}V \longrightarrow V \times \mathbb{S}^1$$

is given by $\psi_V(z,z') = \left(\frac{z}{z'}, \frac{z}{|z|}\right)$ if $z' \neq 0$ and by $\psi_V(z,0) = (\infty, z)$.

So we have that $p : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ is locally trivial. This locally trivial bundle is called the *Hopf fibration*.

Given maps $p : E \longrightarrow X$ and $f : Y \longrightarrow X$, we may construct the *pullback* of p over f, denoted by $f^*(p) : f^*(E) \longrightarrow Y$, where $f^*(E) = \{(y,e) \in Y \times E \mid f(y) = p(e)\}$ and $f^*(p) : f^*(E) \longrightarrow Y$ is the projection. All these maps fit into the so-called *pullback diagram*

$$\begin{tikzcd} f^*(E) \arrow[r, "\widetilde{f}"] \arrow[d, "f^*(p)"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X, \end{tikzcd}$$

where $\widetilde{f} : f^*(E) \longrightarrow E$ is the other projection. I.e. $f^*(p)(y,e) = y$ and $\widetilde{f}(y,e) = y$.

4.2.10 **Proposition**. *If $p : E \longrightarrow X$ is a locally trivial bundle and $f : Y \longrightarrow X$ is continuous, then the pullback of p over f, $f^*(p) : f^*(E) \longrightarrow Y$, is a locally trivial bundle that has the same fiber F as p has.*

Proof: Suppose that $y \in Y$ and that U is a neighborhood of $f(y)$ in X such that there exists a homeomorphism φ_U that makes the triangle

$$\begin{tikzcd} U \times F \arrow[rr, "\varphi_U"] \arrow[dr] & & p^{-1}U \arrow[dl] \\ & U & \end{tikzcd}$$

commute. Put $V = f^{-1}U$. Then V is a neighborhood of y, and the map $\psi_V : V \times F \longrightarrow f^*(p)^{-1}V$ given by $\psi_V(y,e) = (y, \varphi_U(f(y),e))$ is a homeomorphism that makes the triangle

$$\begin{tikzcd} V \times F \arrow[rr, "\psi_V"] \arrow[dr] & & f^*(p)^{-1}V \arrow[dl] \\ & V & \end{tikzcd}$$

commute.

$\square$

4.2.11 **Corollary**. *Given a covering map $p : E \longrightarrow X$ and a map $f : Y \longrightarrow X$, the pullback $f^*(p) : f^*(E) \longrightarrow Y$ is covering map that has the same fiber as p.* □

4.2.12 Example. Assume that $\mathbb{R}$ is the space of real numbers and consider the *exponential map*

$$p : \mathbb{R} \longrightarrow \mathbb{S}^1$$

defined by $p(t) = \mathrm{e}^{2\pi \mathrm{i} t} \in \mathbb{S}^1 \subset \mathbb{C}$. Clearly, we have that $p(t) = p(t')$ if and only if $t' - t \in \mathbb{Z}$. So we have that $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ as abelian groups and as topological spaces. Let us show that it is a locally trivial bundle with fiber $\mathbb{Z}$ (see Figure 4.3). Put $U = \mathbb{S}^1 - \{1\}$, so that we have $p^{-1}U = \mathbb{R} - \mathbb{Z}$. Then there is a homeomorphism ψ_U that makes the triangle

$$\begin{tikzcd}
p^{-1}U \arrow[rr, "\psi_U"] \arrow[dr] & & U \times \mathbb{Z} \arrow[dl] \\
& U &
\end{tikzcd}$$

commute. It is given by $\psi_U(t) = (\mathrm{e}^{2\pi \mathrm{i} t}, [t])$, where $[t] \in \mathbb{Z}$ satisfies $t = [t] + t'$ with $0 < t' < 1$. And its inverse $\varphi_U : U \times \mathbb{Z} \longrightarrow p^{-1}U$ is given by $\varphi_U(\zeta, n) = n + t$, where $\zeta = \mathrm{e}^{2\pi \mathrm{i} t} \in U$ with $0 < t < 1$.

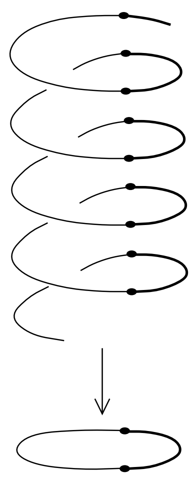

Figure 4.3

Analogously, if we put $V = \mathbb{S}^1 - \{-1\}$, so that

$$p^{-1}V = \mathbb{R} - \left(\mathbb{Z} + \frac{1}{2}\right) = \left\{t \in \mathbb{R} \mid t \neq n + \frac{1}{2};\ n \in \mathbb{Z}\right\},$$

then we define $\psi_V : p^{-1}V \longrightarrow V \times \mathbb{Z}$ by $\psi_V(t) = \left(\mathrm{e}^{2\pi \mathrm{i} t}, \left[t + \frac{1}{2}\right]\right)$. Then its inverse $\varphi_V : V \times \mathbb{Z} \longrightarrow p^{-1}V$ is given by $\varphi_V(\zeta, n) = n + t$ for $\zeta = \mathrm{e}^{2\pi \mathrm{i} t} \in V$ with $-\frac{1}{2} < t < \frac{1}{2}$.

4.2.13 Exercise.

(a) Let $p_i : E_i \longrightarrow X_i$ be a covering map for $i = 1, 2, \ldots, n$. Show that the product

$$p_1 \times \cdots \times p_n : E_1 \times \cdots \times E_n \longrightarrow X_1 \times \cdots \times X_n$$

is a covering map.

(b) Show that in general an infinite product of covering maps is not a covering map. (*Hint:* Let $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ be the exponential map and assume that for each $i = 1, 2, \ldots,$ $p_i = p$. Prove that

$$q = \prod_{i=1}^{\infty} p_i : \prod_{i=1}^{\infty} E_i \longrightarrow \prod_{i=1}^{\infty} X_i\,,$$

where $E_i = \mathbb{R}$ and $X_i = \mathbb{S}^1$ for all i, is not a covering map.

In view of the rich structure of Hurewicz fibrations, it is a pertinent question, under what conditions a locally trivial bundle is a Hurewicz fibration. A quite general answer is given in [10]. In order to state the result, we need some preparation.

4.2.14 Definition. Given an open cover $\mathcal{U} = \{\mathcal{U}_\alpha \mid \alpha \in \mathfrak{J}\}$ of a topological space X, we define a *partition of unity* subordinate to $\mathcal{U}$ as a family of continuous functions $\eta_\alpha : X \longrightarrow I$, $\alpha \in \mathfrak{J}$, such that the following hold:

(i) $\operatorname{supp}(\eta_\alpha) \subseteq U_\alpha$, where $\operatorname{supp}(\eta_\alpha)$ denotes the closed support of η_α, namely the closure of the set $u_\alpha^{-1}(0, 1]$.

(ii) Every point $x \in X$ has a neighborhood V such that the restrictions $\eta_\alpha|_V \equiv 0$ except for finitely many indexes α.

(iii) For each point $x \in X$ the (finite) sum $\sum_{\alpha\in\mathfrak{J}} \eta_\alpha(x) = 1$.

In metric spaces, each open cover admits a subordinate partition of unity as one may show. We have the following.

4.2.15 Definition. A space X is said to be *paracompact* if any open cover of X admits a subordinate partition of unity.

Thus any metric space is paracompact. By a theorem of Miyazaki [37], every CW-complex is paracompact.

We have the next result due to Dold [10].

4.2.16 **Theorem**. *Let $p : E \longrightarrow X$ be a locally trivial bundle and let $\mathcal{U}$ be a trivializing cover. If $\mathcal{U}$ admits a partition of unity, then $p : E \longrightarrow X$ is a Hurewicz fibration.*

As a consequence, we have the following.

4.2.17 **Corollary**. *Let $p : E \longrightarrow X$ be a locally trivial bundle over a paracompact space X. Then $p : E \longrightarrow X$ is a Hurewicz fibration.* □

4.3 TOPOLOGICAL COFIBRATIONS

The concept of cofibration is dual to that of fibration.

4.3.1 DEFINITION. Let $\mathcal{C}$ be some class of k-spaces. We say that an inclusion map $i : A \longrightarrow X$ has the *homotopy extension property* for the class $\mathcal{C}$ (the $\mathcal{C}$-HEP for short) or that it is a *$\mathcal{C}$-cofibration* if given a homotopy $H : A \times I \longrightarrow Y$ and a map $f : X \longrightarrow Y$ such that $f(a) = H(a, 0)$, for any space Y in $\mathcal{C}$, there is a map $\widetilde{H} : X \times I \longrightarrow E$ such that $\widetilde{H}(x, 0) = f(x)$ for all $x \in X$, and $\widetilde{H}(a, t) = H(a, t)$ for all $a \in A$, $t \in I$. In a diagram

$$
\begin{tikzcd}
A \arrow[r, "j_0"] \arrow[d, hook, "j"'] & A \times I \arrow[d, "j \times \mathrm{id}"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, "j_0"'] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "\widetilde{H}"] & \\
& & Y\,.
\end{tikzcd}
\tag{4.3.2}
$$

If $\mathcal{C}$ is the class of all k-spaces, then we say that j is a *cofibration*.

For what follows, we shall require two elementary concepts. Recall that A is a *retract* of X if $A \subset X$ and there is a map $r : X \longrightarrow A$, called *retraction*, such that $r|_A = \mathrm{id}_A$. Furthermore, we say that A is a *strong deformation retract* of X if $A \subset X$ and there exists a homotopy $H : X \times I \longrightarrow X$ such that

$$
\begin{aligned}
H(x,0) &= 0 \quad \text{for all} \quad x \in X\,,\\
H(a,t) &= a \quad \text{for all} \quad a \in A,\ t \in I\,,\\
H(x,1) &\in A \quad \text{for all} \quad x \in X\,.
\end{aligned}
$$

In this case, $r(x) = H(x, 1)$ is a retraction $r : X \longrightarrow A$, which we call *strong deformation retraction*. The homotopy H is called a *deformation*.

In the next we shall follow the papers [47, 48]. The following is a useful characterization of a cofibration.

4.3.3 **Proposition**. *Let $A \subset X$ be closed. Then the inclusion map $j : A \hookrightarrow X$ is a cofibration if and only if there is a retraction $r : X \times I \longrightarrow X \times \{0\} \cup A \times I$.*

Proof: Notice first that $X \times \{0\}$ and $A \times I$ are closed in their union, hence a map $\varphi : X \times \{0\} \cup A \times I \longrightarrow Y$ is continuous if and only if its restrictions to both closed sets are continuous. Assume that we have a retraction r and we have the homotopy extension problem depicted in (4.3.2). The maps f and H determine a well-defined continuous map $K : X \times \{0\} \cup A \times I \longrightarrow Y$. The composite $\widetilde{H} = K \circ r : X \times I \longrightarrow Y$ solves the extension problem. Thus j is a cofibration.

Conversely, assume that $j : A \hookrightarrow X$ is a cofibration and in the extension problem (4.3.2) take $Y = X \times \{0\} \cup A \times I$ as well as $f : X \longrightarrow X \times \{0\} \cup A \times I$ and $H : A \times I \hookrightarrow X \times \{0\} \cup A \times I$ to be the inclusions. Then the extension $r = \widetilde{H} : X \times I \longrightarrow X \times \{0\} \cup A \times I$ is the desired retraction. □

The previous result is indeed stronger. We have the following.

4.3.4 **Proposition**. *Let $A \subset X$ be closed. Then the inclusion map $j : A \hookrightarrow X$ is a cofibration if and only if there is a strong deformation retraction $r : X \times I \longrightarrow X \times \{0\} \cup A \times I$.*

Proof: It only remains to prove that if $j : A \hookrightarrow X$ is a cofibration and $r : X \times I \longrightarrow X \times \{0\} \cup A \times I$ is a retraction, then r is indeed a strong deformation retraction. Consider the homotopy $H : X \times I \times I \longrightarrow X \times I$ given by

$$H(x, s, t) = (r_1(x, ts), tr_2(x, s) + s(1 - t)),$$

where $r(x, t) = (r_1(x, t), r_2(x, t))$, $r1(x, t) \in X$ and $r_2(x, t) \in I$. If $s = 0$, then $r(x, 0) = (x, 0)$, hence $H(x, 0, t) = (x, 0)$, and if $x \in A$, then $r(x, t) = (x, t)$, hence $H(x, s, t) = (x, s)$. Therefore, the homotopy is relative to $X \times \{0\} \cup A \times I$. Finally, $H(x, s, 0) = (x, s)$ and $H(x, s, 1) = r(x, s)$, thus the homotopy starts with the identity and ends with the retraction r. □

4.3.5 EXAMPLE. $\partial I^n \hookrightarrow I^n$ is a cofibration. Namely there is a retraction $r : I^n \times I \longrightarrow I^n \times \{0\} \cup \partial I^n \times I$. Figure 4.4 shows r. It is an exercise to give the algebraic expression for r.

4.3.6 EXERCISE. Consider the inclusion $i : I^n \times \{0\} \cup \partial I^n \times I \hookrightarrow I^n \times I$. Show the following.

(a) i is a cofibration.

(b) i is a homotopy equivalence. (*Hint:* The homotopy $H : I^n \times I \times I \longrightarrow I^n \times I$ given by $H(s, t, \tau) = (1 - \tau)(s, t) + \tau r(s, t)$, where $r(s, t)$ is as above, starts with the identity and ends with $i \circ r$. Furthermore, $r \circ i = \mathrm{id}$.)

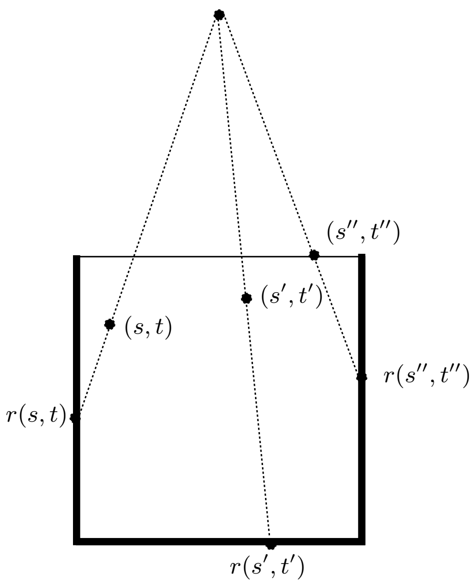

Figure 4.4 The retraction $r : I^n \times I \longrightarrow I^n \times \{0\} \cup \partial I^n \times I$

We shall give several further results which characterize cofibrations. Our first result gives a characterization which shows that tha concept of cofibration has locaal character, i.e. it depends only on a certain neighborhood of the small space in the large one.

4.3.7 **Theorem**. *Take a closed subset A of a space X. Then the inclusion $j : A \hookrightarrow X$ is a cofibration if and only if there exist*

(a) *a neighborhood U of A in X and a deformation $H : U \times I \longrightarrow X$ to A, relative to A, namely such that*

$$H(x,0) = 0 \quad \text{for all} \quad x \in U\,,$$
$$H(a,t) = a \quad \text{for all} \quad a \in A,\, t \in I\,,$$
$$H(x,1) \in A \quad \text{for all} \quad x \in U\,.$$

(b) *a continuous function $\varphi : X \longrightarrow I$ such that $A = \varphi^{-1}(0)$ and $\varphi(x) = 1$ for all $x \notin U$.*

Proof: If j is a cofibration, then by Proposition 4.3.3, there is a retraction $r : X \times I \longrightarrow X \times \{0\} \cup A \times I$ and let r_1 and r_2 be the components of r in X and I, respectively. Define U, H, and φ by

$$U = \{x \in X \mid r_1(x,1) \in A\}\,,$$
$$H = r_1|_{U\times I}\,,$$
$$\varphi(x) = \sup_{t\in I} |t - r_2(x,t)|$$

First notice that U is a neighborhood of A since it can be seen as $r^{-1}(A \times (0,1])$, and $A \times (0,1]$ is open in $X \times \{0\} \cup A \times I$. Further, $H(x,0) = r_1(x,0) = x$, since $r(x,0) = (x,0)$, $H(a,t) = a$, since $r(a,t) = (a,t)$, and $H(x,1) \in A$, since $x \in U$, by definition of U. Moreover, $\varphi(x) \sup_{t \in I} |t - r_2(x,t)| = 0$ if and only if $r_2(x,t) = t$. Thus $t > 0$ and so $x \in A$.

Conversely, assume that we have U, H, and φ as in the statement of the theorem. Since $A \subset X$ is closed, it is enough to construct a retraction $r : X \times I \longrightarrow X \times \{0\} \cup A \times I$. Define it by

$$r(x,t) = \begin{cases} (x,0) & \text{if } \varphi(x) = 1\,, \\ (H(x, 2(1-\varphi(x))t), 0) & \text{if } \frac{1}{2} \leq \varphi(x) < 1\,, \\ H\left(x, \frac{t}{2\varphi(x)}, 0\right)\,, & \text{if } 0 < \varphi(x) \leq \frac{1}{2} \text{ and } 0 \leq t \leq 2\varphi(x)\,, \\ (H(x,1), t - 2\varphi(x))\,, & \text{if } 0 < \varphi(x) \leq \frac{1}{2} \text{ and } 2\varphi(x) \leq t \leq 1\,, \\ (x,t)\,, & \text{if } \varphi(x) = 0\,. \end{cases}$$

It is an easy exercise to show that r is continuous. To see that it is well defined, notice that if $t > 2\varphi(x)$, then since $x \in U$, we have $H(x,1) \in A$; moreover, if $\varphi(x) = 0$, then we must have $x \in A$. To verify that r is a retraction, notice that $r(x,0) = (x,0)$ and $r(a,t) = (H(a,t'),t) = (a,t)$ (for some value of t'). $\square$

The following result relates fibrations and cofibrations. It is a consequence of one of the previous results.

4.3.8 **Theorem**. *Let $p : E \longrightarrow X$ be a Hurewicz fibration and let A be a strong deformation retract of X and there is a function $\varphi : X \longrightarrow I$ such that $A = \varphi^{-1}(0)$. Then given maps $f : X \longrightarrow X$ and $g : A \longrightarrow E$ such that $p \circ g = f \circ j$, where $j : A \hookrightarrow X$ is the inclusion map, there is a map $h : X \longrightarrow E$ that lifts f, i.e. such that $p \circ h = f$, and extends g, i.e., $h \circ j = g$. In a diagram*

(4.3.9)
$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook, "j"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "h"] & E \end{tikzcd}$$

Furthermore, the map h is unique up to homotopy relative to A.

Proof: By assumption, there is a retraction $r : X \longrightarrow A$ and a deformation $H : \mathrm{id}_X \simeq j \circ r$ rel A. If h such that $h \circ j = g$ exists, then $h \circ H : h \simeq h \circ j \circ r = g \circ r$ rel A. This proves the last assertion of the theorem.

Define $K : X \times I \longrightarrow X$ by

$$K(x,t) = \begin{cases} H(x, \frac{t}{\varphi(x)}) & \text{if } t < \varphi(x)\,, \\ H(x,1) & \text{if } t \geq \varphi(x)\,. \end{cases}$$

One easily shows that K is continuous. Since p is a Hurewicz fibration, there is a lifting $\widetilde{K} : X \times I \longrightarrow E$ of $f \circ K : X \times I \longrightarrow X$, namely $p \circ \widetilde{K} = f \circ K$

and $\widetilde{K}(x,0) = gr(x)$ for any $x \in X$. The desired lifting $h : X \longrightarrow E$ is given by $h(x) = \widetilde{K}(x, \varphi(x))$. □

Notice that if we are interested in cofibrations where the space X is a CW-complex and A is a subcomplex, the previous result holds also for Serre fibrations. A consequence of the last result is the next.

4.3.10 **Theorem**. *Let $p : E \longrightarrow X$ be a Hurewicz fibration and $j : A \hookrightarrow X$ a cofibration (where $A \subset X$ is closed). Given a homotopy $H : X \times I \longrightarrow X$ and a map $f : X \times \{0\} \cup A \times I \longrightarrow E$ such that $p \circ f = H|_{X\times\{0\}\cup A\times I}$, there exists a lifting of H, $\widetilde{H} : X \times I \longrightarrow E$ such that $p \circ \widetilde{H} = H$ and $\widetilde{H}|_{X\times\{0\}\cup A\times I} = f$. In a diagram*

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & E \end{tikzcd} \tag{4.3.11}$$

Proof: By Proposition 4.3.4, $X\times\{0\}\cup A\times I$ is a strong deformation retract of $X\times I$. By Theorem 4.3.7, one can construct a function $\psi : X \longrightarrow I$ such that $\psi^{-1}(0) = A$. Define $\varphi : X \times I \longrightarrow I$ by $\varphi(x,t) = t\psi(x)$. Thus $\varphi^{-1}(0) = X \times \{0\} \cup A \times I$. □

4.3.12 **Theorem**. *Suppose that A is closed in X and let $j : A \hookrightarrow X$ be the inclusion map. Then these two statements are equivalent:*

(a) *Given a Hurewicz fibration $p : E \longrightarrow B$ and a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "h"] & B, \end{tikzcd}$$

there exists a lifting $h : X \longrightarrow E$ such that $p \circ h = f$ and $h \circ j = g$.

(b) *The map j is a cofibration and a homotopy equivalence.*

If (a) *and* (b) *hold, then the lifting h of f is unique up to a homotopy relative to $j(A)$.*

Proof: We prove first (b) ⇒ (a). Since j is a cofibration, by Theorem 4.3.8 (a) follows from (b).

Conversely, to prove (a) ⇒ (b) recall 4.1.5 2 that the map $\pi : M(I,Y) \longrightarrow Y$ given by $\pi(\sigma) = \sigma(0)$ is a Hurewicz fibration for any space Y. Let $f : X \longrightarrow Y$ and $H : A \times I \longrightarrow Y$ such that for any $a \in A$, $f(a) = H(a,0)$. Then by Lemma 1.4.20, H corresponds to a map $\widehat{H} : A \longrightarrow M(I,Y)$ such that $\pi \circ \widehat{H} = f|_A$. Hence by 4.3.12, $\widetilde{H}$ exists as desired. Hence $A \hookrightarrow X$ is a cofibration. □

4.3.13 Remark. If in the previous result X is a CW-complex and $j : A \longrightarrow X$ is the inclusion of a subcomplex, then the result holds for a Serre (instead of a Hurewicz) fibration $p : E \longrightarrow B$.

4.4 Topological homotopy groups

In this section we define the homotopy groups of a pointed space and we construct the long exact sequence of homotopy groups determined by a Serre fibration.

We start considering the nth cube I^n, which is the product of n copies of the unit interval $I = [0, 1]$, and its boundary ∂I^n which consists of points such that at least one of their coordinates is ether 0 or 1.

4.4.1 Definition. Let X be a pointed space with base point x_0. We define $\pi_n(X, x_0)$ as the set of homotopy classes of maps of pairs $\alpha : (I^n, \partial I^n) \longrightarrow (X, \{x_0\})$. The homotopies between two such maps of pairs must of course send ∂I^n to the base point x_0.

We define an operation in $\pi_n(X, x_0)$ as follows. Let $\alpha, \beta : (I^n, \partial I^n) \longrightarrow (X, \{x_0\})$ represent two elements; define

$$\alpha * \beta(s_1, s_2, \dots, s_n) = \begin{cases} \alpha(2s_1, s_2, \dots, s_n) & \text{if } 0 \leq s_1 \leq \frac{1}{2}\,, \\ \beta(2s_1 - 1, s_2, \dots, s_n) & \text{if } \frac{1}{2} \leq s_1 \leq 1\,. \end{cases}$$

It is an *exercise* to verify that the homotopy class $[\alpha * \beta]$ depends only on the homotopy classes $[\alpha]$ and $[\beta]$ so that we have a well defined *sum operation* in $\pi_n(X, x_0)$ given by $[\alpha] + [\beta] = [\alpha * \beta]$. Furthermore, define an *identity element* 0 to be the homotopy class of the constant map c_{x_0} which sends I^n to the base point x_0, and an *inverse element* to $[\alpha]$, $-[\alpha] = [\overline{\alpha}]$, where

$$\overline{\alpha}(s_1, s_2, \dots, s_n) = (1 - s_1, s_2, \dots, s_n)\,.$$

The next lemma will be useful in what follows.

4.4.2 **Lemma**. *Given two maps $f, g : I \longrightarrow I$ such that $f(0) = g(0)$ and $f(1) = g(1)$, then $f \simeq g \quad rel\, \partial I$. Thus in particular, if $f(0) = 0$ and $f(1) = 1$, then $f \simeq \mathrm{id}_I \quad rel\, \partial I$ and if $f(0) = 0$ and $f(1) = 0$, then $f \simeq c_0 \quad rel\, \partial I$.*

Proof: Define a homotopy $H : I \times I \longrightarrow I$ by $H(s, t) = (1 - t)f(s) + tg(s)$. Then $H : f \simeq g \quad \text{rel}\, \partial I$. (See Figure 4.5.) □

We shall use Lemma 4.4.2 to prove a general associativity result. We shall write the expression

$$\alpha_1 * \alpha_2 * \cdots * \alpha_k\,,$$

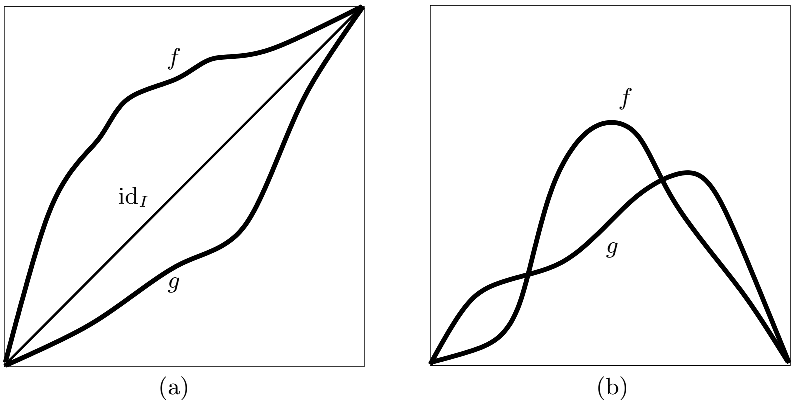

Figure 4.5

without parentheses, which, if it is not stated otherwise, means the element

$$(\alpha_1 * \alpha_2 * \cdots * \alpha_k)(s_1, s_2, \dots, s_n) = \begin{cases} \alpha_1(ks_1, s_2, \dots, s_n) & \text{if } 0 \le s_1 \le \frac{1}{k}, \\ \alpha_2(ks_1 - 1, s_2, \dots, s_n) & \text{if } \frac{1}{k} \le s_1 \le \frac{2}{k}, \\ \quad\vdots & \quad\vdots \\ \alpha_k(ks_1 - k + 1, s_2, \dots, s_n) & \text{if } \frac{k-1}{k} \le s_1 \le 1\,. \end{cases}$$

4.4.3 **Lemma**. *Given maps α_1, α_2, ... , $\alpha_k : (I^n, \partial I^n) \longrightarrow (X, x_0)$ one has*

$$(\alpha_1 * \cdots * \alpha_r) * (\alpha_{r+1} * \cdots * \alpha_k) \simeq \alpha_1 * \cdots * \alpha_k \quad \textit{rel } \partial I \times I^{n-1}$$

Proof: Let $f : I \longrightarrow I$ be the piecewise linear function such that $f(0) = 0$, $f(\frac{1}{2}) = \frac{r}{k}$, and $f(1) = 1$ (see Figure 4.6), which by Lemma 4.4.2 is homotopic to id_I relative to ∂I. We clearly have

$$(\alpha_1 * \cdots * \alpha_r) * (\alpha_{r+1} * \cdots * \alpha_k)(s_1, s_2, \dots, s_n) = (\alpha_1 * \cdots * \alpha_k)(f(s_1), s_2, \dots, s_n)\,.$$

Hence the result. □

4.4.4 **Theorem**. *The set $\pi_n(X, x_0)$ with the sum operation, identity element and inverse elements is a group.*

Proof: By Lemma 4.4.3, the operation is associative. We prove now that $0 + [\alpha] = [\alpha] = [\alpha] + 0$. Let $f : I \longrightarrow I$ and $g : I \longrightarrow I$ be piecewise linear functions such that $f(0) = 0$, $f(\frac{1}{2}) = 0$, and $f(1) = 1$, and $g(0) = 0$, $g(\frac{1}{2}) = 1$, and $g(1) = 1$ (see Figure 4.7). Then $(c_{x_0} * \alpha)(s_1, s_2, \dots, s_n) = \alpha(f(s_1), s_2, \dots, s_n)$ and $\alpha * c_{x_0}(s_1, s_2, \dots, s_n) = \alpha(g(s_1), s_2, \dots, s_n))$, and the result follows from 4.4.2.

We prove now that $[\alpha] + [\overline{\alpha}] = 0 = [\overline{\alpha}] + [\alpha]$. Let now $f : I \longrightarrow I$ and $g : I \longrightarrow I$ be piecewise linear functions such that $f(0) = 0$, $f(\frac{1}{2}) = 1$, and $f(1) = 0$, and $g(0) = 1$, $g(\frac{1}{2}) = 0$, and $g(1) = 1$ (see Figure 4.8). Then $(\alpha * \overline{\alpha})(s_1, s_2, \dots, s_n) = \alpha(f(s_1), s_2, \dots, s_n)$ and $\overline{\alpha} * \alpha(s_1, s_2, \dots, s_n) = \alpha(g(s_1), s_2, \dots, s_n)$. Thus the result follows from 4.4.2, since $f \simeq c_0 : I \longrightarrow I \quad \text{rel } \partial I$ and $g \simeq c_1 \quad \text{rel } \partial I$. □

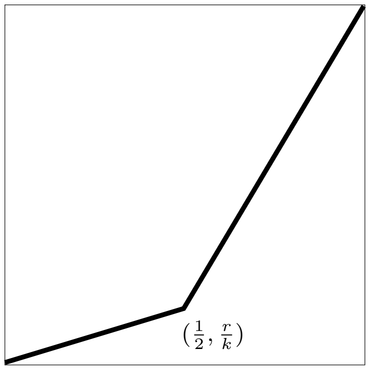

Figure 4.6

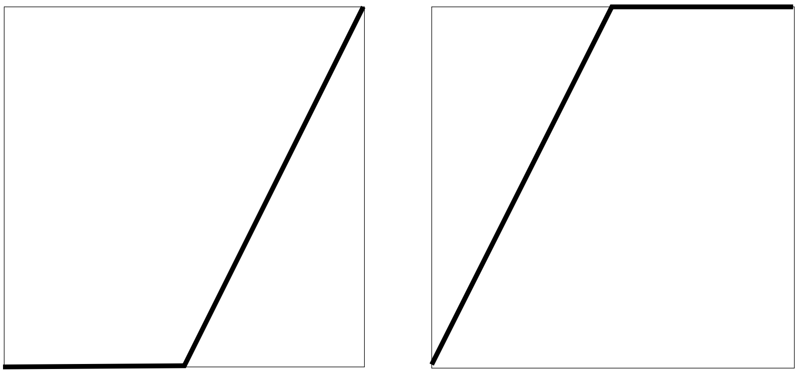

Figure 4.7

We have written $+$ for the group operation. This we shall do only if $n \geq 2$, since in this case the groups are abelian. This follows from the next general result. Before we state it, we shall say two operations in W are *mutually distributive up to homotopy* if

$$(a * b) \bullet (c * d) \simeq (a \bullet c) * (b \bullet d) \text{ for all } a, b, c, d \in W\,.$$

We say that $*$ and $\bullet$ have a *common bilateral identity element up to homotopy* if there is an element $e \in W$ such that

$$a * e \simeq e * a \simeq a \bullet e \simeq e \bullet a \simeq a \text{ for all } a \in W\,.$$

We say that the multiplications are *mutually distributive*, resp. they have a *common bilateral unit* if we can replace $\simeq$ by $=$ in the corresponding conditions.

4.4.5 **Lemma**. *Let W be a space equipped with two continuous multiplications $*$, $\bullet$ such that*

(a) $*$ *and* $\bullet$ *have a common bilateral identity element up to homotopy, and*

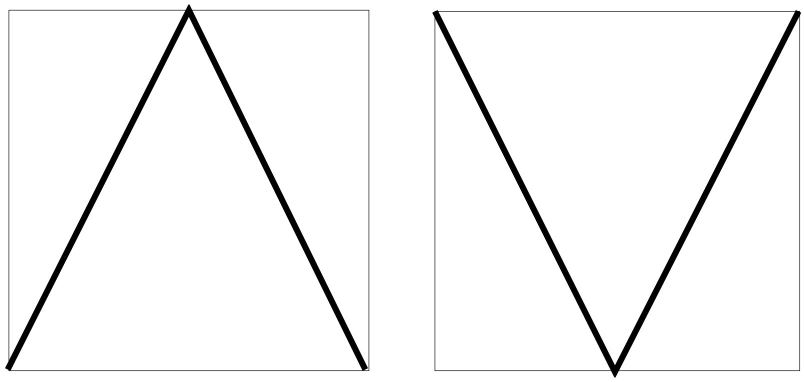

Figure 4.8

(b) $*$ *and* $\bullet$ *are mutually distributive.*

Then $*$ *and* $\bullet$ *are homotopic, as well as being commutative and associative up to homotopy.*

Proof: Take $a, b, c, d \in G$ and let $e \in W$ be the common identity element up to homotopy. Therefore,

$$a * b \simeq (a \bullet e) * (e \bullet b) \simeq (a * e) \bullet (e * b) \simeq a \bullet b\,,$$

and so $*$ and $\bullet$ are homotopic. Moreover,

$$a \bullet b \simeq a * b \simeq (e \bullet a) * (b \bullet e) \simeq (e * b) \bullet (a * e) \simeq b \bullet a \simeq b * a\,,$$

and so the multiplications are commutative up to homotopy. Finally,

$$a * (b * c) \simeq (a \bullet e) * (b \bullet c) \simeq (a * b) \bullet (e * c) \simeq (a * b) * c\,,$$

and so the multiplications are associative up to homotopy. □

In our case, if $n \geq 2$, we can define a second operation on $\pi_n(X, x_0)$ as follows. Let $\alpha, \beta : (I^n, \partial I^n) \longrightarrow (X, \{x_0\})$ represent two elements; define

$$\alpha \bullet \beta(s_1, s_2, \ldots, s_n) = \begin{cases} \alpha(s_1, 2s_2, \ldots, s_n) & \text{if } 0 \leq s_2 \leq \frac{1}{2}\,, \\ \beta(s_1, 2s_2 - 1, \ldots, s_n) & \text{if } \frac{1}{2} \leq s_2 \leq 1\,. \end{cases}$$

The operations $*$ and $\bullet$ are mutually distributive up to homotopy. Namely, let $\alpha, \beta, \gamma, \delta : (I^n, \partial I^n) \longrightarrow (X, \{x_0\})$ represent four elements in $\pi_n(X, x_0)$. Then

$$((\alpha * \beta) \bullet (\gamma * \delta))(s_1, s_2, \ldots, s_n) =$$

$$\begin{cases} (\alpha * \beta)(s_1, 2s_2, \ldots, s_n) = \begin{cases} \alpha(2s_1, 2s_2, \ldots, s_n), & 0 \leq s_1 \leq \frac{1}{2}\,,\ 0 \leq s_2 \leq \frac{1}{2} \\ \beta(2s_1 - 1, 2s_2, \ldots, s_n), & \frac{1}{2} \leq s_1 \leq 1\,,\ 0 \leq s_2 \leq \frac{1}{2} \end{cases} \\ (\gamma * \delta)(s_1, 2s_2 - 1, \ldots, s_n) = \begin{cases} \gamma(2s_1, 2s_2 - 1, \ldots, s_n), & 0 \leq s_1 \leq \frac{1}{2}\,,\ \frac{1}{2} \leq s_2 \leq 1 \\ \delta(2s_1 - 1, 2s_2 - 1, \ldots, s_n), & \frac{1}{2} \leq s_1 \leq 1\,,\ \frac{1}{2} \leq s_2 \leq 1 \end{cases} \end{cases}$$

and on the other hand

$$((\alpha \bullet \gamma) * (\beta \bullet \delta))(s_1, s_2, \dots, s_n) =$$

$$\begin{cases} (\alpha \bullet \gamma)(s_1, 2s_2, \dots, s_n) = \begin{cases} \alpha(2s_1, 2s_2, \dots, s_n), & 0 \le s_1 \le \frac{1}{2}\,,\ 0 \le s_2 \le \frac{1}{2} \\ \gamma(2s_1, 2s_2 - 1, \dots, s_n), & 0 \le s_1 \le \frac{1}{2}\,,\ \frac{1}{2} \le s_2 \le 1 \end{cases} \\ (\beta \bullet \delta)(s_1, 2s_2 - 1, \dots, s_n) = \begin{cases} \beta(2s_1 - 1, 2s_2, \dots, s_n), & \frac{1}{2} \le s_1 \le 1\,,\ 0 \le s_2 \le \frac{1}{2} \\ \delta(2s_1 - 1, 2s_2 - 1, \dots, s_n), & \frac{1}{2} \le s_1 \le 1\,,\ \frac{1}{2} \le s_2 \le 1 \end{cases} \end{cases}$$

Thus clearly both expressions are equal. It is also easy to see that the constant map is a two-sided identity element up to homotopy for both operations. Hence we conclude the following.

4.4.6 **Proposition**. *If $n \ge 2$, then the groups $\pi_n(X, x_0)$ are abelian.* □

4.4.7 Note. If $n = 1$, then $\pi_1(X, x_0)$ is the so-called *fundamental group*. It is not necessarily abelian and thus if we take elements $a, b \in \pi_1(X, x_0)$ we write the operation simply by ab.

4.4.8 Remark. Given a map of pairs $\alpha : (I^n, \partial I^n) \longrightarrow (X, x_0)$, then it defines a pointed map $\overline{\alpha} : I^n/\partial I^n \longrightarrow X$, where the quotient space $I^n/\partial I^n$ has as base point the point onto which ∂I^n collapsed. Fixing a homeomorphism $I^n/\partial I^n \approx \mathbb{S}^n$, then $\overline{\alpha}$ determines a pointed map $\alpha' : \mathbb{S}^n \longrightarrow X$. Conversely, given a pointed map $\alpha' : \mathbb{S}^n \longrightarrow X$, it determines a map of pairs $\alpha : (I^n, \partial I^n) \longrightarrow (X, x_0)$. Thus we can view the homotopy groups $\pi_n(X, x_0)$ as the pointed homotopy sets $[\mathbb{S}^n, X]_*$ and the operation is given as follows. If $\alpha', \beta' : \mathbb{S}^n \longrightarrow X$ are two pointed maps, take the quotient map $q : \mathbb{S}^n \longrightarrow \mathbb{S}^n \vee \mathbb{S}^n$ which collapses the equator $\mathbb{S}^{n-1}$ to a (base) point and define $\alpha' * \beta' = (\alpha', \beta') \circ q$ (see Figure 4.9).

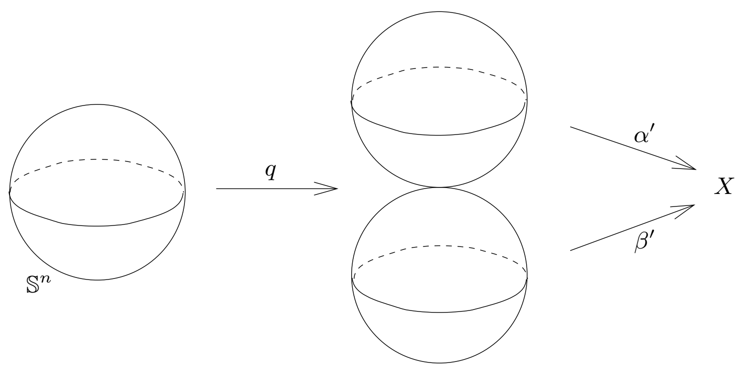

Figure 4.9 The sum of α' and β' in $[\mathbb{S}^n, X]_*$

4.4.9 **Proposition**. *Given two points $x_0, x_1 \in X$ and a path $\gamma : x_0 \simeq x_1$, there is an isomorphism $\varphi_\gamma : \pi_n(X, x_1) \cong \pi_n(X, x_0)$. Consequently, if X is path connected, $\pi_n(X, x_0)$ does not depend on x_0 and we may write $\pi_n(X)$ instead.*

Proof: Consider the canonical inclusion $I^n \hookrightarrow I^{n+1}$ which maps $(s_1, s_2, \ldots, s_n)$ to $(s_1, s_2, \ldots, s_n, 0)$. Then $I^n \subset \partial I^{n+1}$ and we may put $J^n = \partial I^{n+1} - I^n$ (see Figure 4.10, where J^n is upside down).

Let K be the interval $[\frac{1}{4}, \frac{3}{4}]$ and consider the complement . There is a homeomorphism $\eta : I^n \longrightarrow J^n$ which maps K^n to I^n (enlarging and translating), and $I^n - K^n$ to $\partial I^n \times I$ so that every radial segment inside $I^n - K^n$ maps affinely onto I as shown in Figure 4.10).

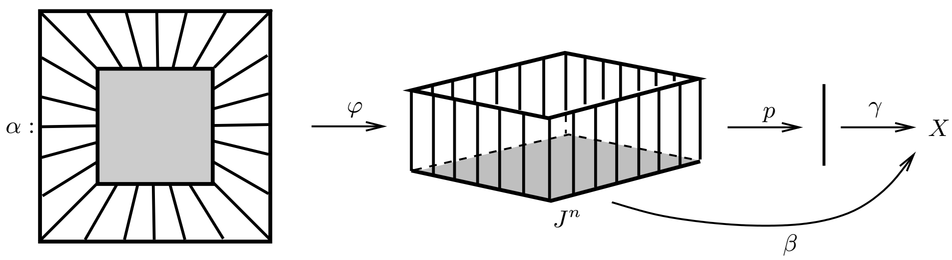

Figure 4.10 The isomorphism $\varphi_\gamma : \pi_n(X, x_1) \longrightarrow \pi_n(X, x_0)$

Given $\beta : (I^n, \partial I^n) \longrightarrow (X, x_1)$, consider the map $\alpha : (I^n, \partial I^n) \longrightarrow (X, x_0)$ given by composing η with the map $\psi : J^n \longrightarrow X$ defined as

$$\psi(s,t) = \begin{cases} \beta(s) & \text{if } (s,1) \in I^n \times \{1\} \\ \gamma(t) & \text{if } (s,t) \in \partial I^n \times I\,. \end{cases}$$

Now define $\varphi_\gamma([\beta]) = [\alpha]$. It is a routine *exercise* to show that φ_γ is an isomorphism. (*Hint:* If one takes $\overline{\gamma}(t) = \gamma(1-t)$, then $\varphi_{\overline{\gamma}}$ is the inverse isomorphism.) $\square$

Figure 4.11 illustrates the effect of φ_γ to a representative β of an element in $\pi_1(X, y)$.

4.4.10 REMARK. One may define another isomorphism $\varphi_\gamma : \pi_n(X, x_1) \longrightarrow \pi_n(X, x_0)$. Assume, as before, that $\beta : (I^n, \partial I^n) \longrightarrow (X, x_1)$ represents an element in $\pi_n(X, x_1)$. Recall 4.3.5, where we show that the inclusion $\partial I^n \hookrightarrow I^n$ is a cofibration. Thus the extension problem depicted in the diagram

$$\begin{tikzcd}
\partial I^n \arrow[r, hook] \arrow[d, hook, "j_0"'] & I^n \arrow[d, hook] \arrow[ddr, bend left, "\beta"] & \\
\partial I^n \times I \arrow[r, hook] \arrow[drr, bend right, "\Gamma"] & I^n \times I \arrow[dr, dashed] & \\
 & & X\,,
\end{tikzcd}$$

where $\Gamma(s,t) = \gamma(t)$, has a solution H. Define $\alpha : I^n \longrightarrow X$ by $\alpha(s) = H(s,1)$. Then $\alpha \simeq \beta$ and α represents an element in $\pi_n(X, x_0)$. Let φ_γ be defined by $\varphi_\gamma([\beta]) = [\alpha]$.

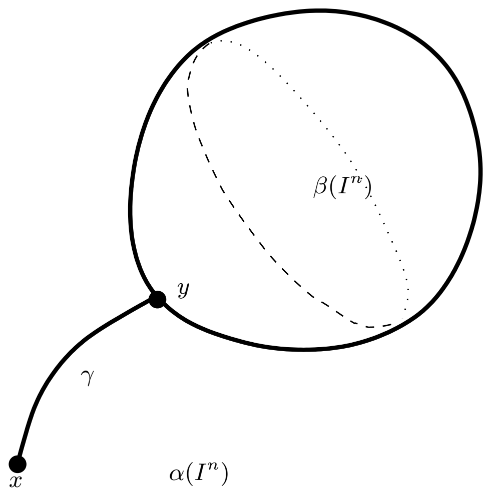

Figure 4.11 The effect of φ_γ on β

4.4.11 Exercise. In the previous exercise prove the following.

(a) Any two α_1 and α_2 obtained as above are homotopic relative to ∂I^n, thus φ_γ is well defined.

(b) φ_γ is bijective.

(c) φ_γ is a group isomorphism.

4.4.12 Definition. Let (X, A, x_0) be a pointed pair of spaces with base point $x_0 \in A$. For $n \geq 1$ we define $\pi_n(X, A, x_0)$ as the set of homotopy classes of maps of triples $(I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, \{x_0\})$, where $J^{n-1} = \partial I^n - I^{n-1}$ is as above and the homotopies map ∂I^n into A and J^{n-1} to the base point x_0.

We define an operation in $\pi_n(X, A, x_0)$ for $n \geq 2$ using the same formulas as for $\pi_n(X, x_0)$, where now the last coordinate s_n does not play the same role as the previous ones, since it can only be different from one if $(s_1, s_2, \ldots, s_{n-1}) \in \partial I^{n-1}$. Hence we may use only either of the coordinates $s_1, s_2, \ldots, s_{n-1}$ to define the operation in $\pi_n(X, A, x_0)$ and we have at least two of them which are mutually distributive if $n \geq 3$. Thus we may prove similarly the following result.

4.4.13 **Theorem**. *If $n \geq 2$ the set $\pi_n(X, A, x_0)$ is a group, and it is abelian if $n \geq 3$.* □

The following is a criterion for a map $\alpha : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, \{x_0\})$ to represent the identity element $0 \in \pi_n(X, A, x_0)$.

4.4.14 **Proposition**. *An element $[\alpha] \in \pi_n(X, A, x_0)$ is equal to zero if and only if α is homotopic relative to ∂I^n to a map $\alpha' : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, \{x_0\})$ such that $\alpha'(I^n) \subset A$.*

Proof: Assume first that $\alpha \simeq \alpha'$ rel ∂I^n, with $\alpha'(I^n) \subset A$. If we now deform I^n to a point, say by a homotopy $H : I^n \times I \longrightarrow In$ given by $H(s, t) = (1 - t)s$, where $s = (s_1, s_2, \ldots, s_n) \in I^n$. Then $\alpha \simeq 0$ and thus $[\alpha] = 0 \in \pi_n(X, A, x_0)$.

Conversely, assume that $[\alpha] = 0$. This means that there is a homotopy of triples $G : I^n \times I \longrightarrow X$ such that $G(s, 0) = \alpha(s)$, $G(s, 1) = x_0$, and $G(\partial I^n \times I) \subset A$, and $G(J^{n-1} \times I) = \{x_0\}$. There is a retraction $r : I^n \times I \longrightarrow I^n \times \{0\} \cup \partial I^n \times I$. Then define $H : I^n \times I \longrightarrow X$ by $H = G \circ r : I^n \times I \longrightarrow A$. Clearly for all $s \in I^n$ we have $H(s, 0) = G(s, 0) = \alpha(s)$, furthermore for all $s \in \partial I^n$ and all $t \in I$ we have $H(s, t) = G(s, t) \subset A$. Thus H is a homotopy relative to ∂I^n which starts with α and ends with a map α' whose image lies in A. Figure 4.4 shows r. □

4.4.15 EXERCISE. Give an explicit formula for the retraction $r : I^n \times I \longrightarrow I^n \times \{0\} \cup \partial I^n \times I$ used in the previous proof and depicted in Figure 4.4.

Given two pointed pairs (X, A, x_0) and (Y, B, y_0) and a base-point-preserving map $f : (X, A, x_0) \longrightarrow (Y, B, y_0)$, by mapping $\alpha : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, \{x_0\})$ to the composite $f \circ \alpha : (I^n, \partial I^n, J^{n-1}) \longrightarrow (Y, B, \{y_0\})$, one obtains a homomorphism

$$f_* : \pi_n(X, A, x_0) \longrightarrow \pi_n(Y, B, y_0)\,.$$

4.4.16 EXERCISE. Show that π_n is a functor from the category of pointed pairs of spaces and continuous maps $\mathfrak{Top}^2_*$ to the category of groups $\mathfrak{Grp}$ (or of abelian groups $\mathfrak{Ab}$ if $n > 2$), i.e. $(\mathrm{id}_{(X,A,x_0)})_* = 1_{\pi_n(X,A,x_0)} : \pi_n(X, A, x_0) \longrightarrow \pi_n(X, A, x_0)$ and if $f : (X, A, x_0) \longrightarrow (Y, B, y_0)$ and $f : (Y, B, y_0) \longrightarrow (Z, C, z_0)$ are base-point-preserving maps, then $(g \circ f)_* = g_* \circ f_* : \pi_n(X, A, x_0) \longrightarrow \pi_n(Z, C, z_0)$.

Notice that if one takes the pointed pair $(X, \{x_0\}, x_0)$, then $\pi_n(X, \{x_0\}, x_0) = \pi_n(X, x_0)$. Consider the inclusions $i : (A, x_0) \hookrightarrow (X, x_0)$ and $j : (X, \{x_0\}, x_0) \hookrightarrow (X, A, x_0)$. Furthermore, given a map of triples $\alpha : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, \{x_0\})$ we may take its restriction $\alpha' : (I^{n-1}, \partial I^{n-1}) \subset (\partial I^n, J^{n-1}) \longrightarrow (A, \{x_0\})$. Then the mapping $[\alpha] \mapsto [\alpha']$ clearly defines a homomorphism $\partial : \pi_n(X, A, x_0) \longrightarrow \pi_{n-1}(A, x_0)$ which can be shown to be a homomorphism. The homomorphisms induced by i and j together with ∂ fit into a long sequence of homomorphisms. When there is no danger of confusion we shall omit the base points from the notation. We have the following.

4.4.17 **Theorem**. *The sequence*

$$\cdots \longrightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \longrightarrow \cdots \longrightarrow \pi_0(X)$$

is exact. Notice that the last three terms of the sequence need not be groups. Nonetheless the exactness makes sense.

Proof: We start checking the exactness at $\pi_n(X)$. First notice that if we take any $[\alpha'] \in \pi_n(A)$, then $j_*i_*([\alpha'] \in \pi_n(X, A)$ is represented by the composite

$$\alpha : (I^n, \partial I^n, J^{n-1}) \xrightarrow{\alpha'} (A, \{x_0\}, \{x_0\}) \hookrightarrow (X, A, \{x_0\})\,.$$

Thus $\alpha(I^n) \subset A$ and by the criterion 4.4.14, $j_*i_*([\alpha']) = [\alpha] = 0 \in \pi_n(X, A)$. Therefore $\mathrm{im}\,(i_*) \subseteq \ker(j_*)$. Conversely, if $\alpha : (I^n, \partial I^n) \longrightarrow (X, \{x_0\})$ is such that $j \circ \alpha : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, \{x_0\})$ is nullhomotopic, then again by 4.4.14 $j \circ \alpha$ is homotopic to a map $\alpha' : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, \{x_0\})$ such that $\alpha'(I^n) \subset A$. Therefore α' determines a map of pairs $\beta : (I^n, \partial I^n) \longrightarrow (A, \{x_0\})$ such that $i_*([\beta]) = [\alpha'] = [\alpha]$. Hence $\ker(i_*) \subseteq \mathrm{im}\,(j_*)$ and the sequence is exact at $\pi_n(X)$.

Take now $[\alpha] \in \pi_n(X)$, $\alpha : (I^n, \partial I^n) \longrightarrow (X, \{x_0\})$. We can consider α as a map $i \circ \alpha : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, \{x_0\})$ whose restriction $\beta : (I^{n-1}, \partial I^{n-1}) \subset (\partial I^n, J^{n-1}) \longrightarrow (A, \{x_0\})$ is constant. Hence $\partial j_*([\alpha]) = 0$ and thus $\mathrm{im}\,(j_*) \subseteq \ker(\partial)$. Conversely, assume that $\alpha : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, \{x_0\})$ is such that its restriction $\alpha' : (I^{n-1}, \partial I^{n-1}) \hookrightarrow (\partial I^n, J^{n-1}) \xrightarrow{\alpha|} (A, \{x_0\})$ is nullhomotopic via a homotopy $K : I^{n-1} \times I \longrightarrow A$ such that $K(s, 0) = \alpha'(s) = \alpha(s, 0)$ and $K(s, 1) = x_0 = K(s', t)$ for all $s \in I^{n-1}$, $s' \in \partial I^{n-1}$, $t \in I$. We can view $I^{n-1} \times I$ as $I^{n-1} \times \{0\} \times I \subset I^n \times I$ and I^n as $I^n \times \{0\} \subset I^n \times I$ and take the retraction $r : I^n \times I \twoheadrightarrow (I^{n-1} \times \{0\} \times I) \cup (I^n \times \{0\})$ given by

$$r(s, t', t) = \begin{cases} (s, t' - t, 0) & \text{if } t \le t'\,, \\ (s, 0, t - t') & \text{if } t \ge t'\,, \end{cases}$$

where $(s, t', t) \in I^{n-1} \times I \times I$ (see Figure 4.12, where (b) represents the retraction seen from the side).

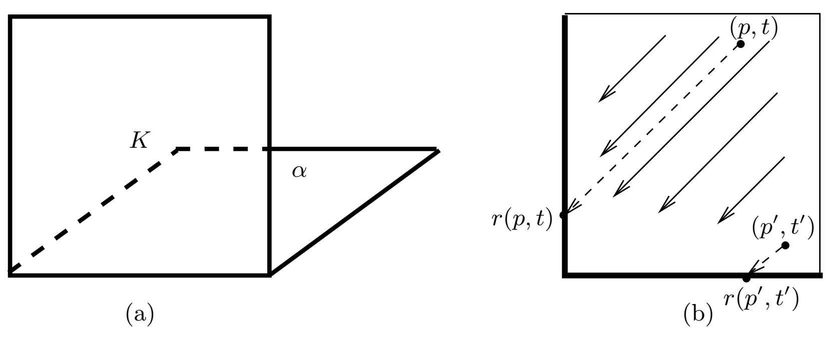

Figure 4.12 The retraction $r : I^n \times I \longrightarrow I^n \times \{0\} \cup I^{n-1} \times I$

Take the composite $K' : I^n \times I \xrightarrow{r} I^n \times \{0\} \cup I^{n-1} \times I \xrightarrow{(\alpha,K)} X$, where the second arrow means that we take α on the bottom $I^n \times \{0\}$, while we take the homotopy K on the vertical wall $I^{n-1} \times I$. Then we consider β to be the end of the homotopy K', namely $\beta : I^n \longrightarrow X$ is given by $\beta(s) = K'(s, 1)$. By definition,

$\beta(\partial I^n) = \{x_0\}$ and thus by restriction it determines $\beta' : (I^n, \partial I^n) \mapsto (A, \{x_0\})$. Hence $j_*([\beta']) = [\alpha]$ and so $\ker(\partial) \subset \operatorname{im}(j_*)$ and the sequence is exact at $\pi_n(X, A)$.

Let us take now an element $[\alpha] \in \pi_n(X, A)$, $\alpha : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, \{x_0\})$. $i_*\partial([\alpha])$ is represented by the restriction

$$\alpha' : (I^{n-1}, \partial I^{n-1}) \hookrightarrow (\partial I^n, J^{n-1}) \xrightarrow{\alpha|} (A, \{x_0\}) \hookrightarrow (X, \{x_0\}),$$

We consider $\alpha : I^n = I^{n-1} \times I \longrightarrow X$ as a homotopy such that $\alpha(s, 0) = \alpha'$, $\alpha(s, 1) = x_0 = \alpha(s', t)$ for all $s \in I^{n-1}$, all $s' \in \partial I^{n-1}$, and all $t \in I$. Thus α is a nullhomotopy of α' relative to ∂I^{n-1} and thus $i_*\partial([\alpha]) = [\alpha'] = 0$. Hence $\operatorname{im}(\partial) \subseteq \ker(i_*)$. Conversely, assume that $\beta : (I^{n-1}, \partial I^{n-1}) \longrightarrow (A, \{x_0\})$ is such that the composite $\beta' : (I^{n-1}, \partial I^{n-1}) \longrightarrow (A, \{x_0\}) \hookrightarrow (X, \{x_0\})$ is nullhomotopic. Let $H : I^{n-1} \times I \longrightarrow X$ be a nullhomotopy, i.e. $H(s, 0) = \beta(s)$, $H(s, 1) = x_0 = H(s', t)$ for all $s \in I^{n-1}$, all $s' \in \partial I^{n-1}$, and all $t \in I$. Since $I^{n-1} \times I = I^n$, H can be seen as a map $\alpha : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, \{x_0\})$, since $\alpha(I^{n-1} \times \{0\}) = \beta(I^{n-1}) \subset A$ and $\alpha(J^{n-1}) = x_0$. Hence $[\beta] = \partial[\alpha]$ and $\ker(i_*) \subseteq \operatorname{im}(\partial)$ and the sequence is exact at $\pi_{n-1}(X)$. □

4.4.18 EXERCISE. Let $B \subset A \subset X$ be a triple of pointed spaces. Show that there is a long exact sequence

$$\cdots \longrightarrow \pi_n(A, B) \xrightarrow{i_*} \pi_n(X, B) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A; B) \longrightarrow \cdots,$$

where $\partial : \pi_n(X, A) \longrightarrow \pi_{n-1}(A) \longrightarrow \pi_{n-1}(A, B)$, the first arrow being the connecting homomorphism of the pair (X, A) and the second being induced by the inclusion $(A, \{x_0\}) \hookrightarrow (A, B)$. (*Hint:* Either combine the long exact sequences of the different pairs or adapt the proof of 4.4.17 to this case. Notice that both sequences are equivalent.)

4.4.19 **Theorem**. *Let K be a pointed simplicial set. There is a natural isomorphism $\pi_q(K) \xrightarrow{\cong} \pi_q(|K|)$.*

4.4.20 **Proposition**. *Given a Serre fibration $p : E \longrightarrow X$ with fiber F over x_0, p induces an isomorphism $p_* : \pi_n(E, F) \longrightarrow \pi_n(X, x_0)$ for all n, that is as a map of pairs, $p : (E, F) \longrightarrow (X, x_0)$ is a* weak homotopy equivalence.

Proof: Given an element $a \in \pi_n(X, x_0)$ represented by a map $\alpha : (I^n, \partial I^n) \longrightarrow (X, x_0)$, consider the lifting problem

$$\begin{tikzcd} J^{n-1} \arrow[r, "\kappa_{x_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \arrow[r, "\alpha"'] \arrow[ur, dashed, "\beta"] & X \end{tikzcd}$$

Since the inclusion $J^{n-1} \hookrightarrow I^n$ is a cofibration and a homotopy equivalence, by Theorem 4.3.12 (and Remark 4.3.13), the problem has a solution $\beta : (I^n, \partial I^n, J^{n-1}) \longrightarrow$

$(E,F,\{x_0\})$. Thus $b=[\beta]\in\pi_n(E,F)$ is such that $p_*(b)=a$ and hence p_* is surjective.

Assume now that $b\in\pi_n(E,F)$ is such that $p_*(b)=0$. If b is represented by a map $\beta:(I^n,\partial I^n,J^{n-1})\longrightarrow(E,F,\{x_0\})$, this means that $\alpha=p\circ\beta:(I^n,\partial I^n)\longrightarrow(X,x_0)$ is nullhomotopic. Hence there is a homotopy $H:I^n\times I\longrightarrow X$ such that $H(s,0)=p\beta(s)$ and $H(\partial I^n\times I\cup I^n\times\{1\})=x_0$. Consider the following lifting problem

$$\begin{tikzcd}
I^n\times\{0\}\cup\partial I^n\times I \arrow[r, "(\beta,\kappa_{e_0})"] \arrow[d, "j"'] & E \arrow[d, "p"] \\
I^n\times I \arrow[r, "H"'] \arrow[ur, dashed, "K"] & X
\end{tikzcd}$$

Hence K is a homotopy of β to β' relative to ∂I^n, such that $\beta'(I^n)\subset F$. Hence, by 4.4.14, $b=[\beta]=0$ and p_* is injective. $\square$

Hence we have the following consequence.

4.4.21 **Theorem**. *Given a Serre fibration $p:E\longrightarrow X$ with fiber F, there is a natural long exact sequence*

$$\cdots\longrightarrow\pi_{n+1}(X)\longrightarrow\pi_n(F)\longrightarrow\pi_n(E)\longrightarrow\pi_n(X)\longrightarrow\pi_{n-1}(F)\cdots.$$

Proof: By Theorem 4.4.17 applied to the pair (E,F) we have a long exact sequence, where we may replace $\pi_n(E,F)$ with $\pi_n(X,x_0)$ for all n, namely

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr] & \pi_{n+1}(E,F) \arrow[r, "\partial"] \arrow[d, "p_*"', "\cong"] & \pi_n(F,e_0) \arrow[r, "i_*"] & \pi_n(E,e_0) \arrow[r, "j_*"] \arrow[dr, "p_*"'] & \pi_n(E,F) \arrow[r, "\partial"] \arrow[d, "p_*"', "\cong"] & \pi_{n-1}(F,e_0) \arrow[r] & \cdots \\
& \pi_{n+1}(X,x_0) \arrow[ur, "\partial\circ p_*^{-1}"'] & & & \pi_n(X,x_0) \arrow[ur, "\partial\circ p_*^{-1}"'] & &
\end{tikzcd}$$

$\square$

4.4.22 **Theorem**. *Let $p:E\longrightarrow X$ be a Hurewicz fibration with fiber $F=p^{-1}(x_0)$ and contractible total space E. Then the holonomy $\theta:\Omega X\longrightarrow F$ is a weak homotopy equivalence.*

Proof: Let $\Gamma_p:E\times_X M(I,X)\longrightarrow M(I,E)$ be a PLM for p. Consider the path fibration $\pi:PX\longrightarrow X$ (see 4.1.5 1) whose fiber over x_0 is ΩX and consider the map $\eta:PX\longrightarrow E$ given by $\eta(\sigma)=\Gamma_p(e_0,\sigma)(1)$. The restriction of η to the loop space ΩX is the holonomy. Thus there is a commutative diagram

$$\begin{tikzcd}
\Omega X \arrow[r, hook, "\iota"] \arrow[d, "\theta"'] & PX \arrow[r, two heads, "\pi"] \arrow[d, "\eta"'] & X \arrow[d, equal] \\
F \arrow[r, hook, "i"'] & E \arrow[r, two heads, "p"'] & X\,.
\end{tikzcd}$$

The vertical arrows fit into the long exact sequences of homotopy groups of π and p, namely, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r, equal] & \pi_{n+1}(PX) \arrow[r, "\pi_*"] \arrow[d, "\eta_*"'] & \pi_{n+1}(X) \arrow[r, "\partial", "\cong"'] \arrow[d, equal] & \pi_n(\Omega X) \arrow[r, "\iota_*"] \arrow[d, "\theta_*"] & \pi_n(PX) \arrow[r, equal] \arrow[d, "\eta_*"'] & 0 \\
0 \arrow[r, equal] & \pi_{n+1}(E) \arrow[r, "p_*"'] & \pi_{n+1}(X) \arrow[r, "\cong", "\partial"'] & \pi_n(F) \arrow[r, "i_*"'] & \pi_n(E) \arrow[r, equal] & 0\,.
\end{tikzcd}$$

Hence ∂ on the top as well as on the bottom is an isomorphism and therefore $\theta_* : \pi_n(\Omega X) \longrightarrow \pi_n(F)$ is an isomorphism for all n. Thus θ is a weak homotopy equivalence.

4.4.23 NOTE. The connecting homomorphisms $\partial : \pi_n(X) \longrightarrow \pi_{n-1}(\Omega X)$ are isomorphisms because they lie between two zero-groups. That the one on the top is an isomorphism is no surprise, since by the exponential law, $\pi_{n-1}(\Omega X) = \pi_0(M_*(\mathbb{S}^{n-1}, M_*(\mathbb{S}^1, X)) = [\mathbb{S}^n, X]_* = \pi_n(X)$.

In this example, by using 4.4.21 and 4.2.5, we get an exact sequence

(4.4.13)
$$\begin{gathered}\cdots \longrightarrow \pi_q(\mathbb{Z}) \longrightarrow \pi_q(\mathbb{R}) \longrightarrow \pi_q(\mathbb{S}^1) \longrightarrow \pi_{q-1}(\mathbb{Z}) \longrightarrow \cdots \\ \cdots \longrightarrow \pi_1(\mathbb{R}) \longrightarrow \pi_1(\mathbb{S}^1) \longrightarrow \pi_0(\mathbb{Z}) \longrightarrow \pi_0(\mathbb{R}).\end{gathered}$$

Since we have

$$\pi_q(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ 0 & \text{if } q \neq 0, \end{cases}$$

and

$$\pi_q(\mathbb{R}) = 0 \quad \text{if} \quad q \geq 0\,,$$

we obtain the next result.

4.4.14 **Theorem**. *The homotopy groups of* $\mathbb{S}^1$ *are given by*

$$\pi_q(\mathbb{S}^1) = \begin{cases} \mathbb{Z} & \text{if } q = 1, \\ 0 & \text{if } q \neq 1. \end{cases}$$ □

That is to say, we have proved that $\mathbb{S}^1$ *is an Eilenberg–Mac Lane space of type* $K(\mathbb{Z}, 1)$ (see Chapter 8).

4.4.15 EXERCISE. Let $p : E \longrightarrow X$ be a covering map where X is path connected and *locally path connected.* To say that X is a locally path-connected space means that for each point $x \in X$ and each neighborhood U of x in X there is a neighborhood $V \subset U$ of x that is path connected. Let X be path connected. Prove that for every map $f : X \longrightarrow X$ and for all points $x_0 \in X$ and $y_0 \in p^{-1}(f(x))$, there exists a unique lifting $\widetilde{f} : X \longrightarrow E$ such that $\widetilde{f}(x_0) = y_0$ if and only if $f_*\pi_1(X, x_0) \subset p_*\pi_1(E, y_0)$. (*Hint:* For each point $x \in X$ let $\alpha : I \longrightarrow X$ be a path such that $\alpha(0) = x_0$ and $\alpha(1) = x$. Using 4.2.8, there exists a unique path $\widetilde{\alpha} : I \longrightarrow E$ such that $\widetilde{\alpha}(0) = y_0$ and $p \circ \widetilde{\alpha} = \alpha$. We then define $\widetilde{f} : X \longrightarrow E$ by $\widetilde{f}(x) = \widetilde{\alpha}(1)$. Using the hypotheses, prove that $\widetilde{f}$ is well defined and continuous.)

For the following exercises recall the definition of the fundamental group $\pi_1(X) = \pi_1(X, x_0)$ given in 4.4.7.

4.4.16 Exercise. Let $p : E \longrightarrow X$ be a covering map.

(a) Prove that we have an *action* of the fundamental group of the base $\pi_1(X, x_0)$ on the fiber $F = p^{-1}(x_0)$ such that if $[\alpha] \in \pi_1(X, x_0)$ and $e \in F$, then $e \cdot [\alpha] = \widetilde{\alpha}(1)$, where $\widetilde{\alpha} : I \longrightarrow E$ is the lifting of α satisfying $\widetilde{\alpha}(0) = e$ (see 4.2.8). In other words, prove that $e \cdot 1 = e$ and that $e \cdot ([\alpha][\beta]) = (e \cdot [\alpha]) \cdot [\beta]$, where $1, [\alpha], [\beta] \in \pi_1(X, x_0)$ (that is, $\pi_1(X, x_0)$ *acts on* F). Moreover, if the space E is path connected, prove that for every $e_1,\ e_2 \in F$ there exists $[\alpha] \in \pi_1(X, x_0)$ such that $e_1 \cdot [\alpha] = e_2$ (that is, the action is *transitive*). (*Hint:* The action is defined by using the unique path-lifting property 4.2.8. In order to prove that it is transitive, for any given e_1 and e_2 take a path $\widetilde{\alpha}$ from e_1 to e_2 and define $\alpha = p \circ \widetilde{\alpha}$.)

(b) Prove that the homomorphism $p_* : \pi_1(E, e_0) \longrightarrow \pi_1(X, x_0)$ is a monomorphism. (*Hint:* If $\widetilde{\alpha} : I \longrightarrow E$ is a closed path in E such that $\widetilde{\alpha}(0) = \widetilde{\alpha}(1) = e_0$ and such that $\alpha = p \circ \widetilde{\alpha} \simeq 0$ in X, then there is a lifting of every nullhomotopy of α, which in turn defines a nullhomotopy of $\widetilde{\alpha}$.)

(c) Assume that the space E is path connected and take $e_0 \in F$. Prove that the function $[\alpha] \mapsto e_0 \cdot [\alpha]$ defines an isomorphism (as sets) between F and the set of (right) cosets of $p_*\pi_1(E, e_0)$ in $\pi_1(X, x_0)$. (*Hint:* One has $e_0 \cdot [\alpha] = e_0 \cdot [\beta]$ if and only if $p_*\pi_1(E, e_0)[\alpha] = p_*\pi_1(E, e_0)[\beta]$.)

(d) Suppose that E is simply connected, that is, $\pi_1(E) = 1$. Conclude that $\pi_1(X, x_0) \cong F$ as sets. A covering map $p : E \longrightarrow X$ such that $\pi_1(E) = 1$ is called a *universal covering map.*

4.4.17 Exercise. Let $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ be the exponential map, namely, $p(t) = \mathrm{e}^{2\pi \mathrm{i} t}$. Prove that p is a universal covering map, so that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ at least as sets. (See Figure 4.13, and compare this with 4.2.12.)

4.4.18 Exercise. Let $p : \mathbb{S}^n \longrightarrow \mathbb{RP}^n$ for $n > 1$ be the canonical projection. Prove that p is a universal covering map whose fiber F consists of two points. Conclude that $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$.

4.4.19 Note. The results stated in Exercises 4.4.16(b) and (c) can be obtained from the long exact homotopy sequence of a Serre fibration (see 4.4.21).

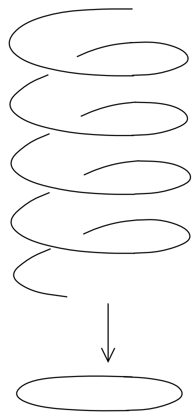

Figure 4.13

4.4.1 The geometric realization of the singular simplicial set

Recall the map $\rho_X : |\mathcal{S}(X)| \longrightarrow X$ defined by $\rho_X([\sigma, s]) = \sigma(s)$, where $\sigma : \Delta^n \longrightarrow X$ and $s \in \Delta^n$, introduced in the previous chapter. The following is a result of Milnor.

4.4.20 **Theorem**. *The map $\rho_X : |\mathcal{S}(X)| \longrightarrow X$ is a weak homotopy equivalence.*

Proof: We must prove that $\rho_{X*} : \pi_n(|\mathcal{S}(X)|) \longrightarrow \pi_n(X)$ is an isomorphism for all n. We start proving the bijection in the case $n = 0$. We must show that ρ_X establishes a one-to-one correspondence between the path-components of $|\mathcal{S}(X)|$ and X. To see that, we may assume without loss of generality that X nonempty and 0-connected and it will be enough to prove that $|\mathcal{S}(X)|$ is also 0-connected. Take a base point $x_0 \in X$ and let $\widetilde{x}_0$ be the unique 0-cell in $|\mathcal{S}(X)|$ which corresponds to the singular simplex $\Delta^0 \longrightarrow X$, $1 \mapsto x_0$. Notice that since every (connected) CW-complex has at least one 0-cell, every path component component of $|\mathcal{S}(X)|$ contains a 0-cell. Now we show that for every 0-cell $\widetilde{x} \neq \widetilde{x}_0$ in $|\mathcal{S}(X)|$ there is a 1-cell therein whose boundary is $\{\widetilde{x}, \widetilde{x}_0\}$. To see this, consider a path in X joining $\rho_X(\widetilde{x})$ and $\widetilde{x}_0$. The path gives rise to a singular 1-simplex with the desired property. Thus $|\mathcal{S}(X)|$ is 0-connected.

Now we prove that

$$\rho_{X*} : \pi_n(|\mathcal{S}(X)|, \widetilde{x}_0) \longrightarrow \pi_n(X, x_0)$$

is an isomorphism for every $n > 0$. First let $f : \mathbb{S}^n \longrightarrow X$ represent some element of $\pi_n(X, x_0)$. We consider the sphere $\mathbb{S}^n$ as the geometric realization $|\partial\Delta^{n+1}|$ (see 2.2.8), such that the base point $e_0 \in |\partial\Delta[n+1]|$ corresponds to some 0-cell in its CW-decomposition. According to Theorem 2.2.10, there is an adjoint to f,

$\widehat{f} : \partial\Delta[n+1] \longrightarrow \mathcal{S}X$. Hence the geometric realization $|\widehat{f}|$ represents an element in $\pi_n(|\mathcal{S}X|, \widetilde{x}_0)$ which is clearly mapped by ρ_X to $\rho_X \circ |\widehat{f}| = f$ (see the proof of 2.2.10). This proves that ρ_{X*} is surjective.

Since ρ_{X*} is a homomorphism, to prove that it is injective it is enough to show that $\ker \rho_{X*}$ is trivial. To do this, we shall prove that if a map

$$f : (|\partial\Delta[n+1]|, e_0) \longrightarrow (|\mathcal{S}X|, \widetilde{x}_0)$$

is such that the composite $\rho_X \circ f$ admits an extension $G : (|\Delta[n+1]|, e_0) \longrightarrow (|\mathcal{S}X|, \widetilde{x}_0)$, then f itself can also be extended over $|\Delta[n+1]|$. Hence we assume f and G given, and let $\widehat{f} : \partial\Delta[n+1] \longrightarrow \mathcal{S}(|\mathcal{S}X|)$ and $\widehat{G} : \Delta[n+1] \longrightarrow \mathcal{S}X$ be their respective adjoint maps according to 2.2.10. Let $H : \mathcal{S}(|\mathcal{S}X|) \times \Delta[1] \longrightarrow \mathcal{S}(|\mathcal{S}X|)$ be a simplicial homotopy relative to $\mathcal{S}X$, which by Proposition 2.4.8 exists. Define a simplicial map

$$\widetilde{f} : \Delta[n+1] \times \Lambda^1[1] \cup \partial\Delta[n+1] \times \Delta[1] \longrightarrow \mathcal{S}(|\mathcal{S}X|)$$

by

$$\widetilde{f}(s, \varepsilon_1\omega) = \alpha_{\mathcal{S}X}\widehat{G}(s) \quad \text{and} \quad \widetilde{f}(\delta_i, t) = H(\widehat{G}(\delta_i), t)$$

for $s \in \Delta[n+1]$, $t \in \Delta[1]_n$, and $i \in \mathbf{n+1}$. The domain of its adjoint is homeomorphic to $|\Delta[n+1]|$ and agrees with f on the boundary. Therefore it is the desired extension of f and $\ker \rho_{X*} = 0$. Hence ρ_{X*} is injective. $\square$

Chapter 5 Elements of homological algebra

In this chapter we give a brief introduction to homological algebra over a principal ideal domain. We shall state results and omit proofs, and we refer the reader to any of the many books on the topic, for instance [25] or [32].

The basic objects of homological algebra are R-modules. Since we are assuming R to be a principal ideal domain, it is commutative by definition. Hence we make no distinction between left and right R-modules. One of the main properties that a principal ideal domain has is that *given a free R-module M and a submodule $N \subseteq M$, then N is also a free module.*

In most of our applications, R will be $\mathbb{Z}$ or a field.

5.1 The functors Tor and Ext

Given two R-modules M and N there are several functors: the well-known *tensor product* $M \otimes_R N$ and *Hom-product* $Hom_R(M, N)$ (see [25] or [32]), and their *derived functors* $\mathrm{Tor}_R(M, N)$ and $\mathrm{Ext}_R(M, N)$. In what follows, we shall define these two.

5.1.1 **Proposition**. *Given an R-module M there is a short exact sequence*

$$0 \longrightarrow F_2 \xrightarrow{\beta} F_1 \xrightarrow{\alpha} M \longrightarrow 0 \tag{5.1.2}$$

such that F_1 and F_2 are free modules. This sequence is called free resolution *of M.*

Proof: Take any set S of generators of M and define F_1 as the free R-module generated by S. If g_s is the free generator of F_1 corresponding to the generator $s \in S$, then define $\alpha(g_s) = s$ and extend α to all of F_1. To finish the proof take $F_2 = \ker(\alpha)$ and let $\beta : F_2 \longrightarrow F_1$ be the inclusion. Since F_2 is free because it is a submodule of F_1 which is free, we obtain the desired short exact sequence. □

5.1.3 Definition. Given two R-modules M and N we define the R-modules $\mathrm{Tor}_R(M, N)$ and $\mathrm{Ext}_R(M, N)$ as follows. By the previous proposition, there is a short exact sequence of R-modules (5.1.2), where F_1 and F_2 are free. Even though

β is injective, the homomorphism $\beta \otimes 1_N$ is not necessarily injective, unless, for instance, N is a free module. Define $\mathrm{Tor}_R(M,N)$ to be the kernel of

$$F_2 \otimes N \xrightarrow{\beta\otimes 1_N} F_1 \otimes N\,.$$

Furthermore, the homomorphism $\beta^* = \mathrm{Hom}(\beta, 1_N)$ is not necessarily surjective, unless, for instance, N is free. Define $\mathrm{Ext}_R(M,N)$ to be the cokernel of

$$\mathrm{Hom}_R(F_1,N) \xrightarrow{\beta^*} \mathrm{Hom}_R(F_2,N)\,.$$

5.1.4 **Proposition**. *The R-modules* $\mathrm{Tor}_R(M,N)$ *and* $\mathrm{Ext}_R(M,N)$ *are well defined, that is, they do not depend on the free resolution* (5.1.2). *Furthermore* $\mathrm{Tor}_R(M,N)$ *is a covariant bifunctor in M and N and* $\mathrm{Ext}_R(M,N)$ *is a contravariant functor in M and covariant in N.*

Proof: Let $\varphi : M \longrightarrow M'$ be any homomorphism of R-modules and consider free resolutions

(5.1.5) $$0 \longrightarrow F_2 \xrightarrow{\beta} F_1 \xrightarrow{\alpha} M \longrightarrow 0$$

(5.1.6) $$0 \longrightarrow F_2' \xrightarrow{\beta'} F_1' \xrightarrow{\alpha'} M' \longrightarrow 0$$

If $g_s \in F_1$ is a free generator, take any element $g_s' \in \alpha'^{-1}(\varphi\alpha(g_s))$ (it exists because α' is surjective) and define f_1 as the homomorphism determined by $f_1(g_s) = g_s'$. Then clearly $\alpha' \circ f_1 = f \circ \alpha$. By restriction, f_1 determines a homomorphism $f_2 : F_2 \longrightarrow F_2'$ such that $\beta' \circ f_2 = f_1 \circ \beta$. Thus we have a commutative diagram

(5.1.7)
$$\begin{tikzcd}
0 \arrow[r] & F_2 \arrow[r, "\beta"] \arrow[d, "f_2"'] & F_1 \arrow[r, "\alpha"] \arrow[d, "f_1"'] & M \arrow[r] \arrow[d, "\varphi"'] & 0 \\
0 \arrow[r] & F_2' \arrow[r, "\beta'"] & F_1' \arrow[r, "\alpha'"] & M' \arrow[r] & 0
\end{tikzcd}$$

Now define homomorphisms

$$\varphi_* = f_2 \otimes 1_n : \mathrm{Tor}_R(M,N) = \ker(\beta\otimes 1_N) \longrightarrow \ker(\beta'\otimes 1_N) = \mathrm{Tor}_R(M',N)\,,$$
$$\varphi^* = f_2^* : \mathrm{Ext}_R(M',N) = \mathrm{coker}\,(\beta'^*) \longrightarrow \mathrm{coker}\,(\beta^*) = \mathrm{Ext}_R(M,N)\,.$$

5.1.8 EXERCISE. Consider a short exact sequence of R-modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and take any R-module D.

(a) Show that $A \otimes_R D \longrightarrow B \otimes_R D \longrightarrow C \otimes_R D \longrightarrow 0$ is exact. Show with an example that the first arrow need not be injective.

(b) Show that $0 \longrightarrow \mathrm{Hom}_R(C,D) \longrightarrow \mathrm{Hom}_R(B,D) \longrightarrow \mathrm{Hom}_R(A,D)$ is exact. Show with an example that the last arrow need not be surjective.

5.1.9 EXERCISE. Consider a short exact sequence $0 \longrightarrow F_2 \xrightarrow{\beta} F_1 \xrightarrow{\alpha} M \longrightarrow 0$ of R-modules, where F_1 and F_2 are free R-modules, and let N be any R-module.

(a) Show that the sequence $0 \longrightarrow \mathrm{Tor}_R(M,N) \longrightarrow F_2 \otimes_R N \longrightarrow F_1 \otimes_R N \longrightarrow M \otimes_R N \longrightarrow 0$ is exact.

(b) Show that the sequence $0 \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(F_1,N) \longrightarrow \mathrm{Hom}_R(F_2,N) \longrightarrow \mathrm{Ext}_R(M,N) \longrightarrow 0$ is exact.

Now we prove that $\mathrm{Tor}_R(M,N)$ and $\mathrm{Ext}_R(M,N)$ are independent of the choices. Assume first that we have instead of diagram (5.1.7) the diagram

$$\begin{tikzcd} 0 \arrow[r] & F_2 \arrow[r, "\beta"] \arrow[d, "f_2'"'] & F_1 \arrow[r, "\alpha"] \arrow[d, "f_1'"'] & M \arrow[r] \arrow[d, "\varphi"'] & 0 \\ 0 \arrow[r] & F_2' \arrow[r, "\beta'"] & F_1' \arrow[r, "\alpha'"] & M' \arrow[r] & 0 \end{tikzcd} \tag{5.1.10}$$

where we just replaced f_1 and f_2 by f_1' and f_2'. Then $\alpha'(f_1 - f_1') = 0$, that is $\mathrm{im}\,(f_1 - f_1') \subseteq \ker(\alpha') = \mathrm{im}\,(\beta')$. Take a free generator g_s of F_1 and its image $f_1(g_s) - f_1'(g_s)$. Since this last lies in $\mathrm{im}\,(\beta')$, take an element $x_s' \in F_2'$ such that $\beta'(x_s') = f_1(g_s) - f_1'(g_s)$ and define $\varphi : F_1 \longrightarrow F_2'$ by $\varphi(g_s) = x_s'$ on the generators. Thus

$$\beta' \circ \varphi = f_1 - f_1' \,.$$

Hence $f_2 \otimes 1_N - f_2' \otimes 1_N = 0$ on $\ker(\beta \otimes 1_N)$ and $f_2^* - f_2'^* = 0$ on $\mathrm{coker}\,(\beta^*)$. Thus

$$f_2 \otimes 1_N \qquad \text{and} \qquad f_2^*$$

do not depend on the choice of f_1 and f_2.

We already saw that given free resolutions of M and M' respectively, there are well-defined homomorphisms

$$\varphi_* : \mathrm{Tor}_R(M,N) \longrightarrow \mathrm{Tor}_R(M',N) \ \text{ and } \ \varphi^* : \mathrm{Ext}_R(M',N) \longrightarrow \mathrm{Ext}_R(M,N)\,.$$

Clearly, by definition, the constructions are functorial (on the resolutions), namely $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$, $1_{M*} = 1_{\mathrm{Tor}_R(M,N)}$ and $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$, $1_M^* = 1_{\mathrm{Ext}_R(M,N)}$. Now we may compare the values of Tor and Ext using two different resolutions of M by taking $\varphi = 1_M$. As a consequence, there are homomorphisms going both ways whose composite is the identity. Thus Tor and Ext do not depend on the resolution and are functors of M. One can also easily see that both are covariant functors of N. □

5.1.11 REMARK. If M is a free R-module, then one may take $F_1 = M$ and $F_2 = 0$. Hence in this case, $\mathrm{Tor}_R(M,N) = \mathrm{Ext}_R(M,N) = 0$. In particular, if R is a field, every module M is free and $\mathrm{Tor}_R(M,N)$ and $\mathrm{Ext}_R(M,N)$ are always trivial.

In what follows, if $R = \mathbb{Z}$, we omit the subindex R in Tor and Ext.

5.1.12 EXERCISE. Show the next.

(a) $\mathrm{Ext}(\mathbb{Z}_n, \mathbb{Z}) \cong \mathbb{Z}_n$.

(b) $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_k$, where k is the greatest common divisor of m and n.

(c) $\mathrm{Tor}_R(M, N\oplus N') \cong \mathrm{Tor}_R(M,N)\oplus\mathrm{Tor}_R(M,N'), \mathrm{Ext}_R(M, N\oplus N') \cong \mathrm{Ext}_R(M,N)\oplus \mathrm{Ext}_R(M,N')$.

(d) $\mathrm{Tor}_R(M \oplus M', N) \cong \mathrm{Tor}_R(M,N) \oplus \mathrm{Tor}_R(M',N)$, $\mathrm{Ext}_R(M \oplus M', N) \cong \mathrm{Ext}_R(M,N) \oplus \mathrm{Ext}_R(M',N)$.

5.2 CHAIN COMPLEXES AND HOMOLOGY

Chain complexes constitute the foreplay to homology. We shall study them in this section.

5.2.1 DEFINITION. A *chain complex* of R-modules D is a family of R-modules D_n together with *boundary homomorphisms* $\partial_n : D^n \longrightarrow D_{n_1}$ such that for all n, $\partial_n \circ \partial_{n+1} = 0$. Thus if we consider $Z_n(D) = \ker(\partial_n)$ and $B_n(D) = \mathrm{im}\,(\partial_{n+1})$, we have that $B_n(D) \subseteq Z_n(D)$ and we define the *homology groups* of D by

$$H_n(D) = Z_n(D)/B_n(D)\,.$$

We shall denote the *homology classes* by $[d] \in H_n(D)$ if $d \in Z_n(C)$.

Let D be a chain complex and let C be a *chain subcomplex* of D, namely, for each n, we have $C_n \subset D_n$ is a submodule and the boundary homomorphisms of C are the restrictions of those of D, i.e. the following square commutes:

$$\begin{tikzcd} C_n \arrow[r, "\partial_n^C"] \arrow[d, hook] & C_{n-1} \arrow[d, hook] \\ D_n \arrow[r, "\partial_n^D"'] & D_{n-1}\,. \end{tikzcd}$$

One has a *chain quotient complex* D/C given by $(D/C)_n = D_n/C_n$ and $\partial_n^{D/C} : (D/C)_n \longrightarrow (D/C)_{n-1}$ induced by ∂_n^D. In what follows we shall write A instead of D/C, so that we have a short exact sequence of chain complexes $0 \longrightarrow C \longrightarrow D \longrightarrow A \longrightarrow 0$.

5.2.2 **Theorem**. *Let $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{q} A \longrightarrow 0$ be a short exact sequence of chain complexes. Then there is a long exact sequence of homology groups*

$$\cdots \longrightarrow H_{n+1}(D) \xrightarrow{q_*} H_{n+1}(A) \xrightarrow{\delta} H_n(C) \xrightarrow{i_*} H_n(D) \xrightarrow{q_*} H_n(A) \longrightarrow \cdots\,.$$

Proof: We have the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C_{n+2} \arrow[r, "\partial_{n+2}"] \arrow[d, "i_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "i_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "i_n"] & C_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+2} \arrow[r, "\partial_{n+2}"] \arrow[d, "q_{n+2}"] & D_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "q_{n+1}"] & D_n \arrow[r, "\partial_n"] \arrow[d, "q_n"] & C_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & A_{n+2} \arrow[r, "\partial_{n+2}"] \arrow[d] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d] & A_n \arrow[r, "\partial_n"] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

We must define d. Consider a cycle $a \in A_{n+1}$, i.e. $\partial_{n+1}(a) = 0$. Since q_{n+1} is surjective, there is an element $d \in D_{n+1}$ such that $q_{n+1}(d) = a$. Take $d' = \partial_{n+1}(d)$. By the commutativity of the corresponding square, $q_n(d') = 0$, therefore, there is an element $c \in C_n$ such that $i_n(c) = d'$. By the commutativity of the corresponding square and the injectivity of i_{n-1}, $\partial_n(c) = 0$, i.e. c is a cycle. Define $\delta([a]) = [c]$. If a_1 and a_2 are two cycles which represent the same class, then $a_1 - a_2 = \partial_{n+2}(a')$ for some $a' \in A_{n+2}$. There is a $d'' \in D_{n+2}$ such that $q_{n+2}(d'') = a'$. Hence $q_{n+1}\partial_{n+2}(d'') = a_1 - a_2$. Now we have that $\partial_{n+1}\partial_{n+2}(d'') = 0$, so that, since i_n is injective, the only element in A_n mapping to 0 is 0. Thus $\delta(a_1 - a_2) = 0$ and so δ is well defined. These facts are depicted in the following diagrams.

$$\begin{tikzcd}
 & c \arrow[d, mapsto] \\
d \arrow[r, mapsto] \arrow[d, mapsto] & d' \\
a &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & 0 \arrow[d, mapsto] \\
d'' \arrow[r, mapsto] \arrow[d, mapsto] & \partial(d'') \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
a' \arrow[r, mapsto] & a_1 - a_2 &
\end{tikzcd}$$

Thus we have the long sequence of the statement. We must show its exactness. First take a homology class $[d] \in H_{n+1}(D)$. Since d is a boundary, i.e. $d = \partial_{n+2}(d'')$, if we take $[a] = q_*([d]) = [q_{n+1}(d)] \in H_{n+1}(A)$, by the description given above to obtain $\delta([a])$ we have that $d' = \partial_{n+1}(d) = \partial_{n+2}(d'') = 0$, and since i_n is injective, $c = 0$. Hence $\delta i_*([d]) = 0$ and so $\mathrm{im}\,(\delta) \subset \ker(i_*)$. Conversely, if $\delta([a]) = 0$, then $c = \partial_{n+1}(c')$. If we take d as before, then $\partial_{n+1}(d - i_{n+1}(c')) = d' - \partial_{n+1}i_{n+1}(c') = d' - i_n\partial_{n+1}(c') = d' - i_n(c) = d' - d' = 0$. Thus $d - i_{n+1}(c')$ is a cycle that determines an element $[d - i_{n+1}(c')] \in H_{n+1}(D)$ such that $q_*([d - i_{n+1}(c')]) = [q_{n+1}(d - i_{n+1}(c'))] = [q_{n+1}(d)] = [a]$. Consequently $\ker(\delta) \subset \mathrm{im}\,(q_*)$ and the sequence is exact at $H_{n+1}(A)$ for all n.

Now take a homology class $[a] \in H_{n+1}(A)$, then $i_*\delta([a]) = i_*([c]) = [i_n(c)]$, where c is as described above. But $i_n(c) = d' = \partial_{n+1}(d)$, hence it is a boundary and so $[i_n(c)] = 0 \in H_n(D)$. Thus $\mathrm{im}\,(\delta) \subset \ker(i_*)$. Conversely, if $i_*([c]) = 0 \in H_n(D)$, then $i_n(c) = \partial_{n+1}(d)$ for some $d \in D_{n+1}$. Thus, if $a = q_{n+1}(d)$, then by definition of δ, $\delta([a]) = [c]$. Hence $\ker(i_*) \subset \mathrm{im}\,(\delta)$ and the exactness at $H_n(C)$ holds for all n.

If we take $[c] \in H_n(C)$, then $q_* i_*([c]) = [q_n i_n(c)] = 0$, since $q_n \circ i - n = 0$. Thus $\mathrm{im}\,(i_*) \subset \ker(q_*)$. Conversely, if $q_*([d]) = 0$, then $q_n(d) = \partial_{n+1}(a)$ for some $a \in A_{n+1}$. Since q_{n+1} is surjective, there is $d' \in D_{n+1}$ such that $q_{n+1}(d') = a$. Consider $q_n(d - \partial_{n+1}(d')) = q_n(d) - q_n\partial_{n+1}(d') = q_n(d) - \partial_{n+1}q_{n+1}(d') = 0$. Therefore, there is a cycle $a' \in A_n$ such that $d - \partial_{n+1}(d') = i_n(a')$ and hence $i_*([a']) = [i_n(a')] = [d - \partial_{n+1}(d')] = [d]$. Thus $\ker(q_*) \subset \mathrm{im}\,(i_*)$ and the sequence is exact at $H_n(D)$ for all n. $\square$

5.2.3 Definition. Let $f, g : C \longrightarrow D$ be two chain homomorphisms between chain complexes. A *chain homotopy* between f and g is a family of homomorphisms $P_n : C_n \longrightarrow D_{n+1}$ such that $\partial^D_{n+1} \circ P_n + P_{n-1} \circ \partial^C_n = f_n - g_n : C_n \longrightarrow D_n$. If such a chain homotopy exists, then we say that f and g are *chain homotopic*. We write $P : f \simeq g$.

5.2.4 **Theorem**. *If $f, g : C \longrightarrow D$ are chain homotopic chain homomorphisms, then $f_* = g_* : H_n(C) \longrightarrow H_n(D)$ for all n. Let P be a chain homotopy between f and g.*

Proof: Recall that given a cycle $c \in C_n$, then $f_*([c]) = [f(c)]$ and $g_*([c]) = [g(c)]$. Since there is $P : f \simeq g$, we have that for each n and each cycle c in C_n, $f_n(c) - g_n(c) = \partial^D_{n+1}P_n(c) + P_{n-1}\partial^C_n(c)$. Thus $f_*([c]) - g_*([c]) = [f_n(c) - g_n(c)] = [\partial^D_{n+1}P_n(c)] + [P_{n-1}\partial^C_n(c)] = 0$. $\square$

5.3 The Künneth and the universal coefficients formulae

Consider a chain complex D of R-modules

$$\cdots \longrightarrow D_{n+1} \xrightarrow{\partial_{n+1}} D_n \xrightarrow{\partial_n} D_{n-1} \longrightarrow \cdots \longrightarrow D_0$$

and define complexes $Z(D)$ and $B(D)$ given at n by $Z_n(D) = \ker(\partial_n)$ and $B_n(D) = \mathrm{im}\,(\partial_{n+1})$ and whose differentials are zero. Then we have a short exact sequence of complexes

$$0 \longrightarrow Z(D) \xrightarrow{i} D \xrightarrow{\partial} B(D) \longrightarrow 0\,. \tag{5.3.17}$$

5.3.18 Definition. Given chain complexes A and B one defines the *tensor product* $A \otimes B$ as the chain complex given at n by

$$(A \otimes B)_n = \bigoplus_{k+l=n} A_k \otimes B_l$$

with the differential $\partial_n : \bigoplus_{k+l=n} A_k \otimes B_l \longrightarrow \bigoplus_{k+l=n-1} A_k \otimes B_l$ given by

$$\partial_n(a \otimes b) = \partial_k(a) \otimes b \oplus (-1)^k a \otimes \partial_l(b)\,.$$

If C and D are chain complexes, we can tensor each term of (5.3.17) with C to obtain a sequence of chain complexes

$$C \otimes_R Z(D) \xrightarrow{1\otimes i} C \otimes_R D \xrightarrow{1\otimes \partial} C \otimes_R B(D) \longrightarrow 0\,. \tag{5.3.19}$$

It is an *exercise* to show that this is an exact sequence of chain complexes as well as to prove the following.

5.3.20 **Lemma**. *If C or D is R-free, then*

$$0 \longrightarrow C \otimes_R Z(D) \xrightarrow{1\otimes i} C \otimes_R D \xrightarrow{1\otimes \partial} C \otimes_R B(D) \longrightarrow 0$$

is an exact sequence of chain complexes. □

5.3.21 Exercise. Let $0 \longrightarrow C \xrightarrow{\alpha} D \xrightarrow{\beta} E \longrightarrow 0$ be an exact sequence of chain complexes and chain homomorphisms. Show that there is a long exact sequence

$$\cdots \xrightarrow{\partial} H_n(C) \xrightarrow{\alpha_*} H_n(D) \xrightarrow{\beta_*} H_n(E) \xrightarrow{\partial} H_{n-1}(C) \longrightarrow \cdots\,,$$

where $\partial[e]$, $[e] \in Z_n(E)/B_n(E)$, is given as follows. Let $e = \beta(d)$. Then $\beta(\partial(d)) = 0$. Hence $\partial(d) = \alpha(c)$. Define $\partial[e] = [c] \in Z_{n-1}(C)/B_{n-1}(C)$. Show furthermore that $c \in Z_{n-1}(C)$ and that the class $[c]$ depends only on the given class $[e]$.

If we apply the previous exercise to the short exact sequence of chain complexes of 5.3.20, we obtain a long exact sequence

$$\cdots \longrightarrow H_n(C \otimes_R Z(D)) \xrightarrow{(1\otimes i)_*} H_n(C \otimes_R D) \xrightarrow{(1\otimes \partial)_*} H_n(C \otimes_R B(D)) \xrightarrow{\partial}$$
$$\longrightarrow H_{n-1}(C \otimes_R Z(D)) \longrightarrow \cdots\,,$$

which will be useful in computing the homology of an arbitrary tensor product $C \otimes D$ of chain complexes.

5.3.22 **Lemma**. *Assume that C is an R-free chain complex with all differentials equal to zero, and that D is an arbitrary chain complex. Then for every n*

$$H_n(C \otimes_R D) = C \otimes_R H_n(D)\,.$$

Proof: The differential of the chain complex $C\otimes D$ is $\pm 1\otimes\partial$ and, since C is R-free, we have short exact sequences

$$0 \longrightarrow C \otimes_R Z(D) \xrightarrow{1\otimes i} C \otimes_R D \xrightarrow{1\otimes \partial} C \otimes_R B(D) \longrightarrow 0$$
$$0 \longrightarrow C \otimes_R B(D) \longrightarrow C \otimes_R D \longrightarrow C \otimes_R D/Z(D) \longrightarrow 0$$

Hence $\operatorname{im}(1\otimes\partial) = C \otimes_R B(D)$ and $\ker(1\otimes\partial) = C \otimes Z(D)$.

On the other hand, since we have a short exact sequence

$$0 \longrightarrow B(D) \longrightarrow Z(D) \longrightarrow H(D) \longrightarrow 0$$

and C is R-free, we obtain a short exact sequence

$$0 \longrightarrow C \otimes_R B(D) \longrightarrow C \otimes_R Z(D) \longrightarrow C \otimes_R H(D) \longrightarrow 0\,.$$

□

Hence, by the previous lemma, if C is R-free, then

$$H(Z(C)\otimes_R D) = Z(C)\otimes_R H(D) \quad \text{and} \quad H(B(C)\otimes_R D) = B(C)\otimes_R H(D)\,.$$

Consider the homomorphism

$$\overline{\partial} : (B(C)\otimes_R H(D))_{k+1} \longrightarrow (Z(C)\otimes_R H(D))_k$$

as defined in Exercise 5.3.21, namely if $c\otimes[d] \in B(C)\otimes_R H(D)$, take $c' \in C$ so that $\partial(c') = c$. Hence $(\partial\otimes 1)(c'\otimes[d]) = c\otimes[d]$.

Now, since $c'\otimes d \in C\otimes_R D$, we have

$$\partial(c'\otimes d) = \partial(c')\otimes d \pm c'\otimes\partial(d) = c\otimes d\,,$$

since $d \in Z(D)$. Thus $c\otimes d$ is in the image of $Z(C)\otimes_R D \longrightarrow C\otimes_R D$ and $\overline{\partial}(c\otimes[d])$ is its homology class in $H(Z(C)\otimes_R D)$. Consequently $\overline{\partial} = j\otimes 1$, where $j : B(C) \longrightarrow Z(C)$ is the inclusion. To obtain $\ker(\overline{\partial}) = \ker(j\otimes 1)$ and $\operatorname{coker}(\overline{\partial}) = \ker(j\otimes 1)$, notice that $0 \longrightarrow B(C) \longrightarrow Z(C) \longrightarrow H(C) \longrightarrow 0$ is a free resolution of $H(C)$. Therefore, by Exercise 5.1.9 we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Tor}_R(H(C),H(D)) \longrightarrow B(C)\otimes_R H(D) \longrightarrow Z(C)\otimes_R H(D) \longrightarrow \\ \longrightarrow H(C)\otimes_R H(D) \longrightarrow 0\,. \end{aligned}$$

The following holds.

5.3.23 **Theorem**. (Algebraic Künneth formula for homology) *If C is an R-free chain complex, then there is a natural exact sequence*

$$0 \longrightarrow H(C)\otimes_R H(D) \xrightarrow{\nu} H(C\otimes_R D) \xrightarrow{\Delta} \operatorname{Tor}_R(H(C),H(D)) \longrightarrow 0\,, \tag{5.3.24}$$

where $\nu([c]\otimes[d]) = [c\otimes d]$ and Δ has degree -1.

Proof: The first part follows from the previous considerations. Furthermore

$$\nu([c]\otimes[d]) = (i\otimes 1)_*(c\otimes[d]) = [i(c)\otimes d] = [c\otimes d]\,.$$

Finally, Δ has degree -1, i.e. it lowers the grading in 1, since $\partial\otimes 1$ has degree -1. The naturality is clear. $\square$

5.3.25 NOTE. A more explicit expression for (5.3.24) is as follows

$$\begin{aligned} 0 \longrightarrow \textstyle\bigoplus_{i+j=n} H_i(C)\otimes_R H_j(D) \longrightarrow H_n(C\otimes_R D) \longrightarrow \\ \longrightarrow \textstyle\bigoplus_{i+j=n-1} \operatorname{Tor}_R(H_i(C),H_j(D)) \longrightarrow 0\,. \end{aligned} \tag{5.3.26}$$

5.3.27 **Corollary**. *If both C and D are R-free chain complexes, then there is an isomorphism*

(5.3.24)

$$H_n(C\otimes_R D) \cong \left(\bigoplus_{i+j=n} H_i(C)\otimes_R H_j(D)\right) \oplus \left(\bigoplus_{i+j=n-1} \operatorname{Tor}_R(H_i(C),H_j(D))\right),$$

which in general is not natural.

Proof: We have to show that the short exact sequence (5.3.26) splits. That is, we construct a homomorphism

$$\gamma : H_n(C \otimes_R D) \cong \left(\bigoplus_{i+j=n} H_i(C) \otimes_R H_j(D) \right)$$

such that $\gamma \circ \nu = 1$.

First notice that if C is R-free, then the short exact sequence $0 \longrightarrow Z(C) \longrightarrow C \longrightarrow B(C) \longrightarrow 0$ splits, since $B(C)$ is also free (*exercise*). Thus for each n, $C_n \cong Z_n(C) \oplus B_{n-1}(C)$.

By the assumption, C and D are free and therefore $Z(C)$ and $Z(D)$ are summands therein. Let $\alpha : C \longrightarrow Z(C)$ and $\beta : D \longrightarrow Z(D)$ be the projections (left-inverse to the inclusions). Composing with the quotient homomorphisms, they induce epimorphisms

$$\varphi : C \twoheadrightarrow H(C) \qquad \textit{and} \qquad D \twoheadrightarrow H(D)\,.$$

Thus we have an epimorphism

$$\varphi \otimes \psi : C \otimes_R D \twoheadrightarrow H(C) \otimes_R H(D)\,.$$

Clearly $\varphi \otimes \psi$ determines a homomorphism γ such that $\gamma \circ \nu = 1$. □

5.3.25 **Theorem**. (Algebraic universal coefficients theorem for homology) *If C is an R-free chain complex and A is an R-module, there is an exact sequence*

$$0 \longrightarrow H_n(C) \otimes_R A \longrightarrow H_n(C \otimes_R A) \xrightarrow{\Delta} \mathrm{Tor}_R(H_{n-1}(C), A) \longrightarrow 0$$

which splits (nonnaturally).

Proof: See A as a chain complex D by taking $D_0 = A$ and $D_n = 0$ for $n \neq 0$. The result follows from 5.3.23 and 5.1.12. □

There is a dual version of the previous. First recall that a *cochain complex* of R-modules C is a family of R-modules C^n together with *boundary homomorphisms* $\delta^n : C^n \longrightarrow C^{n+1}$ such that for all n, $\delta^{n+1} \circ \delta^n = 0$. Thus if we consider $Z^n(C) = \ker(\delta^n)$ and $B^n(C) = \mathrm{im}\,(\delta^{n-1})$, we define the *cohomology groups* of C by

$$H^n(C) = Z^n(C)/B^n(C)\,.$$

We have the following.

5.3.26 **Theorem**. (Algebraic Künneth formula for cohomology) *If C is an R-free cochain complex, then there is a natural exact sequence*

(5.3.33)
$$0 \longrightarrow H^*(C) \otimes_R H^*(D) \xrightarrow{\nu} H^*(C \otimes_R D) \xrightarrow{\Delta} \mathrm{Tor}_R(H^*(C), H^*(D)) \longrightarrow 0\,,$$

where $\nu([c] \otimes [d]) = [c \otimes d]$ and Δ has degree -1. Furthermore, this exact sequence splits, though not naturally. □

5.3.34 Note. A more explicit expression for (5.3.33) is as follows

$$
\begin{aligned}
&0 \longrightarrow \textstyle\bigoplus_{i+j=n} H^i(C) \otimes_R H^j(D) \longrightarrow H^n(C \otimes_R D) \longrightarrow \\
&\qquad \longrightarrow \textstyle\bigoplus_{i+j=n+1} \operatorname{Tor}_R(H^i(C), H^j(D)) \longrightarrow 0\,.
\end{aligned} \tag{5.3.35}
$$

5.3.36 **Corollary**. *If both C and D are R-free cochain complexes, then there is an isomorphism*

(5.3.37)

$$H^n(C \otimes_R D) \cong \left(\bigoplus_{i+j=n} H^i(C) \otimes_R H^j(D) \right) \oplus \left(\bigoplus_{i+j=n+1} \operatorname{Tor}_R(H^i(C), H^j(D)) \right),$$

which in general is not natural. □

5.3.38 **Theorem**. (Algebraic universal coefficients theorem for cohomology) *If C is an R-free cochain complex and A is an R-module, there is an exact sequence*

$$0 \longrightarrow H^n(C) \otimes_R A \longrightarrow H^n(C \otimes_R A) \xrightarrow{\Delta} \operatorname{Tor}_R(H^{n+1}(C), A) \longrightarrow 0$$

which splits (nonnaturally). □

We say that an R-free chain complex C is of *finite type* if each C_n is finitely generated. In this case, the R-modules $C^n = \operatorname{Hom}_R(C_n, R)$ are R-free (and finitely generated). Thus we have an R-free cochain complex such that $\delta_n : C^n \longrightarrow C^{n+1}$ is given by $\alpha \mapsto \alpha \circ \partial_{n+1}$. We denote it by C^*. Thus we have the following.

5.3.39 **Corollary**. *Let C and D be R-free chain complexes of finite type and consider the R-free cochain complexes C^* and D^* given by $C^n = \operatorname{Hom}_R(C_n, R)$ and $D^n = \operatorname{Hom}_R(D_n, R)$. Then there is a natural exact sequence*

$$
\begin{aligned}
&0 \longrightarrow \textstyle\bigoplus_{i+j=n} H^i(C^*) \otimes_R H^j(D^*) \longrightarrow H^n(C^* \otimes_R D^*) \longrightarrow \\
&\qquad \longrightarrow \textstyle\bigoplus_{i+j=n+1} \operatorname{Tor}_R(H^i(C^*), H^j(D^*)) \longrightarrow 0\,.
\end{aligned} \tag{5.3.39}
$$

□

Chapter 6 Dold-Thom topological groups

In this chapter we shall construct and analyze certain topological groups whose homotopy groups yield homology groups...

6.1 The abelian group $F(S;L)$

In what follows, L will denote an abelian group (additive) with the discrete topology and S will be a pointed set, with base point x_0.

6.1.1 Definition. Define

$$F(S;L) = \{u : S \longrightarrow L \mid u(x_0) = 0 \text{ and } \operatorname{supp}(u) \text{ is finite}\}\,,$$

where the *support* of u is given by $\operatorname{supp}(u) = \{x \in S \mid u(x) \neq 0\}$. Given $u, v \in F(S;L)$, define their sum $u + v$ by

$$(u + v)(x) = u(x) + v(x)\,,$$

the negative $-u$ by $(-u)(x) = -u(x)$, and as the zero take the constant function with value zero. Thus, clearly, $F(S;L)$ has the structure of an abelian group.

Given a pointed function $f : S \longrightarrow T$, define $f_* : F(S;L) \longrightarrow F(T;L)$ by $f_*(u)(y) = \sum_{f(x)=y} u(x)$. Since the support of u is finite, this sum is finite and thus f_* is well defined. Obviously, $f_*(u)$ has finite support as well.

6.1.2 **Proposition**. *The function $f_* : F(S;L) \longrightarrow F(T;L)$ is a group homomorphism.*

Proof: Just observe that $f_*(u + v)(y) = \sum_{f(x)=y}(u + v)(x) = \sum_{f(x)=y}(u(x) + v(x)) = \sum_{f(x)=y} u(x) + \sum_{f(x)=y} v(x) = f_*(u)(y) + f_*(v)(y)$. □

In fact, the assignment

$$S \longmapsto F(S;L) \quad \text{and} \quad f : S \longrightarrow T \longmapsto f_* : F(S;L) \longrightarrow F(T;L)$$

is a functor. We have the following.

6.1.3 **Proposition**. *If $f = \mathrm{id}_S : S \longrightarrow S$, then $f_* = 1_{F(S;L)} : F(S;L) \longrightarrow F(S;L)$. Furthermore, if $f : S \longrightarrow T$ and $g : T \longrightarrow U$ are pointed functions, then $(g \circ f)_* = g_* \circ f_* : F(S;L) \longrightarrow F(U;L)$.*

Proof: The first assertion is obvious. For the second, take $u \in F(S;L)$. Then

$$\begin{aligned}(g_* \circ f_*)(u)(z) = g_*(f_*(u))(z) &= \sum_{g(y)=z} f_*(u)(y) = \sum_{g(y)=z} \sum_{f(x)=y} u(x) \\ &= \sum_{gf(x)=z} u(x) = (g \circ f)_*(u)(x)\,.\end{aligned}$$

□

Thus we have a functor $F(-;L) : \mathfrak{Set}_* \longrightarrow \mathfrak{Ab}$, where $\mathfrak{Set}_*$ denotes the category of pointed sets.

6.1.4 Exercise. Show that given a pointed set S, the assignment $L \longmapsto F(S;L)$ is a functor from the category $\mathfrak{Ab}$ to itself.

Given $l \in L$ and $x \in S$, $x \neq x_0$, denote by $lx \in F(S;L)$ the function given by

$$(lx)(x') = \begin{cases} l & \text{if } x' = x\,, \\ 0 & \text{if } x' \neq x\,. \end{cases}$$

Given any $u \in F(S;L)$, it can be written as $u = l_1x_1 + \cdots + l_kx_k$ if $\mathrm{supp}\,(u) = \{x_1, \dots, x_k\}$ and $u(x_i) = l_i$, $i = 1, \dots, k$ and $l_i = u(x_i)$. Alternatively, one may write

$$u = \sum_{x \in S} l_x x\,,$$

understanding that $l_x \neq 0$ only for finitely many indexes x. In other words, $l_x = u(x)$.

In any of these expressions, we have for a pointed function $f : S \longrightarrow T$ that

$$f_*(u) = l_1 f(x_1) + \cdots + l_k f(x_k) \quad \text{and} \quad f_*(u) = \sum_{x \in S} l_x f(x)\,.$$

6.1.5 Exercise. Show that indeed the formulas above are equivalent to the original definition 6.1.1.

6.1.6 Exercise. Show that $F(S;\mathbb{Z})$ is the free abelian group generated by (the set) $S - \{x_0\}$.

In some occasions it will be convenient to write $\sum_{i=1}^k l_i x_i$ for an element $u \in F(S;L)$ where the x_is are not necessarily different and some may even be the base point. Such a sum we call *unreduced* and it means that one should add up all coefficients l_i such that $x_i = x$ for some $x \in S$. Furthermore, if some $x_i = x_0$ (the base point), then we understand the term $l_i x_i$ as zero. If all x_i in the sum are different from each other and from the base point, then we say that the sum is *reduced*.

One may filter $F(S;L)$ as follows:

$$\cdots \subseteq F^{k-1}(S;L) \subseteq F^k(S;L) \subseteq F^{k+1}(S;L) \subseteq \cdots \subseteq F(S;L)\,, \tag{6.1.7}$$

where

$$F^k(S;L) = \left\{ \sum_{i=1}^{k} l_i x_i \subset F(S;L) \mid l_i \in L\,,\;\; x_i \in S\,,\;\; i = 1, \ldots, k \right\}.$$

6.2 The simplicial abelian group $F(K;L)$

Proposition 6.1.3 in the previous section shows that $F(-;L) : \mathfrak{Set}_* \longrightarrow \mathfrak{Ab}$ is a covariant functor. By Proposition 2.1.4, we have the next.

6.2.1 Definition. Given a simplicial pointed set K, namely a contravariant functor $K : \Delta \longrightarrow \mathfrak{Set}_*$, composing with the functor $F(-;L) : \mathfrak{Set}_* \longrightarrow \mathfrak{Ab}$ we have another functor $F(K;L) = F(-;L) \circ K : \Delta \longrightarrow \mathfrak{Ab}$, which is a simplicial abelian group. In other words, the obtained simplicial abelian group $F(K;L)$ is given by $F(K;L)_n = F(K_n;L)$ and is such that if $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in Δ then we put $\mu^{F(K;L)} = \mu_*^K : F(K_n;L) \longrightarrow F(K_m;L)$.

6.2.2 Definition.

(a) Assume that K is a simplicial pointed set and that $Q \subseteq K$ is a simplicial pointed subset, namely Q is a simplicial set and for each n, Q_n is a subset of K_n. Furthermore, the inclusion functions $i_n : Q_n \hookrightarrow K_n$ determine a morphism of simplicial sets $i : Q \hookrightarrow K$. In this case, one can also define a simplicial quotient set K/Q by $(K/Q)_n = K_n/Q_n$, i.e. Q_n collapses to one point in K_n for each n. If $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in Δ then we take the function $\mu^{K/Q} : (K/Q)_n \longrightarrow (K/Q)_m$ induced by the function $\mu^K : K_n \longrightarrow K_m$. The fact that $Q \subset K$ is a simplicial subset guarantees that $\mu^{K/Q}$ is well defined.

(b) Assume that F is a simplicial abelian group and that $G \subseteq F$ is a simplicial subgroup, namely G is a simplicial group and for each n, the group G_n is a subgroup of F_n. Moreover, the inclusion homomorphisms $i : G_n \hookrightarrow F_n$ determine a homomorphism of simplicial groups $i : G \hookrightarrow F$. In this case, one can also define a simplicial quotient group F/G by $(F/G)_n = F_n/G_n$, i.e. the quotient group of cosets of G_n in F_n for each n. If $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in Δ then we take the homomorphism $\mu^{F/G} : (F/G)_n \longrightarrow (F/G)_m$ induced by the homomorphism $\mu^F : F_n \longrightarrow F_m$. The fact that $G \subset F$ is a simplicial subgroup guarantees that $\mu^{F/G}$ is well defined.

Then we have the next.

6.2.3 **Proposition**. *Assume that $Q \subseteq K$ is a simplicial pointed subset. Then there are an inclusion of simplicial groups $F(Q;L) \subseteq F(K;L)$ and an isomorphism of simplicial groups*

$$\varphi : F(K;L)/F(Q;L) \xrightarrow{\cong} F(K/Q;L)\,,$$

such that the following triangle commutes:

$$\begin{tikzcd} & F(K;L) \arrow[dl, "q"'] \arrow[dr, "p_*"] & \\ F(K;L)/F(Q;L) \arrow[rr, "\varphi"'] & & F(K/Q;L)\,, \end{tikzcd}$$

where $p : K \twoheadrightarrow K/Q$ is the quotient morphism which collapses Q_n to a point for each n.

Proof: The composite $F(Q;L) \overset{i_*}{\hookrightarrow} F(K;L) \overset{p_*}{\twoheadrightarrow} F(K/Q;L)$ is clearly zero, where $i : K \hookrightarrow Q$ is the inclusion morphism. Namely, if and $u : Q \longrightarrow L$ is an element of $F(Q;L)$, then $p_*i_*(u) = (p \circ i)_*(u)$, and if $\overline{x} = p(x)$, then

$$(p \circ i)_*(u)(\overline{x}) = \sum_{pi(y)=\overline{x}} u(y) = 0\,,$$

because since $y \in Q$, $pi(y)$ is the base point of K/Q. Hence p_* factors through the epimorphism $p : F(K;L) \twoheadrightarrow F(K;L)/F(Q;L)$ thus yielding the diagram of the statement. Conversely, if $p_*(u) = 0$, this means that for all $x \in K$, $\sum_{p(x)=\overline{x}} u(x) = 0$. This is true if either $u(x) = 0$ for all $x \notin Q$ or $x \in Q$, since in this case $\overline{x} \in K/Q$ is the base point. Thus $u = i_*(v)$, where $v = u|_Q$. □

Since by 2.5.3 the morphism $F(K;L) \twoheadrightarrow F(K;L)/F(Q;L)$ is a Kan fibration, then $F(K;L) \twoheadrightarrow F(K/Q;L)$ is a Kan fibration too, and by Theorem 3.4.3, we obtain the next.

6.2.4 **Proposition**. *There is a natural long exact sequence*

$$\cdots \xrightarrow{i_*} \pi_n(F(K;L)) \xrightarrow{p_*} \pi_n(F(K/Q;L)) \xrightarrow{\partial} \pi_{n-1}(F(Q;L)) \longrightarrow \cdots$$
$$\longrightarrow \pi_1(F(K/Q;L)) \xrightarrow{\partial} \pi_0(F(Q;L)) \xrightarrow{i_*} \pi_0(F(K;L)) \xrightarrow{p_*} \pi_0(F(K/Q;L)) \longrightarrow 0$$

6.3 The topological abelian group $F(X;L)$

Let X be a pointed k-space with base point $x_0 \in X$ and let L be an abelian group. Now we shall consider the group $F(X;L)$ as defined in the first section, but we shall use the topology of X to endow $F(X;L)$ with a topology which makes it into a topological group in $\mathfrak{K}$-$\mathfrak{Top}$.

6.3.1 DEFINITION. Let X be a pointed space and L an abelian group. We topologize $F(X;L)$ in two steps as follows:

(i) Consider the surjective map

$$\pi_k : (L \times X)^k \twoheadrightarrow F^k(X;L)$$

given by $\pi_k(l_1, x_1; \ldots; l_k, x_k) = l_1x_1 + \cdots l_kx_k$, where the kth power on the left is the product in $\mathfrak{K}$-$\mathfrak{Top}$. Endow $F^k(X;L)$ with the identification topology. Since $(L \times X)^k$ is a k-space, by Theorem 1.4.10, the filtration term $F^k(X;L)$ is a k-space.

(ii) Consider the filtration 6.1.7 and endow $F(X;L)$ with the union topology. By Proposition 1.4.11 $F(X;L)$ becomes a k-space.

6.3.2 **Proposition**. *The group $F(X;L)$ is a topological group.*

Proof: We must show that the map $\delta : F(X;L) \times F(X;L) \longrightarrow F(X;L)$ given by $\delta(u,v) = u - v$ is continuous. To do this, consider the commutative square

$$\begin{tikzcd}
(L \times X)^k \times (L \times X)^m \arrow[r, "\widetilde{\delta}"] \arrow[d, "\pi_k \times \pi_m"'] & (L \times X)^{k+m} \arrow[d, "\pi_{k+m}"] \\
F^k(X;L) \times F^m(X;L) \arrow[r, "\delta"'] & F^{k+m}(X;L)\,,
\end{tikzcd}$$

where

$$\widetilde{\delta}((l_1, x_1; \ldots; l_k, x_k), (l'_1, x'_1; \ldots; l'_m, x'_m)) = (l_1, x_1; \ldots; l_k, x_k, -l'_1, x'_1; \ldots; -l'_m, x'_m)\,,$$

which is obviously continuous, since L is discrete. The product map $\pi_k \times \pi_k$ is an identification by 1.4.24, and since $\pi_{k+m} \circ \widetilde{\delta}$ is continuous, so is δ too. $\square$

6.3.3 **Proposition**. *A continuous pointed map $f : X \longrightarrow Y$ defines a continuous group homomorphism $f_* : F(X;L) \longrightarrow F(X;L)$.*

Proof: By 6.1.2, it is enough to verify the continuity of f_*. To do this, notice that the commutative diagram

$$\begin{tikzcd}
(L \times X)^k \arrow[r, "(1_L \times f)^k"] \arrow[d, "\pi_k"'] & (L \times Y)^k \arrow[d, "\pi_k"] \\
F^k(X;L) \arrow[r, "f_*"'] & F^k(Y;L)
\end{tikzcd}$$

implies that $f_* : F^k(X;L) \longrightarrow F^k(Y;L)$ is continuous for every k. But since $F^k(X;L)$ has the union topology, $f_* : F(X;L) \longrightarrow F(Y;L)$ is continuous as well. $\square$

6.3.4 **Proposition**. *If $f_0, f_1 : X \longrightarrow Y$ are continuous pointed maps which are pointed homotopic, then the homomorphisms $f_{0*}, f_{1*} : F(X;L) \longrightarrow F(Y;L)$ are homotopic.*

Proof: A pointed homotopy $H : X \times I \longrightarrow Y$ defines for each k a commutative square

$$\begin{tikzcd} (L \times X)^k \times I \arrow[r, "\widetilde{H}"] \arrow[d, "\pi_k \times \mathrm{id}_I"'] & (L \times Y)^k \arrow[d, "\pi_k"] \\ F^k(X;L) \times I \arrow[r, "H_*"'] & F^k(Y;L)\,, \end{tikzcd}$$

where $\widetilde{H}((l_1, x_1; \dots; l_k, x_k), t) = (l_1, H(x_1, t); \dots; l_k, H(x_k, t))$. Thus $H_* : F^k(X;L) \times I \longrightarrow F^k(Y;L)$ is continuous for every k, and since $F(X;L) \times I$ has the topology of the union of the spaces $F^k(X;L) \times I$, $H_* : F(X;L) \times I \longrightarrow F(Y;L)$ is a continuous homotopy of homomorphisms from f_{0*} to f_{1*}. □

6.3.5 **Corollary**. *If X and Y have the same pointed homotopy type, then so do $F(X;L)$ and $F(Y;L)$. In particular, if X is contractible, then so is $F(X;L)$.* □

6.3.6 **Lemma**. *The geometric realization $|F(K,L)|$ is an abelian topological group such that $[v,t] + [v',t] = [v + v', t]$.*

Proof: Consider the projections $p_i : F(K,L) \times F(K,L) \longrightarrow F(K,L)$, $i = 1, 2$, and the induced maps $|p_i| : |F(K,L) \times F(K,L)| \longrightarrow |F(K,L)|$, and define $\eta : |F(K,L) \times F(K,L)| \longrightarrow |F(K,L)| \times |F(K,L)|$ by

$$\begin{aligned} \eta[(v,v'),t] &= (|p_1|[(v,v'),t], |p_2|[(v,v'),t]) \\ &= ([p_1(v,v'),t], [p_2(v,v'),t]) \\ &= ([v,t],[v',t])\,. \end{aligned}$$

By 2.3.3, η is a homeomorphism. The group structure $+$ in $|F(K,L)|$ is then given by the diagram

$$\begin{tikzcd} {|F(K,L)| \times |F(K,L)|} \arrow[r, "\eta^{-1}"] \arrow[dr, "+"'] & {|F(K,L) \times F(K,L)|} \arrow[d, "|\mu|"] \\ & {|F(K,L)|}\,, \end{tikzcd}$$

where $\mu : F(K,L) \times F(K,L) \longrightarrow F(K,L)$ is the simplicial group structure.

The following result relates the simplicial groups $F(K;L)$ with the topological groups $F(X;L)$.

6.3.7 **Theorem**. *The topological groups $F(|K|,L)|$ and $|F(K,L)|$ are naturally isomorphic.*

Proof: Take

$$\varphi : F(|K|, L) \longrightarrow |F(K, L)|$$

given by

$$\varphi(u) = \sum_{[\sigma,t]\in|K|} [u[\sigma, t]\sigma, t]\,,$$

where $u : |K| \longrightarrow L$, $\sigma \in K_n$, and $t \in \Delta^n$ (thus $u[\sigma, t]\sigma \in F(K_n, L)$). Using Lemma 6.3.6 one shows that φ is a homomorphism. Thus we only need to check that $\varphi|_{F_k(|K|,L)}$ is continuous. Consider the diagram

$$\begin{tikzcd}
(L \times |K|)^k \arrow[r, dashed] \arrow[d] & {|F(K,L)|^k} \arrow[d, "\mathrm{sum}"] \\
F_k(|K|, L) \arrow[r, "\varphi|_{F_k(|K|,L)}"'] & {|F(K,L)|\,,}
\end{tikzcd}$$

where the map on the top is the product of the maps given by the next diagram.

$$\begin{tikzcd}
L \times (\bigsqcup_n (K_n \times \Delta^n) \arrow[r, dashed] \arrow[d] & \bigsqcup_n (F(K_n, L) \times \Delta^n \arrow[d] \\
L \times |K| \arrow[r] & F(|K|, L)\,,
\end{tikzcd}$$

where the top map is given by $(l, \sigma, t) \mapsto (l\sigma, t)$.

To see that φ is an isomorphism of topological groups, we define its inverse $\psi : |F(K, L)| \longrightarrow F(|K|, L)$ as follows. Take $v \in F(K_n, L)$; then

$$\psi[v, t] = \sum_{\sigma\in K_n} v(\sigma[\sigma, t])\,.$$

To see that ψ is well defined, take $v = \sum_{i=1}^r l_i\sigma_i$; then $f_*^K(v) = \sum_{i=1}^r l_i f^K(\sigma_i)$. Thus

$$\psi[f_*^K(v), t] = \sum_{i=1}^r [f^K(\sigma_i), t] = \sum_{i=1}^r [\sigma_i, f_\#(t)] = \psi[v, f_\#(t)]\,.$$

To see that ψ is continuous, consider the diagram

$$\begin{tikzcd}
\bigsqcup_n (F(K_n, L) \times \Delta^n \arrow[r, dashed] \arrow[d] & \mathrm{SP}^\infty F(|K|, L) \arrow[d, "\mathrm{sum}"] \\
{|F(K,L)|} \arrow[r, "\psi"'] & F(|K|, L)\,,
\end{tikzcd}$$

where the top arrow given by $(\sum_i l_i, \sigma_i), t) \mapsto \langle l_1[\sigma_1, t], \dots\rangle$ is obviously continuous.

Moreover, ψ is a homomorphism. Namely, given $[v, t], [v', t'] \in |F(K, L)|$, by Lemma 6.3.6, there exist unique elements, w, w', t'' such that $[v, t] = [w, t'']$,

$[v', t'] = [w', t'']$. Thus

$$\begin{aligned}\psi([v,t]+[v',t'] &= \psi([w,t'']+[w',t'']) \\ &= \psi[w+w',t''] \\ &= \sum_{\sigma}(w+w')(\sigma)[\sigma,t''] \\ &= \psi[w,t'']+\psi[w',t''] = \psi[v,t]+\psi[v',t'] \,.\end{aligned}$$

In generators, we have that $\psi\varphi(l[\sigma,t]) = \psi[l\sigma,t] = l[\sigma,t]$, thus $\psi\circ\varphi$ is the identity. On the other hand, $\varphi\psi[v,t] = \varphi(\sum_{\sigma\in K_n} v(\sigma[\sigma,t])) = \sum_{\sigma\in K_n}[v(\sigma)\sigma,t] = [\sum_{\sigma\in K_n} v(\sigma)\sigma,t] = [v,t]$, where the next to the last equality follows by Lemma 6.3.6.

6.3.1 The exactness property of the group $F(X;L)$

We consider a pair pointed CW-complexes $A \hookrightarrow X$ and an abelian group L. One has the short exact sequence of simplicial groups

$$F(\mathcal{S}(A);L) \hookrightarrow F(\mathcal{S}(X);L) \twoheadrightarrow F(\mathcal{S}(X)/\mathcal{S}(A);L)\,,$$

which by 2.5.3 is a Kan fibration.

Taking the geometric realization, by **??**, we have a Hurewicz fibration

$$|F(\mathcal{S}(A);L)| \hookrightarrow |F(\mathcal{S}(X);L)| \twoheadrightarrow |F(\mathcal{S}(X)/\mathcal{S}(A);L)|\,.$$

By 6.3.7, we have (natural) isomorphisms of topological groups

$$|F(\mathcal{S}(A);L)| \cong F(|\mathcal{S}(A)|;L)\,,$$

$$|F(\mathcal{S}(X);L)| \cong F(|\mathcal{S}(X)|;L)\,,$$

$$|F(\mathcal{S}(X)/\mathcal{S}(A);L)| \cong F(|\mathcal{S}(X)/\mathcal{S}(A)|;L)$$

By Proposition 2.3.5, there is a (natural) homeomorphism

$$|\mathcal{S}(X)/\mathcal{S}(A)| \approx |\mathcal{S}(X)|/|\mathcal{S}(A)|\,,$$

so that the last isomorphism yields $|F(\mathcal{S}(X)/\mathcal{S}(A);L)| \cong F(|\mathcal{S}(X)|/|\mathcal{S}(A)|;L)$. Hence we have a Hurewicz fibration

$$F(|\mathcal{S}(A)|;L) \hookrightarrow F(|\mathcal{S}(X)|;L) \twoheadrightarrow F(|\mathcal{S}(X)|/|\mathcal{S}(A)|;L)\,. \tag{6.3.7}$$

Indeed, we have the following commutative diagram in which the top row is a Hurewicz fibration.

$$\begin{tikzcd} F(|\mathcal{S}(A)|;L) \arrow[r, hook] \arrow[d, "\rho_{A*}"', "\simeq"] & F(|\mathcal{S}(X)|;L) \arrow[r, two heads] \arrow[d, "\rho_{X*}"', "\simeq"] & F(|\mathcal{S}(X)|/|\mathcal{S}(A)|;L) \arrow[d, "\overline{\rho}_{X*}"', "\simeq"] \\ F(A;L) \arrow[r, hook] & F(X;L) \arrow[r, two heads] & F(X/A;L)\,. \end{tikzcd}$$

Thus we obtain the following.

6.3.8 **Theorem**. *Given a pair of CW-complexes $A \subseteq X$, there is a long exact sequence of homotopy groups*

(6.3.9)

$$\cdots \pi_{q+1}(F(X/A;L)) \longrightarrow \pi_q(F(A;L)) \longrightarrow \pi_q(F(X;L)) \longrightarrow \pi_q(F(X/A;L) \longrightarrow \cdots .$$

□

6.3.2 The dimension property of the group $F(X;L)$

Consider a CW-complex X and the sequence $F(X;L) \overset{i_*}{\hookrightarrow} F(CX;L) \overset{p_*}{\twoheadrightarrow} F(\Sigma X;L)$, which by 6.3.8 has a long exact sequence. We have the holonomy $\theta : \Omega F(\Sigma X;L) \longrightarrow F(X;L)$, where ΣX is the reduced suspension of X given by CX/X. We also have the path fibration $\Omega F(X;L) \overset{\iota}{\hookrightarrow} PF(X;L) \overset{\pi}{\twoheadrightarrow} F(X;L)$, which by 4.1.5 1 is a Hurewicz fibration. Since the contractibility of CX implies that $F(CX;L)$ is contractible, similarly to the proof of 4.4.22, we obtain the next.

6.3.10 **Proposition**. *The holonomy $\theta : \Omega F(\Sigma X;L) \longrightarrow F(X;L)$ induces isomorphisms in homotopy groups.* □

6.3.11 **Corollary**. *There are homotopy equivalences $L = F(\mathbb{S}^0;L) \simeq \Omega F(\mathbb{S}^1;L) \simeq \cdots \simeq \Omega^k F(\mathbb{S}^k;L) \simeq \cdots$.*

Chapter 7 The homotopical homology groups

In this chapter we shall define and analyze homotopical homology groups ...

7.1 Ordinary homology

We start this section with the abstract definition of an ordinary homology theory. As a matter of fact, there are two approaches to the topic. One is through the concept of an ordinary reduced homology theory, defined in the category of pointed spaces $\mathfrak{K}\text{-}\mathfrak{Top}_*$, and the other is through the concept of an ordinary (unreduced) homology theory defined in the category of pairs of spaces $\mathfrak{K}\text{-}\mathfrak{Top}_2$.

Given a pair of pointed spaces (X, A), we construct the *reduced cone* of A, denoted by CA, by $A \times I/A \times \{1\} \cup \{x_0\} \times I$, which is naturally a pointed space with the vertex $\{A \times \{1\}\}$ as base point. We may construct also $X \cup CA$ by identifying in the topological sum $X \sqcup CA$ the point $(a, 0) \in CA$ with the point $a \in X$. This is clearly a functor $\mathfrak{K}\text{-}\mathfrak{Top}_2 \longrightarrow \mathfrak{K}\text{-}\mathfrak{Top}_*$ which also sends homotopic maps of pairs to homotopic maps of pointed spaces. It is a convention to define the cone of the empty set as one point, denoted $*$, and thus, if $A = \emptyset$, then $X \cup CA = X^+ = X \sqcup *$.

7.1.1 Definition. An *ordinary reduced homology theory* $\widetilde{\mathcal{H}}_*$ with coefficients in an abelian group L for some category of pointed k-spaces $\mathfrak{K}\text{-}\mathfrak{Top}_*$ consists of a family of covariant functors $\widetilde{\mathcal{H}}_q : \mathfrak{K}\text{-}\mathfrak{Top}_* \longrightarrow \mathfrak{Ab}$ and a natural transformation $\sigma_X : \widetilde{\mathcal{H}}_q(X) \longrightarrow \widetilde{\mathcal{H}}_{q+1}(\Sigma X)$ such that the following axioms hold:

(i) **Homotopy**. If $f_0, f_1 : X \longrightarrow Y$ are homotopic maps of pointed spaces, then

$$f_{0*} = f_{1*} : \widetilde{\mathcal{H}}_q(X) \longrightarrow \widetilde{\mathcal{H}}_q(Y)\,.$$

(ii) **Exactness**. If $i : A \hookrightarrow X$ and $j : X \hookrightarrow X \cup CA$ denote the inclusion maps of pointed spaces, then the following is an exact sequence:

$$\widetilde{\mathcal{H}}_q(A) \xrightarrow{i_*} \widetilde{\mathcal{H}}_q(X) \xrightarrow{j_*} \widetilde{\mathcal{H}}_q(X \cup CA)\,.$$

(iii) **Suspension**. The natural transformation $\sigma_X : \widetilde{\mathcal{H}}_q(X) \longrightarrow \widetilde{\mathcal{H}}_{q+1}(\Sigma X)$ is an isomorphism for all X. It is called the *suspension isomorphism*.

(iv) **Dimension**. For the 0-sphere $\mathbb{S}^0$ one has

$$\widetilde{\mathcal{H}}_q(\mathbb{S}^0) = \begin{cases} L & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \,. \end{cases}$$

Next we define the concept of an ordinary unreduced homology. In what follows, $\mathfrak{K}$-$\mathfrak{Top}_2$ will denote the category of pairs (X, A) of k-spaces and continuous maps of pairs $f : (X, A) \longrightarrow (Y, B)$ and we may consider several smaller categories $\mathfrak{T}$ and their corresponding categories of pairs $\mathfrak{T}_2$ which includes all pairs of the form $(X, \emptyset)$ for X in $\mathfrak{T}$. For simplicity, for these pairs we put $(X, \emptyset) = X$.

7.1.2 Definition. An *ordinary (unreduced) homology theory* $\mathcal{H}_*$ with coefficients in an abelian group L for $\mathfrak{K}$-$\mathfrak{Top}_2$ consists of a family of covariant functors $\mathcal{H}_q$: $\mathfrak{K}$-$\mathfrak{Top}_2 \longrightarrow \mathfrak{Ab}$ and natural transformations $\partial : \mathcal{H}_q(X, A) \longrightarrow \mathcal{H}_{q-1}(A)$ such that the following axioms hold:

(i) **Homotopy**. If $f_0, f_1 : (X, A) \longrightarrow (Y, B)$ are homotopic maps of pairs of spaces, then

$$f_{0*} = f_{1*} : \mathcal{H}_q(X, A) \longrightarrow \mathcal{H}_q(Y, B) \,.$$

(ii) **Exactness**. If $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$ denote the inclusion maps, then the following is an exact sequence:

$$\cdots \longrightarrow \mathcal{H}_{q+1}(X, A) \xrightarrow{\partial} \mathcal{H}_q(A) \xrightarrow{i_*} \mathcal{H}_q(X) \xrightarrow{j_*} \mathcal{H}_q(X, A) \xrightarrow{\partial} \mathcal{H}_{q-1}(A) \longrightarrow \cdots \,.$$

(iii) **Excision**. If (X, A) is a pair of spaces in $\mathcal{T}_2$ and $\overline{U} \subset A^\circ$, then the inclusion map $i : (X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism

$$i_* : \mathcal{H}_q(X - U, A - U) \longrightarrow \mathcal{H}_q(X, A) \,.$$

(iv) **Dimension**. If $*$ denotes the one-point space, then

$$\mathcal{H}_q(*) = \begin{cases} L & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \,. \end{cases}$$

We shall see that both the reduced and the unreduced homology theories are equivalent, in the sense that each of them determines the other.

7.1.3 Definition.

(a) Given an unreduced homology theory $\mathcal{H}_*$ and a pointed space X with base point x_0, put by definition

$$\widetilde{\mathcal{H}}_q(X) = \mathcal{H}_q(X, \{x_0\})\,.$$

(b) Given a reduced homology theory $\widetilde{\mathcal{H}}_*$ and a pair of spaces (X, A), put by definition

$$\mathcal{H}_q(X, A) = \widetilde{\mathcal{H}}_q(X^+ \cup C(A^+))\,.$$

7.1.4 **Proposition**. *If the pointed spaces are well-pointed, then constructions* (a) *and* (b) *above are inverse of each other, up to isomorphism.*

Proof: First we show that construction (a) followed by construction (b) yields the same functor of pointed spaces with which we started:

$$\widetilde{\mathcal{H}'}_q(X) = \mathcal{H}_q(X, \{x_0\}) = \widetilde{\mathcal{H}}_q(X^+ \cup C\{x_0^+\})\,,$$

where we denote by $\widetilde{\mathcal{H}'}_*$ the resulting functor after both constructions. Since X is well pointed, the map $X^+ \cup C\{x_0^+\} \twoheadrightarrow X$ which collapses the cone to the base point x_0 is a homotopy equivalence, so that $\widetilde{\mathcal{H}}_q(X^+ \cup C\{x_0^+\}) \cong \widetilde{\mathcal{H}}_q(X)$.

Conversely we show that construction (b) followed by construction (a) produces the same functor of pairs with which we started

$$\mathcal{H'}_q(X, A) = \widetilde{\mathcal{H}}_q(X^+ \cup C(A^+)) = \mathcal{H}_q(X^+ \cup C(A^+), *)\,,$$

where we denote by $\mathcal{H'}_*$ the resulting functor after both constructions. Since $C(A^+)$ is contractible, a comparison of the long exact sequences of both pairs shows that the inclusion $(X^+ \cup C(A^+), *) \hookrightarrow (X^+ \cup C(A^+), C(A^+))$ induces isomorphisms in homology. By excision, the inclusion $(X^+ \cup Z_{1/2}(A^+), Z_{1/2}(A^+)) \hookrightarrow (X^+ \cup C(A^+), C(A^+))$ also induces isomorphisms in homology. Finally, the vertical collapsing of the cylinders $(X^+ \cup Z_{1/2}(A^+), Z_{1/2}(A^+)) \twoheadrightarrow (X, A)$ is a homotopy equivalence of pairs. Thus we have an isomorphism $\mathcal{H}_q(X^+ \cup C(A^+), *) \cong \mathcal{H}_q(X, A)$. $\square$

7.2 Singular homology

In this section we shall construct the classical singular homology.

Let X be any topological space and A a subspace. We start recalling the singular simplicial set $\mathcal{S}(X) : \Delta \longrightarrow \mathfrak{Set}$ given by

$$\mathcal{S}(X)(\mathbf{n}) = \mathcal{S}_n(X) = \{\alpha : \Delta^n \longrightarrow X \mid \alpha \text{ is continuous}\}$$

with its face and degeneracy maps given by

$$d_i(\alpha) = \alpha \circ \delta_{i\#} : \Delta^{n-1} \longrightarrow X\,, \qquad s_i(\alpha) = \alpha \circ \sigma_{i\#} : \Delta^{n+1} \longrightarrow X\,.$$

We consider the simplicial abelian group obtained from $\mathcal{S}(X)$ applying the functor $F(-;L)$, namely

$$S(X;L) = F(\mathcal{S}(X);L)\,.$$

By 2.5.2 we have that $S(X;L)$ has the structure of a chain complex with boundary $\partial_n : S_n(X;L) = F(\mathcal{S}_n(X);L) \longrightarrow S_{n-1}(X;L) = F(\mathcal{S}_{n-1}(X);L)$. If $L = \mathbb{Z}$ we simply write $S(X)$ instead of $S(X;\mathbb{Z})$. The proof of the following result is straightforward.

7.2.1 **Proposition**.

(a) The abelian group $S_n(X)$ is the free group generated by all singular n-simplices $\alpha : \Delta^n \longrightarrow X$.

(b) There is an isomorphism $S_n(X;L) \cong S_n(X) \otimes L$. □

Furthermore we have a chain subcomplex $S(A;L)$ and therewith we also have a chain quotient complex $S(X,A;L) = S(X;L)/S(A;L)$. Thus we have a short exact sequence of chain complexes

$$0 \longrightarrow S(A;L) \longrightarrow S(X;L) \longrightarrow S(X,A;L) \longrightarrow 0\,.$$

7.2.2 Definition. The homology groups of the chain complex $(S(X,A;L),\partial_n)$ are the so-called *singular homology groups* with coefficients in L which we shall denote by

$$H_n^{\mathcal{S}}(X,A;L) = H_n(S(X,A;L))\,.$$

If we denote

$$H_n^{\mathcal{S}}(X;L) = H_n^{\mathcal{S}}(X,\emptyset;L) \quad \textit{and} \quad H_n^{\mathcal{S}}(A;L) = H_n^{\mathcal{S}}(A,\emptyset;L)\,,$$

then by Theorem 6.3.1 we have that the following.

7.2.3 **Theorem**. *The long homology sequence is exact*

(7.2.4) $$\cdots \longrightarrow H_{n+1}^{\mathcal{S}}(X;L) \xrightarrow{j_*} H_{n+1}^{\mathcal{S}}(X,A;L) \xrightarrow{\partial} H_n^{\mathcal{S}}(A;L) \xrightarrow{i_*} H_n^{\mathcal{S}}(X;L) \longrightarrow \cdots\,,$$

where j_* is induced by the inclusion $j : X = (X,\emptyset) \hookrightarrow (X,A)$ and i_* is induced by the inclusion $i : A \hookrightarrow X$. This is the long exact singular homology sequence of a pair (X,A). □

Thus $H_*^{\mathcal{S}}$ satisfies the **exactness axiom**.

On the other hand, we have the next.

7.2.5 **Theorem**. *If $f, g : (X, A) \longrightarrow (Y, B)$ are homotopic maps of pairs, then they induce the same homomorphisms in homology*

$$f_* = g_* : H\mathcal{S}_n(X, A; L) \longrightarrow H\mathcal{S}_n(Y, B; L) \qquad \textit{for all } n\,.$$

Thus $H_*^{\mathcal{S}}$ satisfies the **homotopy axiom**.

For the *proof* one may use the homotopy between f and g to construct a chain homotopy between f_* and $g_* : S(X, A; L) \longrightarrow S(Y, B; L)$ (see [22]). □

If we consider the 0-sphere $\mathbb{S}^0$ consisting of two points, then $\mathcal{S}_n(\mathbb{S}^0)$ consists also of two elements, thus $S(\mathbb{S}^0; L) = F(\mathbb{S}^0; L) = L$, since one point always goes to 0 while the other may take all values in L. Since $H_n(*; L) = \widetilde{H}_n(\mathbb{S}^0; L)$, we have the following.

7.2.6 **Theorem**.

$$H_n^{\mathcal{S}}(*; L) \cong \begin{cases} L & \textit{if } n = 0\,, \\ 0 & \textit{if } n > 0\,. \end{cases}$$

□

Thus $H_*^{\mathcal{S}}$ satisfies the **dimension axiom**.

7.3 Ordinary homotopical homology

In what follows we shall see how the topological groups $F(X; L)$ define an ordinary reduced homology theory in the category of pointed regular CW-complexes $\mathfrak{RCW}$.

7.3.1 Definition. Let X be a pointed regular CW-complex. Define

$$\widetilde{H}_q(X; L) = \pi_q(F(X; L))\,.$$

Furthermore, we define a natural transformation

$$\sigma_X : \widetilde{H}_q(X; L) \longrightarrow \widetilde{H}_{q+1}(\Sigma X; L)$$

by considering the map $h : F(X; L) \longrightarrow \Omega F(\sigma X; L)$ defined by

$$h(\sum_{x \in X} l_x x)(t) = \sum_{x \in X} l_x (x \wedge t)$$

(see **??**) and recalling that the functor Σ is left adjoint of the functor Ω and that $\Sigma\mathbb{S}^q = \mathbb{S}^{q+1}$. Namely

$$\begin{tikzcd}
\pi_q(F(X;L)) \arrow[rr, dashed, "\sigma_X"] \arrow[d, equal] & & \pi_{q+1}(F(\Sigma X;L)) \arrow[d, equal] \\
{[\mathbb{S}^q, F(X;L)]_*} \arrow[r, "h_{X*}"'] & {[\mathbb{S}^q, \Omega F(\Sigma X;L)]_*} \arrow[r, "\cong"'] & {[\Sigma\mathbb{S}^q, F(X;L)]_*.}
\end{tikzcd}$$

7.3.2 Definition. Let (X, A) be a CW-pair, namely a CW-complex X and a subcomplex A. Define

$$H_q(X, A; L) = \pi_q(F(X/A; L)\,.$$

7.3.3 **Theorem**. *The functors $H_n(-; L)$ together with the connecting homomorphisms $\partial : H_q(X, A; L) \longrightarrow H_{q-1}(A; L)$ constitute an ordinary homology theory in the category of CW-pairs.*

Proof: We have to prove the four axioms. Assume first that $f_0, f_1 : (X, A) \longrightarrow (Y, B)$ are homotopic maps of pairs. Then they determine homotopic pointed maps $f_0, f_1 : X/A \longrightarrow Y/B$ and by Proposition 6.3.4, the induced homomorphisms $f_{0*}, f_{1*} : H_q(X, A; L) \longrightarrow H_q(Y, B; L)$ are equal. Thus the homotopy axiom holds.

Assume now that we have a CW-pair (X, A) and the associated pair (X^+, A^+). From the exact sequence (6.3.9) applied to the pair (X^+, A^+) and replacing it with (X, A) where convenient, we obtain a long exact sequence

$$\cdots \longrightarrow H_{q+1}(X, A; L) \xrightarrow{\partial} H_q(A; L) \xrightarrow{i_*} H_q(X; L) \xrightarrow{j_*} H_q(X, A; L) \xrightarrow{\partial} \cdots\,.$$

Hence the exactness axiom holds.

Assume that we have a pair of CW-complexes (X, A) and U such that $\overline{U} \subset A^\circ$ and $(X - U, A - U)$ is again a CW-pair. Then the quotient spaces X/A and $(X - U)/(A - U)$ coincide and the topological groups $F(X/A; L)$ and $F((X - U)/(A - U); L)$ are equal. Hence the excision axiom holds.

Finally notice that $*^+ = \mathbb{S}^0$. Since $L = F(\mathbb{S}^0; L)$ is discrete, we have that

$$H_q(*; L) = \pi_q(F(\mathbb{S}^0; L)) = \begin{cases} L & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

and therefore the dimension axiom holds too. □

7.3.4 Remark. There is a relative Whitehead theorem (see [2]), which states the following. If $\varphi : (X', A') \longrightarrow (X, A)$ is a weak homotopy equivalence of pairs (i.e., it is a weak homotopy equivalence $X' \longrightarrow X$ and a weak homotopy equivalence $A' \longrightarrow A$), then it induces a weak homotopy equivalence $\overline{\varphi} : X'/A' \longrightarrow X/A$. As a consequence, we have a homotopy equivalence

$$|\mathcal{S}(X)|/|\mathcal{S}(A)| \longrightarrow X/A\,,$$

as used above.

We shall now prove that our homotopical homology theory is additive. To prove that, we need the following concept. Let X_α, $\alpha \in \Lambda$, be a family of pointed G-spaces and let L be a $\mathbb{Z}[G]$-module. Then we have (algebraically) the direct sum $F = \bigoplus_{\alpha\in\Lambda} F(X_\alpha, L)$. In order to furnish it with a convenient topology, take

$$F^n = \{(u_\alpha) \in F \mid \#\{\alpha \in \Lambda \mid u_\alpha \neq 0\} \leq n\}\,.$$

Then obviously $F^n \subset F^{n+1}$, and $\bigcup_n F^n = F$. For each n, there is a surjection

$$\coprod_{\alpha_1,\ldots,\alpha_n\in\Lambda} (X_{\alpha_1} \times L) \times \cdots \times (X_{\alpha_n} \times L) \twoheadrightarrow F^n\,.$$

Endow F^n with the identification topology and F with the topology of the union of the F^ns. One clearly has the following.

7.3.5 **Lemma**. *There is an isomorphism of topological groups*

$$\bigoplus_{\alpha\in\Lambda} F(X_\alpha, L) \cong F(\bigvee_{\alpha\in\Lambda} X_\alpha, L))$$

induced by the inclusions $X_\alpha \hookrightarrow \bigvee X_\alpha$.

Proof: The inverse is given by the restrictions $F(\bigvee X_\alpha, L) \longrightarrow F(X_\alpha, L)$, $u \mapsto u|_{X_\alpha}$. □

Since obviously $\pi_q(\bigoplus_\alpha F(X_\alpha, L)) \cong \bigoplus_\alpha \pi_q(F(X_\alpha, L))$, as a consequence, we have the next.

7.3.6 **Theorem**. *There is an isomorphism* $\widetilde{H}_q(\bigvee_\alpha X_\alpha; L) \cong \bigoplus_\alpha \widetilde{H}_q(X_\alpha; L)$. □

Consider a (pointed) Hurewicz fibration $F \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B$. Let $\Gamma : E \times_B B^I \longrightarrow E^I$ be a path-lifting map for p. Define a map $\theta_\Gamma : \Omega B \longrightarrow F$ by

$$\theta_\Gamma(\lambda) = \Gamma(e_0, \lambda)(1)\,,$$

where $\Omega B = \mathrm{Map}_*(\mathbb{S}^1, B)$. The map θ_Γ is called the *holonomy* of p determined by Γ.

On the other hand, consider the Hurewicz fibration $F(X; L) \overset{i_*}{\hookrightarrow} F(CX; L) \overset{p_*}{\twoheadrightarrow} F(\Sigma X; L)$ and define $h : F(X; L) \longrightarrow \Omega F(\Sigma X; L)$, where ΣX is the reduced suspension of X, given by the smash-product $X \wedge \mathbb{S}^1$, by

$$h(u)(s) = \sum u(x)(x \wedge s)\,.$$

McCord [35] shows that this is a homomorphism and a homotopy equivalence.

7.3.7 **Theorem**. *The maps* $h : F(X; L) \longrightarrow \Omega F(\Sigma X; L)$ *and* $\theta_\Gamma : F(\Sigma X; L) \longrightarrow F(X; L)$ *are homotopy inverse.*

Proof: Consider the Hurewicz fibration $F(X; L) \overset{i_*}{\hookrightarrow} F(CX; L) \overset{p_*}{\twoheadrightarrow} F(\Sigma X; L)$ and for simplicity rename it to $F \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B$.

Consider $\rho : F = F(X; L) \longrightarrow E^I$ given by $\rho(u)(s) = \sum_{x\in X} u(x)[x, s]$, where $[x, s] \in CX = X \wedge I$. Take the adjoint map

$$\widehat{\rho} : F(X; L) \times I \longrightarrow E\,.$$

$\widehat{\rho}(u,0) = 0$ and $p_*\widehat{\rho}(u,s) = p_* \sum u(x)[x,s] = \sum u(x)(x \wedge s)$.

Define $\beta(u,s) = \Gamma(0,h(u))(s)$.

We have $p_*\beta(u,s) = p_*\Gamma(0,h(u))(s) = h(u)(s)$. Consider the homotopy $H : (F(X;L) \times I) \times I \longrightarrow E$ determined by

$$H((u,s),0) = \widehat{\rho}(u,s)\,, \quad H((u,s),1) = \Gamma(0,h(u))(s)\,, \quad H((u,0),t) = 0 \quad \text{and}$$

$$p_*H((u,s),t) = h(u)(s) \quad \forall t\,.$$

Now consider the homotopy $\mathcal{H} : F(X;L) \times I \longrightarrow E$ given by

$$\mathcal{H}(u,t) = H((u,1),t)\,.$$

Then

$$\begin{aligned} \mathcal{H}(u,0) &= H((u,1),0) = \widehat{\rho}(u,1) = u\,, \\ \mathcal{H}(u,1) &= H((u,1),1) = \Gamma(0,h(u))(1) = \theta_\Gamma(h(u))\,, \\ p_*\mathcal{H}(u,t) &= p_*H((u,1),t) = h(u)(1) = 0\,. \end{aligned}$$

Hence one has that $\mathcal{H}$ lands always in the fiber and thus it restricts to $\mathcal{H} : F(X;L) \times I \longrightarrow F(X;L)$. Therefore we have

$$\mathcal{H} : \mathrm{id}_{F(X;L)} \simeq \theta_\Gamma \circ h : (F(X;L),0) \longrightarrow (F(X;L),0)\,.$$

Since h is a homotopy equivalence [35], this proves that θ is the inverse homotopy equivalence. $\square$

Chapter 8 The homotopical cohomology groups

In this chapter we shall define and analyze homotopical homology groups ...

8.1 Eilenberg–Mac Lane spaces

We start considering the spaces $\mathbb{S}^1$ or more generally, $F(\mathbb{S}^q; L)$ introduced in the previous chapter. First notice that

$$\pi_q(\mathbb{S}^1) \cong \begin{cases} \mathbb{Z} & \text{if } q = 1 \\ 0 & \text{if } q \neq 1 \end{cases} \quad \text{and} \quad \pi_q(F(\mathbb{S}^n; L)) \cong \begin{cases} L & \text{if } q = n \\ 0 & \text{if } q \neq n \,. \end{cases}$$

(see [2, 4.5.13] and 7.3.3). This suggests the following.

8.1.1 Definition. A topological space A is said to be an *Eilenberg–Mac Lane space of type* (L, n) or, more briefly, to be a $K(L, n)$, if it satisfies

$$\pi_q(A) \cong \begin{cases} L & \text{if } q = n, \\ 0 & \text{if } q \neq n. \end{cases}$$

Thus we have the next.

8.1.2 **Theorem**. *For any abelian group L and any n, the topological groups $F(\mathbb{S}^n; L)$ are Eilenberg–Mac Lane spaces of type (L, n).* □

8.2 Ordinary cohomology

8.2.1 Definition. Let (X, A) be a CW-pair (which means that X is a CW-complex and $A \subset X$ is a subcomplex), and let L be an abelian group. We define the *nth cohomology group of* (X, A) *with coefficients in* L as

$$H^n(X, A; L) = [X \cup CA; F(\mathbb{S}^n; L)]_* \,, \qquad n \geq 1 \ ,$$

where we are considering pointed homotopy classes (and the base point $*$ of $X \cup CA$ is obvious). If $A = \emptyset$, then $X \cup CA = X^+ = X \sqcup *$. In this case, we write

$H^n(X;L) = [X^+;F(\mathbb{S}^n;L)]_* = [X,F(\mathbb{S}^n;L)]$, where the last expression denotes the free (that is, not pointed) homotopy classes of maps from X to $F(\mathbb{S}^n;L)$. Since $F(\mathbb{S}^n;L)$ has a natural abelian group structure, the sets $H^n(X,A;L)$ have an induced abelian group structure.

8.2.2 Remark. Since $A \hookrightarrow X$ is a cofibration, the quotient map $q : X \cup CA \longrightarrow X/A$ is a homotopy equivalence (see [2, 4.2.3]). Therefore, one can define the cohomology groups by

$$H^n(X,A;L) = [X/A;F(\mathbb{S}^n;L)]_*\,, \qquad n \geq 0\,;$$

(here the base point $*$ of X/A is $\{A\}$).

Notice that in the the case $n = 0$ we have that $F(\mathbb{S}^0;L) = L$ (with the discrete topology).

8.2.3 Exercise. Show that $H^0(X,A;L) \cong \prod L$, with as many factors as there are path-connected components C of X satisfying $C \cap A = \emptyset$. In particular, if X is path connected, then $H^0(X;L) \cong L$.

More generally, we have the following **additivity** property.

8.2.4 Exercise. Let $(X,A) = \coprod_{\lambda\in\Lambda}(X_\lambda,A_\lambda)$. Show that

$$H^n(X,A;L) \cong \prod_\lambda H^n(X_\lambda,A_\lambda;L)\,.$$

(*Hint:* An element $x \in H^n(X,A;L)$ is represented by a pointed map

$$f : \bigvee_\lambda (X_\lambda/A_\lambda) \longrightarrow F(\mathbb{S}^n;L)\,,$$

which in turn, by the universal property of the wedge, corresponds to a family of maps $f_\lambda : X_\lambda/A_\lambda \longrightarrow F(\mathbb{S}^n;L)$, each one of which represents an element $x_\lambda \in H^n(X_\lambda,A_\lambda;L)$.)

If $f : (X,A) \longrightarrow (Y,B)$ is a map of CW-pairs, then the associated map on the quotient spaces $\overline{f} : X/A \longrightarrow Y/B$ induces a homomorphism

$$f^* : H^n(Y,B;L) \longrightarrow H^n(X,A;L).$$

Just as in the case of homology, these cohomology groups and their induced homomorphisms have the following properties.

8.2.5 **Functoriality**. *If $f : (X,A) \longrightarrow (Y,B)$ and $g : (Y,B) \longrightarrow (Z,C)$ are maps of* CW*-pairs, then*

$$(g \circ f)^* = f^* \circ g^* : H^n(Z,C;L) \longrightarrow H^n(X,A;L)\,.$$

Also, if $\mathrm{id}_{(X,A)} : (X,A) \longrightarrow (X,A)$ is the identity, then

$$\mathrm{id}^*_{(X,A)} = 1_{H^n(X,A;L)} : H^n(X,A;L) \longrightarrow H^n(X,A;L)\,.$$

8.2.6 **Homotopy**. *If $f_0 \simeq f_1 : (X, A) \longrightarrow (Y, B)$ (a homotopy of pairs), then*

$$f_0^* = f_1^* : H^n(Y, B; L) \longrightarrow H^n(X, A; L).$$

8.2.7 **Excision**. *Let $(X; X_1, X_2)$ be a* CW*-triad, that is, X_1 and X_2 are subcomplexes of X such that $X = X_1 \cup X_2$. Then the inclusion $j : (X_1, X_1 \cap X_2) \longrightarrow (X, X_2)$ induces an isomorphism*

$$j^* : H^n(X, X_2; L) \longrightarrow H^n(X_1, X_1 \cap X_2; L), \qquad n \geq 0.$$

8.2.8 **Exactness**. *Suppose that (X, A) is a* CW*-pair. Then we have an exact sequence*

$$\cdots \longrightarrow H^q(A; L) \xrightarrow{\delta} H^{q+1}(X, A; L) \longrightarrow H^{q+1}(X; L) \longrightarrow$$
$$\longrightarrow H^{q+1}(A; L) \xrightarrow{\delta} H^{q+2}(X, A; L) \longrightarrow \cdots .$$

Here δ, called the connecting homomorphism, *is a natural homomorphism, which means that given any map of pairs $f : (Y, B) \longrightarrow (X, A)$ the following diagram is commutative:*

$$\begin{tikzcd} H^q(A; L) \arrow[r, "\delta"] \arrow[d, "(f|B)^*"'] & H^{q+1}(X, A; L) \arrow[d, "f^*"] \\ H^q(B; L) \arrow[r, "\delta"'] & H^{q+1}(Y, B; L) . \end{tikzcd}$$

8.2.9 **Dimension**. *For the space $*$ containing exactly one point we have that*

$$H^i(*; L) = \begin{cases} L & \textit{if } i = 0, \\ 0 & \textit{if } i \neq 0. \end{cases}$$

Proof: The properties of Functoriality 8.2.5 and Homotopy 8.2.6 follow immediately from the definitions.

In order to prove property 8.2.7 it is enough to note that the conditions imposed on X, X_1, and X_2 imply that

$$X/X_2 \quad \text{and} \quad X_1/X_1 \cap X_2$$

are homeomorphic.

In order to prove the property of Excision 8.2.8 we first define

$$\delta : H^q(A; L) \longrightarrow H^{q+1}(X, A; L)$$

by using the composite

$$X/A \xrightarrow{p} X^+ \cup CA^+ \xrightarrow{p'} \Sigma A^+ ,$$

where $X^+ \cup CA^+$ is the *unreduced cone* of (X, A) defined alternatively as $X \sqcup A \times I/\sim$, where $X \supset A \ni a \sim (a, 0) \in A \times I$ and $(a, 1) \sim (a', 1)$ in $A \times I$. Analogously,

ΣA^+ is the *unreduced suspension* of A. Here p is the homotopy inverse of the homotopy equivalence defined by the composite

$$X^+ \cup CA^+ \longrightarrow X^+ \cup CA^+/CA^+ \approx X/A\ ,$$

and p' is the quotient map

$$X^+ \cup CA^+ \longrightarrow X^+ \cup CA^+/X^+ \approx \Sigma A^+.$$

So δ is defined by

$$\begin{aligned} H^q(A;L) &= [A^+;F(\mathbb{S}^q;L)]_* \cong [A^+;\Omega F(\mathbb{S}^{q+1};L)]_* \\ &\cong [\Sigma A^+;F(\mathbb{S}^{q+1};L)]_* \xrightarrow{p^*\circ p'^*} [X/A;F(\mathbb{S}^{q+1};L)]_* \\ &= H^{q+1}(X,A;L). \end{aligned}$$

Exactness 8.2.8 is now obtained by applying the exact sequence of Corollary 3.3.10 in [2]. Specifically, since we have as above that $H^q(X;L) = [\Sigma X^+;F(\mathbb{S}^{q+1};L)]_*$, it follows that the piece of that sequence corresponding to the inclusion $i : A \hookrightarrow X$ is given as

$$\begin{aligned} &[\Sigma X^+, F(\mathbb{S}^{q+1};L)] \longrightarrow [\Sigma A^+, F(\mathbb{S}^{q+1};L)] \longrightarrow [C_i, F(\mathbb{S}^{q+1};L)] \longrightarrow \\ &\longrightarrow [X^+, F(\mathbb{S}^{q+1};L)] \longrightarrow [A^+, F(\mathbb{S}^{q+1};L)]\,, \end{aligned}$$

where we omit the base point for simplicity. This in turn changes into

$$\begin{aligned} &H^q(X;L) \longrightarrow H^q(A;L) \longrightarrow H^{q+1}(X,A;L) \longrightarrow \\ &\longrightarrow H^{q+1}(X;L) \longrightarrow H^{q+1}(A;L) \end{aligned}$$

by using the isomorphisms proved above and the fact that $C_i \simeq X/A$.

Grouping together these pieces for $q \geq 0$ we obtain the desired exact sequence.

In order to prove the Dimension property 8.2.9 it suffices to apply the definition of $F(\mathbb{S}^i;L)$. So we have

$$H^i(*;L) = [\mathbb{S}^0, F(\mathbb{S}^i;L)] = \pi_0(F(\mathbb{S}^i;L)) = \begin{cases} L & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases}$$

since $F(\mathbb{S}^i;L)$ is discrete and equal to L if $i = 0$, while it is path connected if $i > 0$. $\square$

All given axioms of **Functoriality**, **Homotopy**, **Exactness** and **Dimension** are the so-called *Eilenberg–Steenrod axioms* for an ordinary (unreduced) cohomology theory.

The next result establishes the so-called *wedge axiom* for cohomology (cf. 8.2.4).

8.2.10 **Proposition**. *If* $X = \bigvee_{\lambda\in\Lambda} X_\lambda$, *then*

$$\widetilde{H}_q(X;L) \cong \prod_{\lambda\in\Lambda} \widetilde{H}^q(X_\lambda;L)\,.$$

Proof: This follows immediately from the definition of the reduced cohomology groups and **??**. □

8.2.11 Exercise. Let $(X, A) = \coprod(X_\lambda, A_\lambda)$. Prove that for all q,

$$H^q(X, A; L) \cong \prod_\lambda H^q(X_\lambda, A_\lambda; L)\,.$$

This is the so-called *additivity axiom* for cohomology.

8.2.12 Exercise. Prove that if $f : (X, A) \longrightarrow (Y, B)$ is a weak homotopy equivalence of pairs of topological spaces, then

$$f_* : H^q(Y, B) \longrightarrow H^q(X, A)$$

is an isomorphism for all q. This is the so-called *weak homotopy equivalence axiom* for cohomology.

These cohomology groups defined for arbitrary pairs of topological spaces obviously satisfy the axioms of functoriality, homotopy, exactness, and dimension, which we have introduced above. But in this case we have the following excision axiom.

8.2.13 **Excision**. (For excisive triads) *Let $(X; A, B)$ be an excisive triad; that is, X is a topological space with subspaces A and B such that $\overset{\circ}{A} \cup \overset{\circ}{B} = X$, where $\overset{\circ}{A}$ and $\overset{\circ}{B}$ denote the interiors of A and B, respectively. Then the inclusion $j : (A, A \cap B) \longrightarrow (X, B)$ induces an isomorphism*

$$H^n(X, B; L) \longrightarrow H^n(A, A \cap B; L), \qquad n \geq 0.$$

Proof: In order to show that we have this property we take a CW-approximation of $A \cap B$, say $\varphi : \widetilde{A \cap B} \longrightarrow A \cap B$, and extend it to an approximation of A, say $\varphi_1 : \widetilde{A} \longrightarrow A$, and to an approximation of B, say $\varphi_2 : \widetilde{B} \longrightarrow B$, in such a way that $\widetilde{A \cap B} = \widetilde{A} \cap \widetilde{B}$. Thus we can define a map $\widetilde{\varphi} : \widetilde{X} = \widetilde{A} \cup \widetilde{B} \longrightarrow A \cup B = X$ such that $\widetilde{\varphi}|\widetilde{A} = \varphi_1$, $\widetilde{\varphi}|\widetilde{B} = \varphi_2$, and $\widetilde{\varphi}|\widetilde{A \cap B} = \varphi$. Using the hypothesis $\overset{\circ}{A} \cup \overset{\circ}{B} = X$ we can now prove that $\widetilde{\varphi}$ is a weak homotopy equivalence; that is, $\widetilde{\varphi}$ is a CW-approximation of X (see [**?**, 16.24]). Using this result it is clear that the excision axiom for excisive triads follows from the excision axiom (8.2.7) for CW-triads. □

8.2.14 Exercise. Prove that the excision axiom for excisive triads is equivalent to the following axiom. Suppose that (X, A) is a pair of spaces and that $U \subset A$ satisfies $\overline{U} \subset \overset{\circ}{A}$. Then the inclusion $i : (X - U, A - U) \longrightarrow (X, A)$ induces an isomorphism $H^n(X, A; L) \cong H^n(X - U, A - U; L)$ for all $n \geq 0$. (It is precisely this version that gives us the name "excision," because it allows us to "excise" from both X and A a piece "well" contained inside of A without altering the cohomology of the pair.)

Since $[\mathbb{S}^n, K(L,q)] = \pi_n(K(L,q))$ holds, the next result follows.

8.2.15 **Proposition**. *Suppose that $n > 0$. Then we have*

$$H^q(\mathbb{S}^n; L) = \begin{cases} L & \text{if } q = 0, n, \\ 0 & \text{if } q \neq 0, n. \end{cases} \qquad \square$$

Let X be a pointed space with base point x_0. Then for every $n \geq 0$ the inclusion $i : * \longrightarrow X$ defined by $i(*) = x_0$ induces an epimorphism

$$i^* : H^n(X; L) \longrightarrow H^n(*; L),$$

which is split by the monomorphism

$$r^* : H^n(*; L) \longrightarrow H^n(X; L)$$

induced by the unique map $r : X \longrightarrow *$.

8.2.16 Definition. We call $\widetilde{H}^n(X; L) = \ker(i^*)$ the *nth reduced cohomology group* of the pointed space X with coefficients in the group L.

So there is a short exact sequence

$$0 \longrightarrow \widetilde{H}^n(X; L) \longrightarrow H^n(X; L) \longrightarrow H^n(*; L) \longrightarrow 0$$

that splits, and therefore

$$H^n(X; L) = \widetilde{H}^n(X; L) \oplus H^n(*; L).$$

Consequently, by the dimension axiom 8.2.9, we have

$$H^n(X; L) = \begin{cases} \widetilde{H}^0(X; L) \oplus L & \text{if } n = 0, \\ \widetilde{H}^n(X; L) & \text{if } n \neq 0. \end{cases}$$

From now on, if it does not cause confusion, we shall write only $H^n(X)$ (respectively, $\widetilde{H}^n(X)$) instead of $H^n(X; L)$ (respectively, $\widetilde{H}^n(X; L)$).

8.2.17 Exercise. Prove that if X is a pointed space with base point x_0, then for every n we have

$$\widetilde{H}^n(X) = H^n(X, x_0).$$

(Hint: The exact sequence of the pair (X, x_0) decomposes into short exact sequences

$$0 \longrightarrow H^n(X, x_0) \longrightarrow H^n(X) \longrightarrow H^n(x_0) \longrightarrow 0$$

that split.)

8.2.18 Exercise. Assume that X is contractible. Prove that

$$H^{q-1}(A) \cong H^q(X,A)$$

if $q > 1$, and

$$\widetilde{H}^0(A) \cong H^1(X,A)\,.$$

8.2.19 Exercise. Take $A \subset B \subset X$ and assume that the inclusion $A \hookrightarrow B$ is a homotopy equivalence. Prove that the inclusion of pairs $(X,A) \hookrightarrow (X,B)$ induces an isomorphism

$$H^q(X,B) \longrightarrow H^q(X,A)$$

for all q.

The dimension axiom implies that the one-point space, or more generally any contractible space, has trivial reduced cohomology. Specifically, we have the next assertion.

8.2.20 **Proposition**. *Let D be a contractible space. Then we have $\widetilde{H}^n(D) = 0$ for all n.* □

Proposition 8.2.15 can be rewritten in terms of reduced cohomology as follows.

8.2.21 **Proposition**. *Suppose that $n > 0$. Then we have*

$$\widetilde{H}^q(\mathbb{S}^n; L) = \begin{cases} L & \textit{if } q = n, \\ 0 & \textit{if } q \neq n. \end{cases}$$ □

8.2.22 Exercise. Let X be a pointed space with base point x_0. Prove that $\widetilde{H}^q(X;\mathbb{Z}) = [X; K(\mathbb{Z},q)]_*$ and thereby conclude that

$$\widetilde{H}^q(X;\mathbb{Z}) \cong \widetilde{H}^{q+1}(\Sigma X;\mathbb{Z}).$$

(Hint: Apply the exact homotopy sequence to $X \xrightarrow{f} * \longrightarrow C_f = \Sigma X$.)

8.2.23 Exercise. Suppose that $\alpha_k : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ is the map given in Definition **??**. Prove that $\alpha_k^* : \widetilde{H}^n(\mathbb{S}^n;\mathbb{Z}) \longrightarrow \widetilde{H}^n(\mathbb{S}^n;\mathbb{Z})$ corresponds to multiplication by k. (Hint: Prove this by applying the previous exercise and using induction on n.) More generally, verify that the result remains true for any coefficient group L (where multiplication by k is to be understood by viewing L as a module over the integers $\mathbb{Z}$).

8.2.24 Exercise. Prove the following assertions:

(a) All the arrows in the sequence

$$H^r(X,A) \longrightarrow H^r(\{1\}\times(X,A)) \xleftarrow{j^*}$$
$$\longleftarrow H^r(\mathbb{S}^0\times X\cup\mathbb{D}^1\times A, \{0\}\times X\cup\mathbb{D}^1\times A) \xrightarrow{\delta}$$
$$\longrightarrow H^{r+1}(\mathbb{D}^1\times X, \mathbb{S}^0\times X\cup\mathbb{D}^1\times A) = H^{r+1}((\mathbb{D}^1,\mathbb{S}^0)\times(X,A))$$

are isomorphisms, where j is the obvious inclusion. We call the composition of these isomorphisms

$$\alpha : H^r(X,A;L) \longrightarrow H^{r+1}((\mathbb{D}^1,\mathbb{S}^0)\times(X,A);L)$$

the *suspension isomorphism*.

(b) The suspension isomorphism defined in part (a) is a natural isomorphism, that is, it commutes with the homomorphisms induced by maps of pairs.

(c) This suspension isomorphism is in a sense another version of the homomorphism of Exercise 8.2.22. Explain.

Chapter 9 Products in homotopical homology and cohomology

In this chapter we shall define different products in homotopical homology and cohomology and analyze their properties.

9.1 A pairing of the Dold–Thom groups

The main construction needed for products is given as follows. Assume given two spaces X and Y and m,

9.1.1 **Lemma**. *The map $\varepsilon : F(X;L) \longrightarrow L$ given by $\sum_{i=1}^m l_i x_i \mapsto \sum_{i=1}^m l_i$ is well defined and continuous. In particular, $\varepsilon : F(\mathbb{S}^0;L) \longrightarrow L$ is an isomorphism.*

Proof: This follows easily from the fact that the restriction $\varepsilon_n : F_n(X;L) \longrightarrow L$ of ε is continuous, since its composite with the identification $(X \times L)^n \longrightarrow F_n(X;L)$ is obviously continuous. $\square$

The functor F has a well-defined continuous pairing,

$$\mu_{X,Y} : F(X;L) \times F(Y;M) \longrightarrow F(X \wedge Y, L \otimes M)\,, \tag{9.1.2}$$

given by

$$\left(\sum_{i=1}^m l_i x_i, \sum_{j=1}^n m_j y_j\right) \longmapsto \sum_{(i,j)=(1,1)}^{(m,n)} (l_i \otimes m_j)(x_i \wedge y_j)\,.$$

If, in particular, $L = M = R$ is a commutative ring with 1, with $m : R \otimes R \longrightarrow R$ as the ring multiplication, then composing (9.1.2) with m_*, we obtain another pairing,

$$\mu_{X,Y} : F(X;R) \times F(Y;R) \longrightarrow F(X \wedge Y;R)\,. \tag{9.1.3}$$

Using (9.1.2), one obtains products in homology and cohomology. We shall be interested in the following pairings. First recall the homeomprphism $\mathbb{S}^m \wedge \mathbb{S}^n \approx \mathbb{S}^{m+n}$. Thus one has the Eilenberg–Mac Lane space pairing

$$\mu_{m,n} : F(\mathbb{S}^m;L) \times F(\mathbb{S}^n;M) \longrightarrow F(\mathbb{S}^{m+n};L \otimes M)\,. \tag{9.1.4}$$

which for any two pointed spaces X and Y induces a pairing

$$\times : [X,*;F(\mathbb{S}^m;L),0]_* \times [Y;F(\mathbb{S}^n;M)]_* \longrightarrow [X\wedge Y;F(\mathbb{S}^{m+n};L\otimes M)]_*\,,$$

given by $[\alpha]\otimes[\beta]\mapsto[\gamma]$, where

$$\gamma(x\wedge y) = \mu_{m,n}(\alpha(x),\beta(y))\,.$$

This can be translated, by definition of the cohomology groups, into

$$\times : \widetilde{H}^m(X;L)\otimes\widetilde{H}^n(Y;M) \longrightarrow \widetilde{H}^{m+n}(X\wedge Y;L\otimes M)\,. \tag{9.1.5}$$

One can easily verify that this pairing is well defined. If, in particular, $L=M=R$ is a commutative ring with 1, we obtain the pairing,

$$\times : \widetilde{H}^m(X;R)\otimes\widetilde{H}^n(Y;R) \longrightarrow \widetilde{H}^{m+n}(X\wedge Y;R)\,. \tag{9.1.6}$$

9.1.7 Definition. The product $\times$ given in (9.1.5) and (9.1.6) is called the *exterior product* or *cross product* in cohomology. Sometimes we write $\times$*-product.*

If $X=Y$ and we take the diagonal map $\Delta : X\longrightarrow X\wedge X$, we obtain a pairing

$$\smile : \widetilde{H}^m(X;R)\otimes\widetilde{H}^n(X;R) \xrightarrow{\times} \widetilde{H}^{m+n}(X\wedge X;R) \xrightarrow{\Delta^*} \widetilde{H}^{m+n}(X;R)\,. \tag{9.1.8}$$

9.1.9 Definition. The product $\smile$ given in (9.1.8) is called the *interior product* or *cup product* in cohomology. Sometimes we write $\smile$*-product.*

There is a close relationship between the cross product and the cup product (in the unreduced case). We saw in (9.1.8) how to obtain the cup product from the cross product. one can obtain the cross product from the cup product too. We put both relationships in the following, result, where we take $H^q(W;R)=\widetilde{H}^q(W^+;R)$.

9.1.10 **Proposition**.

(a) *The composite* $H^m(X;R)\otimes H^n(X;R) \xrightarrow{\times} H^{m+n}(X\times X;R) \xrightarrow{\Delta^*} H^{m+n}(X;R)$, *where* $\Delta : X\longrightarrow X\times X$ *is the diagonal map, yields the cup product. In other words, if* $a\in H^m(X;R)$ *and* $a'\in H^n(X;R)$, *then*

$$a\smile a' = \Delta^*(a\times a')\in H^{m+n}(X;R)\,.$$

(b) *The composite* $H^m(X;R)\otimes H^n(Y;R) \xrightarrow{\pi_1^*\otimes\pi_2^*} H^m(X\times Y;R)X;R)\otimes H^m(X\times Y;R)X;R) \xrightarrow{\cup} H^{m+n}(X\times Y;R)$, *where* $X \xleftarrow{\pi_1} X\times Y \xrightarrow{\pi_2} Y$ *are the projections, yields the cross product. In other words, if* $a\in H^m(X;R)$ *and* $b\in H^n(Y;R)$, *then*

$$a\times b = \pi_1^*(a)\smile\pi_2^*(b)\in H^m(X\times Y;R)X;R)\,.$$

We come back to the pairing given in (9.1.2) and put $X = \mathbb{S}^q$ and $Y = X$. Taking smash products and the pairing, we obtain

$$\begin{tikzcd}
{[X^+, F(\mathbb{S}^q; R)]_* \times [\mathbb{S}^k, F(X^+; R)]_*} \arrow[r] \arrow[dr, dashed, "\kappa"'] & {[X^+ \wedge \mathbb{S}^k, F(\mathbb{S}^q; R) \wedge F(X^+; R)]_*} \arrow[d, equal] \\
& {[\Sigma^k X^+, F(\Sigma^q X^+; R)]_* \,.}
\end{tikzcd}$$

Composing κ with the homomorphism

$$[\Sigma^k X^+, F(\Sigma^q X^+; R)]_* \longrightarrow [\mathbb{S}^k, F(\Sigma^q X^+; R)]_*$$

induced by the pointed inclusion $\mathbb{S}^0 \longrightarrow X^+$ that sends -1 to some point x_{-1} in the path-connected space X, we obtain

$$H^q(X; R) \otimes H_k(X; R) \longrightarrow \widetilde{H}_k(\Sigma^q X^+; R)\,.$$

If $q \leq k$, using σ^{-q}, we desuspend q times to obtain

$$\frown : H^q(X; R) \otimes H_k(X; R) \longrightarrow H_{k-q}(X; R)\,.\,. \tag{9.1.9}$$

9.1.10 Definition. The product $\frown$ given in (9.1.9) is called the *cap product* in homology. Sometimes we write $\frown$*-product*.

Composing κ now with the homomorphism

$$[\Sigma^k X^+, F(\Sigma^q X^+; R)]_* \longrightarrow [\Sigma^k X^+, F(\mathbb{S}^q; R)]_*$$

induced by the pointed projection $X^+ \longrightarrow \mathbb{S}^0$ that sends all points of the path-connected space X to -1 we obtain

$$H^q(X; R) \otimes H_k(X; R) \longrightarrow \widetilde{H}^q(\Sigma^k X^+; R)\,.$$

If $k \leq q$, using σ^{-k}, we desuspend k times to obtain

$$\frown : H^q(X; R) \otimes H_k(X; R) \longrightarrow H_{q-k}(X; R)\,.\,. \tag{9.1.10}$$

9.1.11 Definition. The product $\frown$ given in (9.1.9) is called the *cap product* in cohomology. Sometimes we write $\frown$*-product*.

Recall that $H_0(*; R) = [\mathbb{S}^0; F(\mathbb{S}^0; R)]_* = R$. If $k = q$ take the composite

$$\eta : H^q(X; R) \otimes H_q(X; R) \longrightarrow H_0(X; R) \longrightarrow H_0(*; R) = R\,,$$

where the last arrow is induced by the pointed projection $X^+ \longrightarrow \mathbb{S}^0$ and the equality follows from the bijection $\varepsilon : F(\mathbb{S}^0, R) \longrightarrow R$ given in Lemma 9.1.1.

9.1.12 Definition. The homomorphism η is the *Kronecker product* $\langle -, - \rangle$, namely

$$\langle \alpha, \beta \rangle = \eta(\alpha \otimes \beta)\,.$$

9.2 Properties of the products

In this section we shall prove the properties of the products. Before that, we extend the products to the relative case.

9.2.1 Products for pairs of spaces

Let us extend the definition of the cup product to pairs of spaces. Recall that given a pair of spaces (W, D) we can associate to it a pointed space by taking the mapping cone of the inclusion $D \hookrightarrow W$, $W \cup CD$. We defined $H^q(W, D; R) = \widetilde{H}^q(W \cup CD; R)$. One can prove that given two pairs of spaces (X, A) and (Y, B), the cone of the inclusion $A \times Y \cup X \times B \hookrightarrow X \times Y$ satisfies

$$(X \times Y) \cup C(A \times Y \cup X \times B) \approx (X \cup CA) \wedge (Y \cup CB)\,.$$

Since the product of pairs is given by $(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B)$, we have the following

$$\begin{tikzcd}
H^m(X,A;R) \otimes H^n(Y,B;R) \arrow[r, dashed, "\times"] \arrow[d, equal] & H^{m+n}((X,A)\times(Y,B);R) \arrow[dd, equal] \\
\widetilde{H}^m(X \cup CA;R) \otimes \widetilde{H}^n(Y \cup CB;R) \arrow[d, "\times"'] & \\
\widetilde{H}^{m+n}((X \cup CA) \wedge (Y \cup CB);R) \arrow[r, "\cong"'] & \widetilde{H}^{m+n}((X \times Y) \cup C(A \times Y \cup X \times B);R)
\end{tikzcd}$$

which defines the *exterior product for pairs* or *cross product for pairs* in cohomology

$$\times : H^m(X, A; R) \otimes H^n(Y, B; R) \longrightarrow H^{m+n}((X, A) \times (Y, B); R)\,.$$

We can now extend the interior product to pairs as follows. Take any $A, A' \subset X$. Then the diagram

$$\begin{tikzcd}
H^m(X,A;R) \otimes H^n(X,A';R) \arrow[r, dashed, "\smile"] \arrow[d, "\times"'] & H^{m+n}(X, A \cup A';R) \\
H^{m+n}((X,A)\times(X,A');R) \arrow[r, equal] & H^{m+n}(X \times X, A \times X \cup X \times A';R) \arrow[u, "\Delta^*"']
\end{tikzcd}$$

defines the *interior product for pairs* or *cup product for pairs*

$$\smile : H^m(X, A; R) \otimes H^n(X, A'; R) \longrightarrow H^{m+n}(X, A \cup A'; R)$$

We start with the cup product.

9.2.2 Properties of the cup product

Before we start with the properties,

9.2.1 **Naturality**. *If $f : (X;A,A') \longrightarrow (Y;B,B')$ is a map of triads (which means that $f(A) \subset B$ and $f(A') \subset B'$), then for all $y \in H^p(Y,B)$ and all $y' \in H^q(Y,B')$ we have that*

$$f^*(y \smile y') = f^*(y) \smile f^*(y') \in H^{p+q}(X, A \cup A')\,.$$

9.2.2 **Associativity**. *For all*

$$x \in H^p(X,A)\,, \quad x' \in H^q(X,A')\,, \quad \textit{and} \quad x'' \in H^r(X,A'')$$

we have that

$$x \smile (x' \smile x'') = (x \smile x') \smile x'' \in H^{p+q+r}(X, A \cup A' \cup A'')\,.$$

9.2.3 **Units**. *Suppose that $1_X \in H^0(X)$ is the element represented by the constant map $X \longrightarrow K(R,0) = R$ that sends the entire space X to the element $1 \in R$. Then for all $x \in H^q(X,A)$ we have that*

$$1_X \smile x = x \smile 1_X = x \in H^q(X,A).$$

9.2.4 **Stabilility**. *The following diagram is commutative:*

$$\begin{tikzcd}
H^p(A) \otimes H^q(X,A') \arrow[r, "\mathrm{id}\otimes i^*"] \arrow[dddd, "\delta\otimes\mathrm{id}"'] & H^p(A) \otimes H^q(A, A \cap A') \arrow[d, "\smile"] \\
& H^{p+q}(A, A \cap A') \\
& H^{p+q}(A \cup A', A') \arrow[u, "j^*"'] \arrow[dd, "\delta"] \\
& \\
H^{p+1}(X,A) \otimes H^s(X,A') \arrow[r, "\smile"] & H^{p+q+1}(X, A \cup A').
\end{tikzcd}$$

Here i and j are inclusions. Moreover, j^* actually turns out to be an excision isomorphism.

In particular, for the case $A' = \emptyset$, we obtain the formula

$$\delta(a \smile i^*x) = \delta a \smile x \in H^{p+q+1}(X,A)$$

for $a \in H^p(A)$ and $x \in H^q(X)$.

9.2.5 **Commutativity**. *For all*

$$x \in H^p(X,A) \quad \textit{and} \quad x' \in H^q(X,A')$$

we have that

$$x \smile x' = (-1)^{pq} x' \smile x \in H^{p+q}(X, A \cup A')\,.$$

The *proof* of these properties, except commutativity, basically reduces to the uniqueness up to homotopy of the maps between Moore spaces that realize the given group homomorphisms. We leave the details of the proof to the reader in the following exercise, and only prove commutativity.

9.2.6 EXERCISE. Establish the properties of naturality, associativity, units, and stability of the cup product in cohomology.

We shall prove commutativity in the particular case of $R = \mathbb{Z}$. In order to do it, we need some preparation. We start with the convention that in what follows we take

$$\mathbb{S}^p = \underbrace{\mathbb{S}^1 \wedge \cdots \wedge \mathbb{S}^1}_{p} \qquad \text{where} \qquad \mathbb{S}^1 = I/\partial I$$

and we write its elements as $\bar{t}_1 \wedge \cdots \wedge \bar{t}_p$, where $\bar{t} \in \mathbb{S}^1$ represents the class of $t \in I$.

9.2.7 **Lemma**. *Let $\alpha_{i\,j} : \mathbb{S}^p \longrightarrow \mathbb{S}^p$ be the map that interchanges two factors, namely*

$$\alpha_{i\,j}(\bar{t}_1 \wedge \cdots \wedge \bar{t}_i \wedge \cdots \wedge \bar{t}_j \wedge \cdots \wedge \bar{t}_p) = \bar{t}_1 \wedge \cdots \wedge \bar{t}_j \wedge \cdots \wedge \bar{t}_i \wedge \cdots \wedge \bar{t}_p\,,$$

and let $\beta : \mathbb{S}^p \longrightarrow \mathbb{S}^p$ be the map given by

$$\beta(\bar{t}_1 \wedge \cdots \wedge \bar{t}_p) = \overline{1 - t_1} \wedge \bar{t}_2 \wedge \cdots \wedge \bar{t}_p\,.$$

Then $\alpha_{i\,j} \simeq \beta$ for all $i \neq j$.

9.2.8 **Proposition**. *Put $\alpha = \alpha_{i\,j}$. Then $\alpha_* : \mathrm{SP}\,\mathbb{S}^p \longrightarrow \mathrm{SP}\,\mathbb{S}^p$ is such that $\alpha_* = -\mathrm{id}_{\mathrm{SP}\,\mathbb{S}^p}$.*

Proof: By 9.2.7, $\alpha_* \simeq \beta_*$, and by the proof of **??**, $\beta_* \simeq -\mathrm{id}_{\mathrm{SP}\,\mathbb{S}^p}$. □

Define $R : I \times I \longrightarrow I$ and $T : I \times I \longrightarrow I$ by $R(s,t) = (1-t)s$ and $T(s,t) = (1-t)s + t$ and let $H : \mathrm{SP}\,\mathbb{S}^p \times I \longrightarrow \mathrm{SP}\,\mathbb{S}^p$ be given as follows. Write the elements of $\mathrm{SP}\,\mathbb{S}^p = \mathrm{SP}(\mathbb{S}^1 \wedge \mathbb{S}^{p-1})$ as $[(\bar{t}_1 \wedge \tau_1), \ldots, (\bar{t}_k \wedge \tau_k)]$, where $\bar{t}_i \in \mathbb{S}^1$ and $\tau_i \in \mathbb{S}^{p-1}$. Then

$$\begin{aligned}
H([(\bar{t}_1 \wedge \tau_1), \ldots, (\bar{t}_k \wedge \tau_k)], s) &= [\overline{R(s,t_1)} \wedge \tau_1, \ldots, \overline{R(s,t_k)} \wedge \tau_k] - \\
&\quad - [\overline{T(s,t_1)} \wedge \tau_1, \ldots, \overline{R(s,t_k)} \wedge \tau_k] \\
H([(\bar{t}_1 \wedge \tau_1), \ldots, (\bar{t}_k \wedge \tau_k)], 0) &= [\bar{0} \wedge \tau_1, \ldots, \bar{0} \wedge \tau_k] - [\overline{t_1} \wedge \tau_1, \ldots, \overline{t_k} \wedge \tau_k] \\
&= -[\overline{t_1} \wedge \tau_1, \ldots, \overline{t_k} \wedge \tau_k] \\
H([(\bar{t}_1 \wedge \tau_1), \ldots, (\bar{t}_k \wedge \tau_k)], 1) &= [\overline{1 - t_1} \wedge \tau_1, \ldots, \overline{1 - t_k} \wedge \tau_k] \\
&\quad - [\bar{1} \wedge \tau_1, \ldots, \bar{1} \wedge \tau_k] \\
&= [\overline{1 - t_1} \wedge \tau_1, \ldots, \overline{1 - t_k} \wedge \tau_k]\,,
\end{aligned}$$

where the minuses can be written, since $\mathrm{SP}\,\mathbb{S}^p$ is an H-group by **??**. Hence $H_0 = -\mathrm{id}$ and $H_1 = \beta_*$. Thus $-\mathrm{id} \simeq \beta_* \simeq \alpha_*$. □

9.2.9 **Corollary**. *Let* $\rho : \mathbb{S}^{p+q} = \mathbb{S}^p \wedge \mathbb{S}^q \longrightarrow \mathbb{S}^q \wedge \mathbb{S}^p = \mathbb{S}^{p+q}$ *be the map given by* $\rho(s \wedge s') = s' \wedge s$. *Then* $\rho_* : \mathrm{SP}\,\mathbb{S}^{p+q} \longrightarrow \mathrm{SP}\,\mathbb{S}^{p+q}$ *is homotopic to* $(-1)^{pq}\mathrm{id}$.

Proof: By the previous proposition, if one exchanges two coordinates, by α, then $\alpha_* \simeq -\mathrm{id}$. Since ρ is a composite of pq such alphas, then the result follows. $\square$

We now prove the commutativity property of the cup-product.

9.2.10 **Theorem**. *Given* $x \in \widetilde{H}^p(X)$ *and* $x' \in \widetilde{H}^q(X)$ *one has* $x \smile x' = (-1)^{pq} x' \smile x \in \widetilde{H}^{p+q}(X)$, *where the groups are taken with* Z*-coefficients.*

Proof: Since the coefficients are $\mathbb{Z}$, the Eilenberg–Mac Lane spaces are given by $K(\mathbb{Z}, n) = \mathrm{SP}\,\mathbb{S}^n$ for all $n \in \mathbb{N}$.

Then $x = [\alpha]$ and $x' = [\beta]$, where $\alpha : X \longrightarrow \mathrm{SP}\,\mathbb{S}^p$ and $\beta : X \longrightarrow \mathrm{SP}\,\mathbb{S}^q$ and the cup product is given by $x \smile x' = [\gamma]$, where $\gamma : X \longrightarrow \mathrm{SP}\,\mathbb{S}^{p+q}$ is defined by the composite

$$\gamma : X \xrightarrow{\Delta} X \wedge X \xrightarrow{\alpha\wedge\beta} \mathrm{SP}\,\mathbb{S}^p \wedge \mathrm{SP}\,\mathbb{S}^q \xrightarrow{\mu_{p,q}} \mathrm{SP}\,\mathbb{S}^{p+q}$$

i.e., $x \smile x' = [\mu_{p,q} \circ (\alpha \wedge \beta) \circ \Delta]$. The following is a commutative diagram.

$$\begin{tikzcd}
 & X \wedge X \arrow[r, "\alpha\wedge\beta"] & \mathrm{SP}\,\mathbb{S}^p \wedge \mathrm{SP}\,\mathbb{S}^q \arrow[r, "\mu_{p,q}"] & \mathrm{SP}\,\mathbb{S}^{p+q} \arrow[dd, "\rho_*"] \\
X \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & & \\
 & X \wedge X \arrow[r, "\beta\wedge\alpha"'] & \mathrm{SP}\,\mathbb{S}^q \wedge \mathrm{SP}\,\mathbb{S}^p \arrow[r, "\mu_{q,p}"'] & \mathrm{SP}\,\mathbb{S}^{q+p},
\end{tikzcd}$$

where ρ is as in Corollary 9.2.9, which is homotopic to $(-1)^{pq}\mathrm{id}$. This shows that

$$\mu_{q,p} \circ (\beta \wedge \alpha) \circ \Delta = \rho \circ \mu_{p,q} \circ (\alpha \wedge \beta) \circ \Delta \simeq (-1)^{pq} \mu_{p,q} \circ (\alpha \wedge \beta) \circ \Delta$$

and taking homotopy classes we obtain

$$x \smile x' = [\mu_{q,p} \circ (\beta \wedge \alpha) \circ \Delta] = (-1)^{pq}[\mu_{p,q} \circ (\alpha \wedge \beta) \circ \Delta] = (-1)^{pq} x' \smile x\,.$$

$\square$

Chapter 10 Transfer for ramified covering maps

Given a map with certain properties, we may roughly say that a transfer is a homomorphism in homology or cohomology, which goes in the opposite direction as the homomorphism induced by the given map. Several properties are expected from the transfer. In this chapter we shall define the transfer for ramified covering maps. This will include also a transfer for ordinary covering maps. We follow [3].

10.1 Transfer for ordinary covering maps

First recall that a *covering map* is a (surjective) map $p : E \longrightarrow X$ such that each point $x \in X$ has an *evenly covered* open neighborhood U, namely such that $p^{-1}U = \coprod_{i\in\mathfrak{I}} \widetilde{U}_i$ and the restriction $p_i = p|_{\widetilde{U}_i} : \widetilde{U}_i \longrightarrow U$ is a homeomorphism. We say that p is n-fold if $\mathfrak{I} = \mathbf{n} = \{1, 2, \ldots, n\}$.

10.1.1 The pretransfer

In what follows, given a space Y, we shall denote by Y^+ the pointed space $X \sqcup \{*\}$ and given a map $g : Y \longrightarrow Z$, we shall denote by $g^+ : Y^+ \longrightarrow Z^+$ the pointed map given by $g^+(y) = g(y)$ if $y \in Y$ and $g^+(*) = *$. Covering maps are usually nonpointed maps; this way we turn them into pointed maps, although they are no longer covering maps. Indeed they become ramified covering maps as we shall see in the next section.

10.1.1 Definition. Let L be a discrete abelian group and let $p : E \longrightarrow X$ be an n-fold covering map. Define the *pretransfer* $\tau_p : F(X^+; L) \longrightarrow F(E^+; L)$ by

$$\tau_p(u)(e) = u(p^+(e)) \,.$$

Recall that given the product $E^n = E \times \cdots \times E$, dividing it by the action of the symmetric group Σ_n acting by permutation of the coordinates, one obtains the nth *symmetric product* $\mathrm{SP}^n E$; i.e. $\mathrm{SP}^n E = E^n/\Sigma_n$. We denote its elements by $\langle e_1, \ldots, e_n\rangle$. If $p : E \longrightarrow X$ is an n-fold covering map, then there is map $\varphi_p : X \longrightarrow \mathrm{SP}^n E$, associated to p, which is defined by

$$\varphi_p(x) = \langle e_1, \ldots, e_n\rangle \,,$$

where $p^{-1}(x) = \{e_1, \dots, e_n\}$. We have the next.

10.1.2 **Proposition**. *The map $\varphi_p : X \longrightarrow \mathrm{SP}^n E$ is continuous.*

Proof: Take $x \in X$ and consider a neighborhood V of $\varphi_p(x) = \langle e_1, \dots, e_n\rangle$. We may assume that there is an evenly covered neighborhood U of X such that the neighborhood $\widetilde{U}_1 \times \cdots \times \widetilde{U}_n$ of the point $(e_1, \dots, e_n) \in E^n$ maps into V. This is possible, since the cubic neighborhoods of a point in E^n can be taken as small as one wishes. Then clearly $\varphi_p(U) \subset V$ and thus φ_p is continuous in x. $\square$

As a consequence, we obtain the following.

10.1.3 **Proposition**. *The pretransfer map $\tau_p : F(X^+; L) \longrightarrow F(E^+; L)$ is a continuous homomorphism.*

Proof: We first show that τ_p is a homomorphism. Take $u, v \in F(X^+ : L)$ and recall that $(u + v)(x) = u(x) + v(x) \in L$. Hence we have

$$\tau_p(u + v)(e) = (u + v)(p^+(e)) = u(p^+(e)) + v(p^+(e)) = \tau_p(u)(e) + \tau_p(v)(e)\,.$$

Hence $\tau_p(u + v) = \tau_p(u) + \tau_p(v)$.

We can define the pretransfer on generators. If $l \in L$, $e \in E$, then we have that $\tau_p : F(X^+, L) \longrightarrow F(E^+, L)$ is given by

$$t_p(lx) = \sum_{i=1}^{n} le_i\,,$$

where $p^{-1}(x) = \{e_1, \dots, e_n\}$.

Consider the map $\delta : L \times X^+ \longrightarrow F_n(E^+, L)$ given by $\delta(l, x) = \tau_p(lx)$ and $\alpha : L \times X^+ \longrightarrow (L \times (E^+)^n)/\Sigma_n$ given by

$$\alpha(l, x) = \langle (l, e_1), \dots, (l, e_n)\rangle\,,$$

where $p^{-1}(x) = \{e_1, \dots, e_m\}$. Let $j_l : (E^+)^n/\Sigma_n \longrightarrow (L \times E^+)^n/\Sigma_n$ be given by $j_l\langle e_1, \dots, e_n\rangle = \langle (l, e_1), \dots, (l, e_n)\rangle$. If $i_l : X^+ \longrightarrow L \times X^+$ is the inclusion at the level l, then $\alpha \circ i_l = j_l \circ \varphi_p$. Hence α is continuous. Notice that the identification $(L \times E^+)^n \longrightarrow F_n(E^+, L)$ factors as the composite

$$(L \times E^+)^n \twoheadrightarrow (L \times E^+)^n/\Sigma_n \overset{\rho_n}{\twoheadrightarrow} F_n(E^+, L)\,,$$

where $\rho_n\langle (l_1, e_1), \dots, (l_n, e_n)\rangle = \sum_{i=1}^n l_i e_i$. Therefore, ρ_n is continuous.

Since $\rho_n \circ \alpha = \delta$, δ is continuous. In order to see that $\tau_p|_{F_r(X,L)}$ is continuous, consider the diagram

$$\begin{tikzcd}
(L \times X^+)^r \arrow[r, "\delta^r"] \arrow[d, two heads] & (F_n(E^+, L))^r \arrow[d, "\mathrm{sum}"] \\
F_r(X^+, L) \arrow[r, "\tau_p|_{F_r(X^+,L)}"'] & F(E^+, L)\,,
\end{tikzcd}$$

where sum is given by the operation on $F(E^+, L)$, which is continuous. Hence τ_p is continuous, as desired. □

We shall show that this pretransfer has all expected properties. We start with the essential property, which we call *normality*.

10.1.4 **Proposition**. *The composite $p_* \circ \tau_p : F(X^+; L) \longrightarrow F(X^+; L)$ is multiplication by n.*

Proof: Take $u \in F(X^+; L)$. Then

$$p_*\tau_p(u)(x) = \sum_{e \in p^{-1}(x)} \tau_p(u)(e) = \sum_{e \in p^{-1}(x)} u(p(e)) = \sum_{e \in p^{-1}(x)} u(x) = nu(x)\,,$$

since there are n elements $e \in p^{-1}(x)$. Thus $p_*\tau_p(u) = nu$ □

Another important property is the so-called *pullback* property. Consider an n-fold covering map $p : E \longrightarrow X$ and a map $f : Y \longrightarrow X$ and take the induced covering map $f^*(p) : f^*(E) \longrightarrow Y$, where $f^*(E) = \{(y, e) \in Y \times E \mid f(y) = p(e)\}$ and $f^*(p)$ is the projection on Y, i.e. $f^*(p)(y, e) = y$. Thus we have a *pullback diagram*

$$\begin{tikzcd} f^*(E) \arrow[r, "\widetilde{f}"] \arrow[d, "f^*(p)"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X\,, \end{tikzcd}$$

where $\widetilde{f} : f^*(E) \longrightarrow E$ is the other projection, namely $\widetilde{f}(y, e) = y$. By 4.2.11, we know that $f^*(p)$ is a covering map.

10.1.5 **Proposition**. *Given a map $f : Y \longrightarrow X$, then $\widetilde{f}^+_* \circ \tau_{f^*(p)} = \tau_p \circ f^+_*$, i.e., the following is a commutative diagram:*

$$\begin{tikzcd} F(f^*(E)^+; L) \arrow[r, "\widetilde{f}^+_*"] & F(E^+; L) \\ F(Y^+; L) \arrow[r, "f^+_*"'] \arrow[u, "\tau_{f^*(p)}"] & F(X^+; L)\,, \arrow[u, "\tau_p"'] \end{tikzcd}$$

Proof: First notice that $\tau_{f^*(p)} : F(Y^+; L) \longrightarrow F(f^*(E)^+; L)$ is given for $v \in F(Y^+; L)$ by

$$\tau_{f^*(p)^+}(v)(y, e) = v(f^*(p)^+(y, e)) = v(y)\,.$$

Therefore we obtain

$$(\widetilde{f}^+_*(\tau_{f^*(p)}(v)))(e) = \sum_{\widetilde{f}^+(y,e')=e} \tau_{f^*(p)}(v)(y, e')) = \sum_{f^+(y)=p^+(e)} v(y)\,.$$

On the other hand, we have

$$(\tau_p(f_*^+(v)))(e) = f_*^+(v)(p^+(e)) = \sum_{f^+(y)=p^+(e)} v(y)\,.$$

$\square$

10.1.6 Remark. This pullback property is in fact a *naturality* property in the following sense. We can consider the category of finite covering maps $\mathfrak{Cov}$, whose objects are covering maps $p : E \longrightarrow X$ and the morphisms $p' \longrightarrow p$, where $p' : E' \longrightarrow X'$ is another covering map, are pairs of maps $(\widetilde{f}; f)$, so that they fit into a *general pullback* diagram

$$\begin{tikzcd} E' \arrow[r, "\widetilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ X' \arrow[r, "f"'] & X \end{tikzcd},$$

namely a diagram where E' is homeomorphic to $f^*(E)$ such that the triangle

$$\begin{tikzcd} E' \arrow[rr, "\varphi", "\approx"'] \arrow[dr, "p'"'] & & f^*(E) \arrow[dl, "f^*(p)"] \\ & X' & \end{tikzcd}$$

commutes and $\widetilde{f} = \mathrm{proj}_E \circ \varphi$. The composition is the obvious one. There are two functors $\mathcal{E}, \mathcal{X} : \mathfrak{Cov} \longrightarrow \mathfrak{Top}_*$ such that given an object $p : E \longrightarrow X$, $\mathcal{E}(p) = E^+$, $\mathcal{X}(p) = X^+$ and given a morphism $(\widetilde{f}; f)$, $\mathcal{E}(\widetilde{f}; f) = \widetilde{f}^+$ and $\mathcal{X}(\widetilde{f}; f) = f^+$. In this context the pullback property states that *the pretransfer is a natural transformation*

$$\tau : F(\mathcal{X}; L) \stackrel{\cdot}{\longrightarrow} F(\mathcal{E}; L)\,,$$

where $F(\mathcal{X}; L)$ denotes the composite functor $F(-; L) \circ \mathcal{X}$ and similarly $F(\mathcal{E}; L) = F(-; L) \circ \mathcal{X}$.

Another property of the pretransfer is the *units* property.

10.1.7 **Proposition**. *For the identity covering map* $\mathrm{id}_X : X \longrightarrow X$*, the pretransfer is the identity* $1_{F(X^+;L)} : F(X^+; L) \longrightarrow F(X^+; L)$*.*

Proof: The pretransfer $\tau_{\mathrm{id}_X} : F(X^+; L) \longrightarrow F(X^+; L)$ is given by $\tau_{\mathrm{id}_X}(u)(x) = u(\mathrm{id}_{X^+}(x)) = u(x)$, thus $\tau_{\mathrm{id}_X} = 1_{F(X^+;L)}$. $\square$

10.1.8 Exercise. Show that the last property is a special case of the normality property 10.1.4.

The pretransfer has also an *additivity* property. Recall the additivity property of the Dold-Thom groups 7.3.5, we have

$$F(X \wedge Y; L) \cong F(X; L) \oplus F(Y; L)\,. \tag{10.1.9}$$

Notice that $(X \sqcup Y)^+ = X^+ \wedge Y^+$.

10.1.10 **Proposition**. *Given an m-fold covering map $p : E \longrightarrow X$ and an n-fold covering map $q : F \longrightarrow X$, the map $(p,q) : E \sqcup F \longrightarrow X$ is an $m+n$-fold covering map such that its pretransfer*

$$\tau_{(p,q)} : F((E \sqcup F)^+; L) \longrightarrow F(X^+; L)$$

is given by $\tau_{(p,q)}(u,v) = \tau_p(u) + \tau_q(v)$, where $(u,v) \in F(E^+; L) \oplus F(F^+; L)$ and we put $F((E \sqcup F)^+; L) = F(E^+; L) \oplus F(F^+; L)$.

Proof: $\tau_{(p,q)}(u,v)(e) = u(p^+(e))$ for $e \in E^+$ and $\tau_{(p,q)}(u,v)(e') = u(q^+(e'))$ for $e' \in F^+$. Hence $\tau_{(p,q)}(u,v) = \tau_p(u) + \tau_q(v)$. □

There is also a *multiplicativity* property. Recall the pairing given in 9.1.2 and we have the next result, whose proof we leave to the reader as an *exercise*.

10.1.11 **Proposition**. *Given an m-fold covering map $p : E \longrightarrow X$ and an n-fold covering map $q : F \longrightarrow X$, the map $p \times q : E \times F \longrightarrow X \times X$ is an mn-fold covering map such that its transfer makes the following diagram commute:*

$$\begin{tikzcd}
F(X^+; L) \times F(X^+; L) \arrow[r, "\mu_{X,X}"] \arrow[d, "\tau_p \times \tau_q"'] & F((X \times X)^+; L) \arrow[d, "\tau_{p \times q}"] \\
F(E^+; L) \times F(F^+; L) \arrow[r, "\mu_{E,F}"'] & F((E \times F)^+; L)\,.
\end{tikzcd}$$

□

In some other sense, the transfer is a contravariant functor. Namely, if one takes the category of all spaces but restricts the morphisms to finite covering maps $p : Y \longrightarrow X$, then the assignment $p \mapsto \tau_p$ is a functor. One part was proved in 10.1.7.

10.1.12 **Proposition**. *Given an m-fold and an n-fold covering maps $p : Y \longrightarrow X$ and $q : Z \longrightarrow Y$, the composite $p \circ q : Z \longrightarrow X$ is an mn-fold covering map such that*

$$\tau_{p \circ q} = \tau_q \circ \tau_p : F(X^+; L) \longrightarrow F(Z^+; L)\,.$$

Proof: Take $u \in F(X^+; L)$ and $z \in Z$. On the one hand we have

$$\tau_{p \circ q}(u)(z) = u((p \circ q)(z)) = u(p(q(z))\,,$$

while on the other we have

$$(\tau_q \circ \tau_p)(u)(z) = \tau_p(u)(q(z)) = u(p(q(z)))\,,$$

thus they are equal as desired. □

10.1.2 The transfer in homology

Since the pretransfer $\tau_p : F(X^+; L) \longrightarrow F(E^+; L)$ for a covering map $p : E \longrightarrow X$ is a continuous homomorphism, it induces a homomorphism in homotopy groups $\tau_{p*} : \pi_q(F(X^+; L)) \longrightarrow \pi_q(F(E^+; L))$. In other words we have a homomorphism

$$t_p : H_q(X; L) \longrightarrow \widetilde{H}_q(E; L)$$

called the *transfer*.

10.1.13 **Theorem**. *Given a covering map $p : E \longrightarrow X$, its transfer $t_p : H_q(X; L) \longrightarrow H_q(E; L)$ has the following properties:*

(a) ***Normality**: Given an n-fold covering map $p : E \longrightarrow X$, then*

$$p_* \circ t_p : \widetilde{H}_q(X; L) \longrightarrow \widetilde{H}_q(X; L) \quad \textit{is multiplication by} \quad n\,.$$

(b) ***Naturality**: Given a morphism of covering maps $(\widetilde{f}, f) : p' \longrightarrow p$, where $p : E \longrightarrow X$ and $p' : E' \longrightarrow X'$ are n-fold covering maps, one has*

$$\widetilde{f}_* \circ t_{p'} = t_p \circ f_* : \widetilde{H}_q(X') \longrightarrow \widetilde{H}_q(E; L)\,.$$

10.2 Ramified covering maps

We start with the definition of a ramified covering map as given in [44]. As in the previous section, we shall need the concept of *nth symmetric product* of a space defined in 135.

10.2.1 Definition. An *n-fold ramified covering map* is a continuous map $p : E \longrightarrow X$ together with a *multiplicity function* $\mu : E \longrightarrow \mathbb{N}$ such that the following hold:

(i) The fibers $p^{-1}(x)$ are finite (discrete) for all $x \in X$.

(ii) For each $x \in X$, $\sum_{e \in p^{-1}(x)} \mu(e) = n$.

(iii) The map $\varphi_p : X \longrightarrow \mathrm{SP}^n E$ given by

$$\varphi_p(x) = \langle \underbrace{e_1, \ldots, , e_1}_{\mu(e_1)}, \ldots, \underbrace{e_m, \ldots, , e_m}_{\mu(e_m)} \rangle\,,$$

where $p^{-1}(x) = \{e_1, \ldots, e_m\}$, is continuous.

10.2.2 Remark. Given an n-fold ramified covering map $p : E \longrightarrow X$ with multiplicity function μ, one can construct an n-fold ramified covering map $p^+ : E^+ \longrightarrow X^+$, where $Y^+ = Y \sqcup \{*\}$ for any space Y and p^+ extends p by defining $p^+(*) = *$ and the multiplicity function μ^+ extends μ by setting $\mu^+(*) = n$. More generally, given a (closed) subspace $A \subset X$, one can construct an n-fold ramified covering map $p' : E' \longrightarrow X/A$, where $E' = E/p^{-1}A$, p' is the map between quotients and the multiplicity function μ' coincides with μ off $p^{-1}A$ and is extended by setting $\mu'(*) = n$, if $*$ is the base point onto which $p^{-1}A$ collapses.

Another useful construction is the following. Let $\overline{E} = E \sqcup X$ and $\overline{p} : \overline{E} \longrightarrow X$ be such that $\overline{p}|_E = p$ and $\overline{p}|_X = \mathrm{id}_X$. Then $\overline{p}$ is an $(n+1)$-fold ramified covering map with the multiplicity function $\overline{\mu} : \overline{E} \longrightarrow \mathbb{N}$ given by $\overline{\mu}|_E = \mu$ and if x lies in the added X, the $\overline{\mu}(x) = 1$.

On the other hand, given a map $f : Y \longrightarrow X$, as above 137, one can construct the *induced n-fold ramified covering map* $f^*(p) : f^*(E) \longrightarrow Y$ by taking the pullback $f^*(E) = \{(y, e) \in Y \times E \mid f(y) = p(e)\}$ and $f^*(p) = \mathrm{proj}_Y$. The *induced multiplicity function* $f^*(\mu) : f^*(E) \longrightarrow \mathbb{N}$ is given by $f^*(\mu)(y, e) = \mu(e)$.

10.2.3 Exercise. For each of the constructions given above prove that the corresponding maps fy are continuous.

10.2.4 Examples. Typical examples of ramified covering maps are the following:

1. Standard covering maps with finitely many leaves, as it is shown in 10.1.2.

2. Orbit maps $E/\Gamma' \longrightarrow E/\Gamma$ for actions of a finite group Γ on a space E and $\Gamma' \subset \Gamma$. They can be considered as $[\Gamma : \Gamma']$-fold ramified covering maps by taking $\mu(e\Gamma') = [\Gamma_e : \Gamma'_e]$, where Γ_e and Γ'_e denote the isotropy subgroups of $e \in E$ for the action of Γ and the restricted action of Γ', and $[\Gamma : \Gamma']$ and $[\Gamma_e : \Gamma'_e]$ denote the corresponding indexes. In fact, Dold [11] proves that all ramified covering maps are of this form for $\Gamma = \Sigma_n$ and $\Gamma' = \Sigma_{n-1}$.

3. *Branched covering maps* on manifolds, namely open maps $p : M^d \longrightarrow N^d$, where M^d and N^d are orientable closed topological manifolds of dimension d, p has finite fibers and its degree is n. Indeed, in[6] it is proven that p is of the form $E/\Gamma' \longrightarrow E/\Gamma$, with $[\Gamma : \Gamma'] = n$, so that, by 2., p is in fact an n-fold ramified covering map. An interesting special case of this is given in [38] and [24], who show that for any closed orientable 3-manifold M^3, there is a branched covering map $p : M^3 \longrightarrow \mathbb{S}^3$ of degree 3.

4. It will be of particular interest to consider the following example. Let B be a space and and consider the twisted product $B^n \times_{\Sigma_n} \overline{n}$, where $\overline{n} = \{1, 2, \ldots, n\}$ and $\times_{\Sigma_n}$, given by identifying in the product a pair $(b_1, b_2, \ldots, b_n; i)$ with the

pair $(b_{\sigma(1)}, b_{\sigma(2)}, \dots, b_{\sigma(n)}; \sigma(i))$.Denote the elements of the quotient space by $\langle b_1, b_2, \dots, b_n; i\rangle$. Take the map

$$\pi_B : B^n \times_{\Sigma_n} \overline{n} \longrightarrow \overset{n}{\mathrm{SP}}\, B\,,$$

given by $\pi_B\langle b_1, b_2, \dots, b_n; i\rangle = \langle b_1, b_2, \dots, b_n\rangle$. Then π_B is an n-fold ramified covering map with multiplicity function $\mu_B : B^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathbb{N}$ given by $\mu_B\langle b_1, b_2, \dots, b_n; i\rangle = \#\{j \mid b_j = b_i\}$ (see [44]).

10.3 The homology transfer

We shall define now the homology transfer. Our spaces in this section will be compactly generated weak Hausdorff spaces. As for the case of ordinary covering maps, we shall define a pretransfer and after applying the homotopy-group functors, we shall obtain the transfer. We shall study both at once.

10.3.1 Definition. Let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function μ. Define the *pretransfer*

$$\tau_p : F(X; L) \longrightarrow F(E; L) \quad \text{by} \quad \tau_p(u) = \widetilde{u}\,,$$

where $\widetilde{u}(e) = \mu(e)u(p(e))$. In other words, if $u = \sum_{i=1}^n l_i x_i \in F(X; L)$, then

$$\tau_p(u) = \sum_{\substack{p(e)=x_i\\ i=1,\dots,n}} \mu(e) l_i e\,.$$

10.3.2 Remark. The pretransfer $\tau_p : F(X; L) \longrightarrow F(E; L)$ is clearly a homomorphism of topological groups and it is thus convenient to see what it does to generators. Namely, if lx is a generator, then the pretransfer satisfies

$$\tau_p(lx)(e) = \mu(e)lx(p(e)) = \begin{cases} \mu(e)l & \text{if } p(e) = x, \text{ i.e., if } e \in p^{-1}(x) \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the only points where $\tau_p(lx)$ is nonzero are the elements of $p^{-1}(x) = \{e_1, e_2, \dots, e_r\}$, that is,

$$\tau_p(lx)(e_1) = \mu(e_1)l\,,\ \tau_p(lx)(e_2) = \mu(e_2)l\,, \dots,\ \tau_p(lx)(e_r) = \mu(e_r)l\,,$$

and thus

$$\tau_p(lx) = \mu(e_1)le_1 + \mu(e_2)le_2 + \cdots + \mu(e_r)le_r\,.$$

We shall prove below that τ_p is continuous. Hence, on homotopy groups, the map τ_p induces the *homology transfer*

$$t_p : \widetilde{H}_q(X; L) \longrightarrow \widetilde{H}_q(E; L)\,.$$

We have the following.

10.3.3 **Proposition**. *Let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function $\mu : E \longrightarrow \mathbb{N}$, where E and X are pointed spaces. Then the pretransfer $\tau_p : F(X;L) \longrightarrow F(E;L)$ is continuous.*

Proof: Since $F(X;L)$ has the topology of the union of the subspaces

$$\cdots \subset F_r(X;L) \subset F_{r+1}(X;L) \subset \cdots \subset F(X;L)\,,$$

τ_p is continuous if and only if the restriction $\tau_p|_{F_r(X;L)}$ is continuous for each $r \in \mathbb{N}$. Since $X \times Y$ has the k-topology, we have a quotient map $q_r : (L \times X)^r \twoheadrightarrow F_r(X;L)$ for each r. Define $\delta : L \times X \longrightarrow F_n(X;L)$ by $\delta(l,x) = \tau_p q_1(l,x) = \tau_p(lx)$, and $\alpha : L \times X \longrightarrow (L \times X)^n/\Sigma_n$ by

$$\alpha(l,x) = [\underbrace{(l,e_1),\ldots,(l,e_1)}_{\mu(e_1)},\ldots,\underbrace{(l,e_m),\ldots,(l,e_m)}_{\mu(e_m)}]\,,$$

where $p^{-1}(x) = \{e_1,\ldots,e_m\}$. For each $l \in L$, let $i_l : X \longrightarrow L \times X$ be given by $i_l(x) = (x,l)$, and let $j_l : E^n/\Sigma_n \longrightarrow (L \times X)^n/\Sigma_n$ be given by $j_l[e_1,\ldots,e_n] = [(l,e_1),\ldots,(l,e_n)]$. Then $\alpha \circ i_l = j_l \circ \varphi_p$, where $\varphi_p : X \longrightarrow E^n/\Sigma_n$. Since j_l and φ_p are continuous and L is discrete, α is continuous.

The quotient map q_n factors through the quotient map $q'_n : (L \times X)^n \twoheadrightarrow (L \times X)^n/\Sigma_n$, yielding the following commutative diagram,

$$\begin{tikzcd}
(L \times X)^n \arrow[r, two heads, "q'_n"] \arrow[d, two heads, "q_n"'] & (L \times X)^n/\Sigma_n \arrow[dl, two heads, "\rho_n"] \\
F_n(X;L)\,, &
\end{tikzcd}$$

where ρ_n is also a quotient map.

Now, δ makes the following diagram commute,

$$\begin{tikzcd}
& (L \times X)^n/\Sigma_n \arrow[d, two heads, "\rho_n"] \\
L \times X \arrow[ur, "\alpha"] \arrow[r, "\delta"'] & F_n(X;L)\,,
\end{tikzcd}$$

therefore, δ is continuous.

In order to show that $\tau_p|_{F_r(X;L)}$ is continuous, consider the diagram

$$\begin{tikzcd}
(L \times X)^r \arrow[r, "\delta^r"] \arrow[d, two heads, "q_r"'] & F_n(E;L) \times \cdots \times F_n(E;L) \arrow[d, "\sum_{i=1}^r"] \\
F_r(X;L) \arrow[r, "\tau_p|_{F_r(X;L)}"'] & F(E;L)\,,
\end{tikzcd}$$

where $\sum_{i=1}^r$ is the operation in $F(E;L)$, which is a topological abelian group in $\mathfrak{K}$-$\mathfrak{Top}$, and hence it is continuous. Since also δ is continuous, and q_r is a quotient map, $\tau_p|_{F_r(X;L)}$ is continuous.

10.3.4 **Corollary**. *Let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function $\mu : E \longrightarrow \mathbb{N}$, where E and X are pointed* CW*-complexes. Then there is a homology transfer $t_p : \widetilde{H}_q(X;L) \longrightarrow \widetilde{H}_q(E;L)$.* □

10.3.5 REMARK. Besides the transfer t_p defined above, for every integer k there is another homology transfer ${}_k\tau$ given by $({}_k\tau)_p(\xi) = k \cdot \tau_p(\xi)$, $\xi \in H_q(X;L)$. This transfer, in turn, is determined by the pretransfer $({}_kt)_p : F(X;L) \longrightarrow F(E;L)$ given by $({}_kt)_p(u) = k \cdot \tau_p(u)$, $u \in F(X;L)$.

10.3.6 EXAMPLE. For the ramified covering map $\pi_B : B^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n B$ of 10.2.4, the homology transfer is given as follows. We first compute

$$\tau_{\pi_B} : F(\overset{n}{\mathrm{SP}}\, B;L) \longrightarrow F(B^n \times_{\Sigma_n} \overline{n};L)$$

on the generators. Set

$$b = (\underbrace{b_1,\ldots,b_1}_{i_1},\underbrace{b_2,\ldots,b_2}_{i_2},\ldots,\underbrace{b_r,\ldots,b_r}_{i_r}) \in B^n\,,$$

where $i_1 + i_2 + \cdots i_r = n$. Then

$$\pi_B^{-1}\langle b\rangle = \{\langle b_1, i_1\rangle, \langle b_2, i_1 + i_2,\rangle \ldots, \langle b_r, n\rangle\}\,.$$

Therefore,

$$\begin{aligned}
\tau_{\pi_B}(l\langle x\rangle) &= \mu\langle b, i_1\rangle l\langle b, i_1\rangle + \mu\langle b, i_1+i_2\rangle l\langle b, i_1+i_2\rangle + \cdots + \\
&\quad + \mu\langle b, i_1+i_2+\cdots+i_r\rangle l\langle b, i_1+i_2+\cdots i_r\rangle \\
&= i_1 l\langle b, i_1\rangle + i_2 l\langle b, i_1+i_2\rangle + \cdots + i_r l\langle b, i_1+i_2+\cdots i_r\rangle \\
&= l\,\underbrace{\langle b, i_1\rangle + \langle b, i_1\rangle + \cdots \langle b, i_1\rangle}_{i_1} + \\
&\quad + \underbrace{\langle b, i_1+i_2\rangle + \langle b, i_1+i_2\rangle + \cdots + \langle b, i_1+i_2\rangle}_{i_2} + \cdots + \\
&\quad + \underbrace{\langle b, n\rangle + \langle b, n\rangle + \cdots + \langle b, n\rangle}_{i_r} \\
&= l\langle b, 1\rangle + \cdots + \langle b, i_1\rangle + \langle b, i_1+1\rangle + \ldots \langle b, i_1+i_2\rangle + \cdots + \\
&\quad + \langle b, i_1+i_2+\cdots+i_{r-1}+\cdots+1\rangle + \langle b, n\rangle \\
&= l\langle b, 1\rangle + l\langle b, 2\rangle + \cdots + l\langle b, n\rangle\,,
\end{aligned}$$

hence

$$\tau_{\pi_B}(l\langle b_1,\ldots,b_n\rangle) = l\langle b_1,\ldots,b_n;1\rangle) + \cdots + l\langle b_1,\ldots,b_n;n\rangle)\,. \tag{10.3.7}$$

Thus, in general, if $\beta = \sum_{i=1}^k l_i\langle b_1^i,\ldots,b_n^i\rangle$, then

$$\tau_{\pi_B}(\beta) = \sum_{(i,l)=(1,1)}^{(k,n)} l_i\langle b_1^i,\ldots,b_n^i;l\rangle\,,$$

since by varying l from 1 to n, the fiber elements over $\langle b_1^i,\ldots,b_n^i\rangle$, namely $\langle b_1^i,\ldots,b_n^i;l\rangle$, are repeated $\mu_B\langle b_1^i,\ldots,b_n^i;l\rangle$ times.

10.3.8 REMARK. Given an n-fold ramified covering map $p : E \longrightarrow X$ with multiplicity function $\mu : E \longrightarrow \mathbb{N}$, and a (closed) subspace $A \subset X$, we have the *restricted ramified covering map* $p_A : E_A \longrightarrow A$, $E_A = p^A$, and the *quotient ramified covering map* $\overline{p} : \overline{E} \longrightarrow X/A$, as described in Remark 10.2.2. The following diagram obviously commutes:

$$\begin{tikzcd} E_A \arrow[r, hook] \arrow[d, "p_A"'] & E \arrow[r, two heads] \arrow[d, "p"'] & \overline{E} \arrow[d, "\overline{p}"] \\ A \arrow[r, hook] & X \arrow[r, two heads] & X/A\,. \end{tikzcd}$$

Thus the diagram above yields

$$\begin{tikzcd} F(A;L) \arrow[r] \arrow[d, "\tau_A"'] & F(X;L) \arrow[r] \arrow[d, "t"'] & F(X/A;L) \arrow[d, "\bar{t}"] \\ F(E_A;L) \arrow[r] & F(E;L) \arrow[r] & F(\overline{E};L)\,, \end{tikzcd}$$

where the horizontal arrows are obvious and τ_A, t, and $\bar{t}$ are the corresponding pretransfers. Therefore, using $\bar{t}$, we have a *relative homology transfer* $t_p : H_q(X, A; L) \longrightarrow H_q(E, E_A; L)$, and by the commutativity of the diagram, also this transfer maps the long exact sequences of (X, A) into the long exact sequence of (E, E_A), provided that the inclusion $A \hookrightarrow X$ is a closed cofibration (in general it is also true by constructing an adequate ramified covering over $X \cup CA$).

The following theorems establish the fundamental properties of the transfer.

10.3.9 **Theorem**. *The composite*

$$p_* \circ t_p : \widetilde{H}_q(X; L) \longrightarrow \widetilde{H}_q(X; L)$$

is multiplication by n.

The *proof* follows immediately from the following proposition.

10.3.10 **Proposition**. *If $p : E \longrightarrow X$ is an n-fold ramified covering map, then the composite*

$$F(X; L) \xrightarrow{\tau_p} F(E; L) \xrightarrow{p_*} F(X; L)$$

is multiplication by n.

Proof: If $u = \sum_{i=1}^{n} l_i x_i \in F(X;L)$, then

$$\begin{aligned} p_*\tau_p(u) &= p_*\tau_p\Big(\sum_{i=1}^{n} l_i x_i\Big) \\ &= \sum_{p(e)=x_i,\, i=1,\ldots,n} \mu(e) l_i x_i \\ &= \sum_{i=1}^{n} l_i x_i \sum_{p(e)=x_i} \mu(e) \\ &= n \sum_{i=1}^{n} l_i x_i = n \cdot u\,. \end{aligned}$$

The invariance under pullbacks is given by the following.

10.3.11 **Theorem**. *Let $p : E \longrightarrow X$ be an n-fold ramified covering map and assume that $f : X \longrightarrow Y$ is continuous. Then the following diagram commutes:*

$$\begin{tikzcd} H_q(Y;L) \arrow[r, "t_{f^*(p)}"] \arrow[d, "f_*"'] & H_q(f^*(E);L) \arrow[d, "\widetilde{f}_*"] \\ H_q(X;L) \arrow[r, "t_p"'] & H_q(E;L)\,, \end{tikzcd}$$

where $f^*(p) : f^*(E) \longrightarrow Y$ is the n-fold ramified covering map induced by $p : E \longrightarrow X$ over f.

As for the previous theorem, the *proof* follows immediately from the next proposition.

10.3.12 **Proposition**. *If $p : E \longrightarrow X$ is an n-fold ramified covering map and $F : X \longrightarrow Y$ is continuous, then the following diagram commutes.*

$$\begin{tikzcd} F(Y;L) \arrow[r, "\tau_{F^*(p)}"] \arrow[d, "F_*"'] & F(F^*(E);L) \arrow[d, "\widetilde{F}_*"] \\ F(X;L) \arrow[r, "\tau_p"'] & F(E;L)\,. \end{tikzcd}$$

Proof: Let $v = \sum_{i=1}^{n} g_i y_i \in F(Y;L)$. Then $\tau_{F^*(p)}(v) \in F(F^*(E);L)$ is such that

$$\begin{aligned}
\widetilde{F}_*\left(\tau_{F^*(p)}(v)\right) &= \widetilde{F}_*\left(\sum_{\substack{F_*(p)(y,e)=y_i\\ i=1,\dots,n}} F_*(\mu)(y,e)g_i(y,e)\right)\\
&= \sum_{\substack{F_*(p)(y,e)=y_i\\ i=1,\dots,n}} \mu(e)g_i\widetilde{F}(y,e)\\
&= \sum_{\substack{p(e)=F(y_i)\\ i=1,\dots n}} \mu(e)g_i e\\
&= \tau_p\left(F_*(v)\right)\,.
\end{aligned}$$

Another property of the homology transfer that is useful is the following.

10.3.13 **Proposition**. *Let $f : Y \longrightarrow X$ be continuous and consider the commutative diagram*

$$\begin{tikzcd}
Y^n \times_{\Sigma_n} \overline{n} \arrow[r, "f^n \times_{\Sigma_n} 1_{\overline{n}}"] \arrow[d, "\pi_Y"'] & X^n \times_{\Sigma_n} \overline{n} \arrow[d, "\pi_X"] \\
\mathrm{SP}^n\, Y \arrow[r, "\mathrm{SP}^n f"'] & \mathrm{SP}^n\, X\,.
\end{tikzcd}$$

Then the following diagram commutes:

$$\begin{tikzcd}
F(Y^n \times_{\Sigma_n} \overline{n};L) \arrow[r, "(f^n \times_{\Sigma_n} 1_{\overline{n}})_*"] & F(X^n \times_{\Sigma_n} \overline{n};L) \\
F(\mathrm{SP}^n\, Y;L) \arrow[r, "(\mathrm{SP}^n f)_*"'] \arrow[u, "\tau_{\pi_Y}"] & F(\mathrm{SP}^n\, X;L)\,. \arrow[u, "\tau_{\pi_X}"']
\end{tikzcd}$$

The *proof* is fairly routinary and follows easily using the description of the transfers given in Example 10.3.6. □

Another property of the transfer is the following homotopy invariance.

10.3.14 **Theorem**. *Let $p : E \longrightarrow X$ be an n-fold ramified covering map. If $f_0, f_1 : Y \longrightarrow X$ are homotopic maps, then*

$$\widetilde{f}_{0*} \circ t_{p_0} = \widetilde{f}_{1*} \circ t_{p_1} : \widetilde{H}_q(Y;L) \longrightarrow \widetilde{H}_q(E;L)\,.$$

The proof is an immediate consequence of the next proposition.

10.3.15 **Proposition**. *Let $p : E \longrightarrow X$ be an n-fold ramified covering map. If $f_0, f_1 : Y \longrightarrow X$ are homotopic maps, then*

$$\widetilde{f}_{0*} \circ \tau_{p_0} = \widetilde{f}_{1*} \circ \tau_{p_1} : F(Y;L) \longrightarrow F(E;L)\,.$$

Proof: If $H : Y \times I \longrightarrow X$ is a homotopy from f_0 to f_1, then $\widehat{H} : F(Y;L) \times I \longrightarrow F(X;L)$ given by $\widehat{H}(v,t) = \sum_{y \in Y} v(y)H(y,t)$ is a (continuous) homotopy from f_{0*} to f_{1*}. Thus, applying Proposition 10.3.12, we get

$$f_{0*} \circ \tau_{p_0} = \tau_p \circ f_{0*} \simeq \tau_p \circ f_{1*} = f_{1*} \circ \tau_{p_1} .$$

$\square$

In 10.3.10 we computed the composite $p_* \circ \tau_p$. The opposite composite $\tau_p \circ p_*$ is also interesting. An immediate computation yields the following.

10.3.16 **Proposition**. *Let $p : E \longrightarrow X$ by an n-fold ramified covering map with multiplicity function μ. Then the composite*

$$F(E;L) \xrightarrow{p_*} F(X;L) \xrightarrow{\tau_p} F(E;L)$$

is given by

$$\tau_p p_*(v)(e) = \sum_{p(e')=p(e)} \mu(e')v(e') ,$$

for any $v \in F(E;L)$. $\square$

In the case of an action of a finite group Γ on E and $X = E/\Gamma$, we have the following consequence.

10.3.17 **Corollary**. *For $v \in F(E;L)$ one has $\tau_p p_*(v)(e) = \sum_{\gamma \in \Gamma} v(\gamma e)$. Therefore, the composite*

$$F(E/\Gamma;L) \xrightarrow{p_*} F(E;L) \xrightarrow{\tau_p} F(E/\Gamma;L)$$

is given by $\tau_p p_*(v) = \sum_{\gamma \in \Gamma} \gamma_*(v)$.

Proof: Just observe that the element γe is repeated in the sum $\mu(e) = |\Gamma_e|$ times.

The two previous results yield the following in homology.

10.3.18 **Theorem**. *Let $p : E \longrightarrow X$ by an n-fold ramified covering map with multiplicity function μ. Then the composite*

$$H_q(E;L) \xrightarrow{p_*} H_q(X;L) \xrightarrow{t_p} H_q(E;L)$$

is given by $t_p p_*(y) = y'$, where $y' = [v'] \in \pi_q(F(E;L))$, and

$$v'(s)(e) = \sum_{p(e')=p(e)} \mu(e')v(s)(e')$$

where $y = [v] \in \pi_q(F(E;L))$ and $s \in \mathbb{S}^q$. $\square$

10.3.19 **Corollary**. *For an action of a finite group Γ on E and $X = E/\Gamma$ one has that the composite*

$$H_q(E;L) \xrightarrow{p_*} H_q(E/\Gamma;L) \xrightarrow{t_p} H_q(E;L)$$

is given by $t_p p_*(y) = \sum_{\gamma\in\Gamma} \gamma_*(y)$.

10.3.20 Remark. Considering an action of H on E and a subgroup $K \subset H$, one has different ramified covering maps as depicted in

$$\begin{tikzcd} & E \arrow[dl, "q'_\Gamma"'] \arrow[dr, "q_\Gamma"] & \\ E/\Gamma' \arrow[rr, "q_\Gamma^{\Gamma'}"'] & & E/\Gamma\,. \end{tikzcd}$$

One may easily compute several combinations of the maps induced by these covering maps and their transfers.

Another interesting property of the transfer is the relationship given by computing the transfer of the composition of two ramified covering maps. Before giving it we need the following.

10.3.21 Definition. Let $p : Y \longrightarrow X$ be an n-fold ramified covering map, with multiplicity function $\mu : Y \longrightarrow \mathbb{N}$ and let $q : Z \longrightarrow Y$ be an m-fold ramified covering map, with multiplicity function $\nu : Z \longrightarrow \mathbb{N}$. Then the composite $p \circ q : Z \longrightarrow X$ is an mn-fold ramified covering map, with multiplicity function $\xi : Z \longrightarrow \mathbb{N}$ given by $\xi(z) = \nu(z)\mu(q(z))$. In order to verify that this composite is indeed an mn-fold ramified covering map, consider the *wreath product* $\Sigma_n \int \Sigma_m$, defined as the semidirect product of Σ_n and $(\Sigma_m)^n$, where Σ_n acts on $(\Sigma_m)^n$ by permuting the n factors. We have an action $(Z^m \times \cdots \times Z^m) \times \Sigma_n \int \Sigma_m \longrightarrow Z^m \times \cdots \times Z^m$ given by $(\zeta_1, \dots, \zeta_n) \cdot (\sigma, \tau_1, \dots, \tau_n) = (\zeta_{\sigma(1)} \cdot \tau_1, \dots, \zeta_{\sigma(n)} \cdot \tau_n)$, where $\zeta_i \in Z^m$. Then we have the following diagram, where all maps are open

$$\begin{tikzcd} Z^m \times \cdots \times Z^m \arrow[r, "q\times\cdots\times q"] \arrow[d, "\pi"'] & Z^m/\Sigma_m \times \cdots \times Z^m/\Sigma_m \arrow[d, "\pi'"] \\ (Z^m)^n/\Sigma_n \int \Sigma_m \arrow[r, dashed] & \mathrm{SP}^n(\mathrm{SP}^m Z)\,. \end{tikzcd}$$

One may easily show that π is compatible with $\pi' \circ (q \times \cdots \times q)$. Therefore, there is a homeomorphism $X^{mn}/\Sigma_n \int \Sigma_m \approx \mathrm{SP}^n(\mathrm{SP}^m Z)$ and hence one has a canonical quotient map $\rho : \mathrm{SP}^n(\mathrm{SP}^m Z) \longrightarrow \mathrm{SP}^{mn} Z$. Then one can easily verify that $\varphi_{p\circ q} = \rho \circ \mathrm{SP}^n(\varphi_q) \circ \varphi_p : X \longrightarrow \mathrm{SP}^n(\mathrm{SP}^m Z) \xrightarrow{\rho} \mathrm{SP}^{mn} Z$. Thus $\varphi_{p\circ q}$ is continuous.

The homology transfer behaves well with respect to composite ramified covering maps.

10.3.22 **Theorem**. *The following hold:*

$$\tau_{p\circ q} = \tau_q \circ \tau_p : F(X;L) \xrightarrow{\tau_p} F(Y;L) \xrightarrow{\tau_q} F(Z;L)\,;$$

$$t_{p\circ q} = t_q \circ t_p : H_k(X;L) \xrightarrow{t_p} H_k(Y;L) \xrightarrow{t_q} H_k(Z;L)\,.$$

Proof: As before, the second formula follows from the first. So, if $u \in F(X;L)$, $v \in F(Y;L)$, $w \in F(Z;L)$, then $v = \tau_p(u)$ if $v(y) = \mu(y)u(p(y))$, and $w = \tau_q(v)$ if $w(z) = \nu(z)v(q(z))$. Hence $(\tau_q\tau_p(u))(z) = \tau_q(\nu(z)v(q(z)) = \nu(z)\mu(q(z))u(pq(z)) = \xi(z)u((p\circ q)(z)) = t_y p \circ q(u)(z)$.

10.3.23 **Corollary**. *Given an n-fold ramified covering map $p : E \longrightarrow X$ with multiplicity function μ and an integer l, there is an ln-fold ramified covering map $p_l : E \longrightarrow X$ such that $p_l = p$ and $\mu_l(e) = l\mu(e)$, $e \in E$. Then $\tau_{p_l} = l\tau_p : F(X;L) \longrightarrow F(E;L)$ and $t_{p_l} = lt_p : H_k(X;L) \longrightarrow H_k(E;L)$.*

Proof: Consider the l-fold ramified covering map $q : E \longrightarrow E$ such that $q = \mathrm{id}_E$ and $\nu(e) = l$ for all $e \in E$. Hence $p_l = p \circ q$. Then apply Theorem 10.3.22.

10.3.24 Remark. The ln-fold covering map p_l obtained from p is a sort of spurious ramified covering map, since the multiplicity of p is artificially multiplied by l. It is interesting to observacion that the previous result shows that the transfer of this new ramified covering map p_l is just the corresponding multiple of the transfer of the original ramified covering map p. Thus on this sort of artificial ramified covering maps, the transfer remains essentially unchanged.

10.4 The cohomology transfer

In this section we define the cohomology transfer and prove some of its properties.

10.4.1 Definition. Let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function μ, where E and X are compactly generated weak Hausdorff spaces of the same homotopy type of CW-complexes. Define its *cohomology transfer*

$$\tau^p : H^q(E;L) = [E, F(\mathbb{S}^q;L)] \longrightarrow [X, F(\mathbb{S}^q;L)] = H^q(X;L)$$

by $\tau^p([\widetilde{\alpha}]) = [\alpha]$, where $\alpha(x) = \sum_{p(e)=x} \mu(e)\widetilde{\alpha}(e)$, $x \in X$. To see that the map α is continuous and that its homotopy class depends only on the homotopy class of $\widetilde{\alpha}$, observe that α is given by the composite

$$\alpha : X \xrightarrow{\varphi_p} \overset{n}{\mathrm{SP}}\, E \xrightarrow{\mathrm{SP}^n \widetilde{\alpha}} \overset{n}{\mathrm{SP}}\, F(\mathbb{S}^q, G) \longrightarrow F(\mathbb{S}^q, G)\,,$$

where the last map is given by the group structure on $F(\mathbb{S}^q;L)$; namely

$$\langle u_1, \ldots, u_n \rangle \mapsto \sum_{x\in\mathbb{S}^q} u_1(x) + \cdots + \sum_{x\in\mathbb{S}^q} u_n(x)\,.$$

Using the fact that X has the homotopy type of a CW-complex, similar arguments to those used in the proof of 10.3.3 show that α is continuous.

We might write τ_n^p instead of τ^p when we wish to remark the multiplicity n of the ramified covering map p.

10.4.2 REMARK. We might assume that E and X are paracompact spaces instead of compactly generated weak Hausdorff spaces of the same homotopy type of a CW-complex. In this case, the same definition yields a transfer that is a homomorphism between Čech cohomology groups

$$\tau^p : \check{H}^q(E;L) \longrightarrow \check{H}^q(X;L)\,,$$

(see observacion **??**), provided that L is an at most countable coefficient group.

10.4.3 NOTE. In order to define the cohomology transfer, the only property of the Eilenberg-Mac Lane spaces given by $F(\mathbb{S}^q;L)$ required, is the fact that they are (weak) topological abelian groups.

Similarly to the homology transfer, the cohomology transfer has the following fundamental properties.

10.4.4 **Theorem**. *The composite*

$$t_n^p \circ p^* : H^k(X;L) \longrightarrow H^k(X;L)$$

is multiplication by n.

Proof: If $[\alpha] \in [X, F(\mathbb{S}^k;L)]$, then $\tau^p p^*(\alpha) = \tau^p(\alpha \circ p) : X \longrightarrow F(\mathbb{S}^k;L)$, and $\tau^p(\alpha \circ p)(x) = \sum_{p(e)=x} \mu(e)\alpha p(e) = \left(\sum_{p(e)=x} \mu(e)\right)\alpha(x) = n \cdot \alpha(x)$. Thus $\tau^p p^*([\alpha]) = n \cdot [\alpha]$.

10.4.5 **Theorem**. *Let* $p : E \longrightarrow X$ *be an* n*-fold ramified covering map and assume that* $F : Y \longrightarrow X$ *is continuous. Then the following diagram commutes:*

$$\begin{tikzcd}
H^q(E;L) \arrow[r, "\tau^p"] \arrow[d, "\widetilde{F}^*"'] & H^q(X;L) \arrow[d, "F^*"] \\
H^q(F^*(E);L) \arrow[r, "\tau^{F^*(p)}"'] & H^q(Y;L)\,,
\end{tikzcd}$$

where $F^*(p) : F^*(E) \longrightarrow Y$ *is the* n*-fold ramified covering map induced by* $p : E \longrightarrow X$ *over* F.

Proof: Let $\widetilde{\alpha} : E \longrightarrow F(\mathbb{S}^q;L)$ represent an element in $H^q(E;L)$. Then the map

$$y \longmapsto \sum_{F_*(p)(y,e)=y} F^*(\mu)(y,e)\widetilde{\alpha}(y,e) = \sum_{p(e)=F(y)} \mu(e)\widetilde{\alpha}(y,e)$$

that represents $\tau^{F^*(p)}\widetilde{F}^*(\widetilde{\alpha})$, clearly represents also $F^*\tau^p([\widetilde{\alpha}]) \in H^q(Y;L)$.

In 10.4.4 we computed the composite $\tau^p \circ p^*$. The opposite composite $p^* \circ \tau^p$ is also interesting. As it was the case for the homology transfer, an immediate computation yields the following results for the cohomology transfer.

10.4.6 **Proposition**. *Let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function μ. Then the composite*

$$H^q(E;L) \xrightarrow{\tau^p} H^q(X;L) \xrightarrow{p^*} H^q(E;L)$$

is given as follows. Take $[\varphi] \in H^q(E;L) = [E, F(\mathbb{S}^q;L)]$, then $p^*\tau^p[\varphi]$ is represented by the map $\varphi' : E \longrightarrow F(\mathbb{S}^q;L)$ given by

$$\varphi'(e) = \sum_{p(e')=p(e)} \mu(e')e' \, .$$

□

In the case of an action of a finite group Γ on E and $X = E/\Gamma$, we have the following consequence.

10.4.7 **Corollary**. *If $\xi \in H^q(E;L)$, then*

$$p^*\tau^p(\xi) = \sum_{\gamma\in\Gamma} \gamma^*(\xi) \in H^q(E;L), .$$

Proof: Just observe that in the sum the element $\gamma^*(\xi)$ is repeated $\mu(e) = |\Gamma_e|$ times.

Generalizations and further properties of the cohomology transfer are studied in [**?**].

10.5 Some applications of the transfers

First we start considering a standard n-fold covering map $p : E \longrightarrow X$. In this case, the pretransfer (and thus also the transfer in homology) has a particularly nice definition. Since the multiplicity function $\mu : E \longrightarrow \mathbb{N}$ is constant $\mu(e) = 1$, the transfer $\tau_p : F(X;L) \longrightarrow F(E;L)$ is given by

$$t_y p(u)(e) = u(p(e)) \, . \tag{10.5.1}$$

This fact has a nice consequence.

10.5.2 **Theorem**. *Let Γ be a finite group acting freely on a Hausdorff space E. Then the orbit map $p : E \longrightarrow E/\Gamma$ is a standard covering map, and its pretransfer induces an isomorphism*

$$t_y p : F(E/\Gamma;L) \xrightarrow{\cong} F(E;L)^\Gamma \, ,$$

where the second term represents the fixed points under the induced Γ*-action on* $F(E;L)$*. Consequently, the pretransfer yields an isomorphism*

$$H_q(E/\Gamma;L) \xrightarrow{\cong} \pi_q(F(E;L)^\Gamma)\,,$$

for all q*.*

Proof: We assume that the projection $p : E \longrightarrow E/\Gamma$ maps the base point to the base point. The pretransfer τ_p is a monomorphism. Namely, if $\tau_p(u) = 0$, then, by (10.5.1), $u(p(e)) = \tau_p(u)(e) = 0$ for all $e \in E$. Since p is surjective, $u = 0$.

On the other hand, obviously $\tau_p(u) \in F(E;L)^\Gamma$ for all $u \in F(E/\Gamma;L)$. To see that it is an epimorphism, take any $v \in F(E;L)^\Gamma$. Then $v(e) = v(e\gamma)$ for all $\gamma \in \Gamma$, and thus v determines a well-defined element $u \in F(E/\Gamma;L)$ by $u(e\Gamma) = v(e)$. Then clearly $\tau_p(u) = v$.

In what follows, we use the fundamental properties 10.3.9 and 10.3.19, and 10.4.4 and 10.4.7 of both the homology and the cohomology transfers to prove some results about the homology and cohomology of orbit maps between orbit spaces of the action of a topological group Γ and a subgroup Γ' of finite index on a compactly generated weak Hausdorff space of the same homotopy type of a CW-complex (and a corresponding result in Čech cohomology for a paracompact space).

Before starting we need to recall Dold's definition of an n-fold ramified covering map [11]. It is a finite-to-one map $p : E \longrightarrow X$ together with a continuous map $\psi_p : X \longrightarrow \mathrm{SP}^n E$ such that

(i) for every $e \in E$, e appears in the n-tuple $\psi_p(p(e)) = \langle e_1, \dots, e_n\rangle$, and

(ii) $\mathrm{SP}^n(p)\psi_p(x) = \langle x, \dots, x\rangle \in \mathrm{SP}^n X$.

This definition is equivalent to Smith's (see 10.2.1), by setting $\varphi_p = \psi_p$ and defining $\mu(e)$ as the number of times that e is repeated in $\psi_p(p(e))$.

We have the following interesting result.

10.5.3 **Proposition**. *Let* Γ *be a topological group acting on a space* Y *on the right and let* $\Gamma' \subset \Gamma$ *be a subgroup of finite index* n*. Then the orbit map* $p : Y/\Gamma' \longrightarrow Y/\Gamma$ *is an* n*-fold ramified covering map.*

Proof: There is a commutative diagram

$$\begin{tikzcd}
Y \times \Gamma \arrow[r] \arrow[d] & Y \arrow[d] \\
Y \times (\Gamma/\Gamma') \arrow[r, "\nu"'] & Y/\Gamma'\,,
\end{tikzcd}$$

where the top map is the action and the vertical maps are the quotient maps. Take the adjoint map of ν, $\eta : Y \longrightarrow M(\Gamma/\Gamma', Y/\Gamma')$. The function space $M(\Gamma/\Gamma', Y/\Gamma')$ has a right Γ-action given as follows. For $f : \Gamma/\Gamma' \longrightarrow Y/\Gamma$, take $(f \cdot \gamma)[\gamma_1] = f(\gamma[\gamma_1]) = f[\gamma\gamma_1]$. The map η is then Γ-equivariant and thus induces a map

$$\overline{\eta} : Y/\Gamma \longrightarrow M(\Gamma/\Gamma', Y/\Gamma')/\Gamma\,.$$

On the other hand, if we identify Γ/Γ' with the set $\underline{n} = \{1, \ldots, n\}$, then we have a homeomorphism

$$M(\Gamma/\Gamma', Y/\Gamma')/\Gamma \approx M(\underline{n}, Y/\Gamma')/\Sigma_n = \overset{n}{\mathrm{SP}}(Y/\Gamma')\,.$$

Let $\psi_p : Y/\Gamma \longrightarrow \mathrm{SP}^n(Y/\Gamma')$ be $\overline{\eta}$ followed by the previous homeomorphism. Then ψ_p satisfies conditions (i) and (ii) and thus p is an n-fold ramified covering map.

We apply the results 10.3.10 and 10.3.17 that we have for the pretransfer to the n-fold ramified covering described above to obtain the following.

10.5.4 **Proposition**. *Let Y be a space with an action of a topological group Γ and let $\Gamma' \subset \Gamma$ be a subgroup of finite index n. Assume that R is a ring where the integer n is invertible. Then $p_* : F(Y/\Gamma', R) \longrightarrow F(Y/\Gamma, R)$ is a split (continuous) epimorphism. Moreover, if Γ is finite and its order m is invertible in R, then the kernel of p_* is the complement in $F(Y/\Gamma', R)$ of the invariant subgroup $F(Y/\Gamma', R)^\Gamma$ under the induced action of Γ. Thus in this case*

$$F(Y/\Gamma, R) \cong F(Y/\Gamma', R)^\Gamma\,;$$

in particular, if Γ is finite and Γ' is trivial, then $m = n$ and

$$F(Y/\Gamma, R) \cong F(Y, R)^\Gamma\,.$$

Proof: By 10.3.10 applied to the n-fold ramified covering $p : Y/\Gamma' \longrightarrow Y/\Gamma$, $p_* \circ t_Y p : F(Y/\Gamma', R) \longrightarrow F(Y/\Gamma', R)$ is multiplication by n, hence it is an isomorphism, and consequently p_* is a split epimorphism. Moreover, if Γ is finite of order m, by 10.3.17, we have that $\tau_p \circ p_* : F(Y/\Gamma', R)^\Gamma \longrightarrow F(Y/\Gamma', R)^\Gamma$ is multiplication by m. So, if m is invertible in R, then $p_* : F(Y/\Gamma', R)^\Gamma \longrightarrow F(Y/\Gamma, R)$ is an isomorphism.

As an immediate consequence of the result above, or applying 10.3.9 and 10.3.19, we obtain the following two well-known results (cf. [44, 2.5], [**?**], [9]).

10.5.5 **Theorem**. *Let Y be a space with an action of a topological group Γ and let $\Gamma' \subset \Gamma$ be a subgroup of finite index n. Assume that R is a ring where the integer n is invertible. Then $p_* : H_q(Y/\Gamma'; R) \longrightarrow H_q(Y/\Gamma; R)$ is a split epimorphism.*

Moreover, if Γ *is finite and its order* m *is invertible in* R*, then the kernel of* p_* *is the complement of* $H_q(Y/\Gamma'; R)^\Gamma$ *in* $H_q(Y/\Gamma'; R)$*. Thus in this case*

$$H_q(Y/\Gamma; R) \cong H_q(Y/\Gamma'; R)^\Gamma ;$$

and in particular,

$$H_q(Y/\Gamma; R) \cong H_q(Y; R)^\Gamma .$$

□

Similarly, 10.4.4 and 10.4.7 one has for cohomology the following.

10.5.6 **Theorem**. *Let* Y *be a space with an action of a topological group* Γ *and let* $\Gamma' \subset \Gamma$ *be a subgroup of finite index* n*. Assume that* R *is a ring where the integer* n *is invertible. Then* $p^* : H^q(Y/\Gamma; R) \longrightarrow H^q(Y/\Gamma'; R)$ *is a split monomorphism. Moreover, if* Γ *is finite and its order* m *is invertible in* R*, then the image of* p^* *is* $H^q(X; R)^\Gamma$*. Thus in this case*

$$H^q(Y/\Gamma; R) \cong H^q(Y/\Gamma'; R)^\Gamma ;$$

and in particular,

$$H^q(Y/\Gamma; R) \cong H^q(Y; R)^\Gamma .$$

□

10.5.7 Remark. One may take a paracompact space Y with an action of a topological group Γ and obtain for Čech cohomology an analogous result, namely $p^* : \check{H}^q(Y/\Gamma; R) \longrightarrow \check{H}^q(Y/\Gamma'; R)$ is a split monomorphism, and

$$\check{H}^q(Y/\Gamma; R) \cong \check{H}^q(Y/\Gamma'; R)^\Gamma .$$

A nice application of the previous ideas is the following generalization of a well-known result of Grothendieck [**?**] (in the case $Y = E\Gamma$).

10.5.8 **Theorem**. *Let* Γ *be a compact Lie group and let* Γ_1 *be the component of* $1 \in \Gamma$*. Let* R *be a ring where* $n = [\Gamma, \Gamma_1]$ *is an invertible element. For an action of* Γ *on a topological space* Y*, one has*

$$H_q(Y/\Gamma; R) \cong H_q(Y/\Gamma_1; R)^{\Gamma/\Gamma_1},$$

$$H^q(Y/\Gamma; R) \cong H^q(Y/\Gamma_1; R)^{\Gamma/\Gamma_1},$$

$$\check{H}^q(Y/\Gamma; R) \cong \check{H}^q(Y/\Gamma_1; R)^{\Gamma/\Gamma_1},$$

the last two according to what kind of a space Y *is.* □

10.6 Duality between the homology and cohomology transfers

In this section we compare the homology transfer with the cohomology transfer.

Given an n-fold ramified cover $p : E \longrightarrow X$ with multiplicity function $\mu : E \longrightarrow \mathbb{N}$, we can extend it to the n-fold ramified covering map $p^+ : E^+ \longrightarrow X^+$ as explained in observacion 10.2.2. Consider the cohomology transfer

$$\tau^p : H^q(E;G) = \widetilde{H}^q(E^+;G) \longrightarrow \widetilde{H}^q(X^+;G) = H^q(X;G)\,,$$

and consider also the homology transfer

$$t_p : H_q(X;G) = \widetilde{H}_q(X^+;G) \longrightarrow \widetilde{H}_q(E^+;G) = H_q(E;G)$$

as given in Definition 10.3.1.

10.6.1 **Theorem**. *Let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function $\mu : E \longrightarrow \mathbb{N}$ and E path connected, and let $t_p : H_q(X;R) \longrightarrow H_q(E;R)$ and $\tau^p : H^q(E;R) \longrightarrow H^q(X;R)$ be its homology and cohomolgy transfers. If $\xi \in H_q(X;G)$ and $\widetilde{\xi} \in H^q(E;G)$, then*

$$\langle t_p(\xi), \widetilde{\xi}\rangle_E = \langle \xi, \tau^p(\widetilde{\xi})\rangle_X \in R\,,$$

*for the Kronecker products for E and X, respectively, and R a commutative ring with 1 (see (**??**)).*

Proof: We have to prove the commutativity of the following diagram:

$$\begin{tikzcd}
{[X^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^q, F(X^+, R)]_*} \arrow[r, "\frown"] & {[X^+, F(\mathbb{S}^0, R)]_*} \arrow[dr] & \\
{[E^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^q, F(X^+, R)]_*} \arrow[u, "\tau^p \times 1"] \arrow[d, "1 \times \tau_p"'] & & R \\
{[E^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^q, F(E^+, R)]_*} \arrow[r, "\frown"'] & {[E^+, F(\mathbb{S}^0, R)]_*} \arrow[ur] &
\end{tikzcd}$$

By the naturality of the construction of the pretransfers and the definition of the $\frown$-product (see Proposition **??**), it is fairly easy to check that this commutativity follows from the commutativity of the following:

$$\begin{tikzcd}
{[X^+, F(X^+, R)]_*} \arrow[r] & {[\mathbb{S}^0, F(X^+, R)]_*} \arrow[dr] & \\
{[E^+, F(X^+, R)]_*} \arrow[u, "\tau^p"] \arrow[d, "\tau_p"'] & & R \\
{[E^+, F(E^+, R)]_*} \arrow[r] & {[\mathbb{S}^0, F(E^+, R)]_*} \arrow[ur] &
\end{tikzcd}$$

Let $\delta : E^+ \longrightarrow F(X^+, R)$ be given by $\delta(e) = \sum_{i=1}^{m(e)} r_i(e)x_i(e)$, $e \in E$. Chasing this element δ along the top of the diagram, one easily verifies that it maps to the element

$$d = \sum_{p(e)=x_{-1}} \mu(e) \sum_{i=1}^{m(e)} r_i(e)\,,$$

while chasing it along the bottom of the diagram, it maps to the element

$$d' = \sum_{i=1}^{m(e_{-1})} r_i(e_{-1}) \sum_{p(e_i)=x_i(e_{-1})} \mu(e_i) = n \sum_{i=1}^{m(e_{-1})} r_i(e_{-1})\,.$$

Call $\rho(e) = \sum_{i=1}^{m(e)} r_i(e)$. Since $\rho = \varepsilon \circ \delta$, by 9.1.1 this defines a continuous map $\rho : E \longrightarrow R$, but since E is path connected and R is discrete, ρ is constant with value $r_\delta \in R$. Hence

$$d = \sum_{p(e)=x_{-1}} \mu(e)\rho(e) = n \cdot r_\delta \quad \text{and} \quad d' = n\rho(e_{-1}) = n \cdot r_\delta\,.$$

Thus $d = d'$ and the diagram commutes.

For simplicity, in what follows we omit the coefficient ring R in homology and cohomology. For the Kronecker product $\langle -,- \rangle_Y : H^q(Y) \otimes H_q(Y) \longrightarrow R$ there are induced homomorphisms $\Phi_Y : H^q(Y) \longrightarrow \mathrm{Hom}(H_q(Y), R)$ and $\Psi_Y : H_q(Y) \longrightarrow \mathrm{Hom}(H^q(Y), R)$ for every space Y, given by $\Phi(y)(\eta) = \langle y, \eta \rangle_Y$ and $\Psi(\eta)(y) = \langle y, \eta \rangle_Y$, $y \in H^q(Y)$, $\eta \in H_q(Y)$.

10.6.2 **Corollary**. *The following diagrams commute*

$$\begin{tikzcd} H^q(E) \arrow[r, "\Phi_E"] \arrow[d, "\tau^p"'] & \mathrm{Hom}(H_q(E), R) \arrow[d, "\mathrm{Hom}(\tau_p,1)"] \\ H^q(X) \arrow[r, "\Phi_X"'] & \mathrm{Hom}(H_q(X), R)\,, \end{tikzcd} \qquad \begin{tikzcd} H_q(X) \arrow[r, "\Psi_X"] \arrow[d, "\tau_p"'] & \mathrm{Hom}(H^q(X), R) \arrow[d, "\mathrm{Hom}(\tau^p,1)"] \\ H_q(E) \arrow[r, "\Psi_E"'] & \mathrm{Hom}(H^q(E), R)\,, \end{tikzcd}$$

the one on the right-hand side only if $\tau^p : H^q(E) \longrightarrow H^q(X)$ is a homomorphism (which is rather seldom the case). □

10.6.3 Remark. Under suitable conditions Φ or Ψ are isomorphisms, in whose case one of the transfers determines the other.

10.7 Comparison with Smith's transfer

In this section we show that the transfer defined in [44] coincides with ours if we take $\mathbb{Z}$-coefficients. To that end, we first recall his definition of the transfer. It makes use of a result of Moore, that we state below. Recall that the *weak product*

$\widetilde{\prod}_{n=1}^{\infty} X_n$ of a family of pointed spaces is the colimit over n of the directed system of spaces

$$X_1 \hookrightarrow X_1 \times X_2 \hookrightarrow X_1 \times X_2 \times X_3 \hookrightarrow \cdots ,$$

where the inclusions are given by letting the last coordinate be the base point. Moore's result, as it appears in [**?**], is as follows.

10.7.1 **Theorem**. (Moore) *A connected space X is weakly homotopy equivalent to the weak product $\widetilde{\prod}_{n\geq 1} K(\pi_n(X), n)$ of Eilenberg-Mac Lane spaces if and only if the Hurewicz homomorphism $h_n : \pi_n(X) \longrightarrow \widetilde{H}_n(X;\mathbb{Z})$ is a split monomorphism for all $n \geq 1$.* □

Suppose that $\rho_n : \widetilde{H}_n(X) = \widetilde{H}_n(X;\mathbb{Z}) \longrightarrow \pi_n(X)$ is a left inverse of h_n. The Kronecker product defined in Section 2 determines an epimorphism

$$\widetilde{H}^n(X;\pi_n(X)) \longrightarrow \mathrm{Hom}(\widetilde{H}_n(X), \pi_n(X)) .$$

Let $[\xi_n] \in \widetilde{H}^n(X;\pi_n(X)) = [X, K(\pi_n(X), n)]_*$ be some preimage of ρ_n. Then the family of pointed maps (ξ_n) defines the weak homotopy equivalence of the previous theorem.

10.7.2 **Corollary**. *If X is a connected topological abelian monoid of the same homotopy type of a* CW*-complex, then there is a homotopy equivalence $X \longrightarrow \widetilde{\prod}_{n\geq 1} K(\pi_n(X), n)$.*

Proof: Since X is a topological abelian monoid, there is a retraction $r : F(X;\mathbb{N}) \longrightarrow X$ given by the retractions

$$r_n : F_n(X;\mathbb{N}) \longrightarrow X , \qquad r_n(x_1 + x_2 + \cdots + x_n) = x_1 + x_2 + \cdots + x_n .$$

Recall, on the other hand, that by the Dold-Thom theorem one has an isomorphism $\pi_n(\mathrm{SP}^\infty X) \cong \widetilde{H}_n(X)$, so that the inclusion $i : X \hookrightarrow \mathrm{SP}^\infty X$ defines the Hurewicz homomorphism (see [**?**]). Since $r \circ i = \mathrm{id}_X$, the homomorphism $\rho_n = r_* : \widetilde{H}_n(X) = \pi_n(F(X;\mathbb{N})) \longrightarrow \pi_n(X)$ provides a left inverse of the Hurewicz homomorphism h_n. Hence, by Moore's theorem, we obtain the result.

10.7.3 Remark. Note that in the proof above, it is enough to assume that X is a *weak* topological abelian monoid, i.e., that the product in X is continuous on compact sets.

For any space E, the space $F(E;\mathbb{N})$ is a weak topological abelian monoid. Thus we have the following.

10.7.4 **Corollary**. *For a connected space E of the same homotopy type of a* CW*-complex, there is a natural homotopy equivalence $w_E : F(E;\mathbb{N}) \longrightarrow K(\widetilde{H}_*(E)) = \widetilde{\prod}_{n=1}^{\infty} K(\widetilde{H}_n(E), n)$.* □

The definition of Smith's transfer is as follows. Given an n-fold ramified cover $p : E \longrightarrow X$ with multiplicity function $\mu : E \longrightarrow \mathbb{N}$, consider the following composite:

$$\widehat{p} : X \xrightarrow{\varphi_p} F_n(E;\mathbb{N}) \longrightarrow F(E;\mathbb{N}) \xrightarrow{\simeq} K(\widetilde{H}_*(E))\,.$$

This map defines a family of elements $[\widehat{p}] \in \widetilde{H}^*(X;\widetilde{H}_*(E))$. On the other hand, the Kronecker product determines a homomorphism

$$\psi : \widetilde{H}^*(X;\widetilde{H}_*(E)) \longrightarrow \mathrm{Hom}(\widetilde{H}_*(X),\widetilde{H}_*(E))\,.$$

Smith's transfer is the image $p_\sharp : \widetilde{H}_*(X) \longrightarrow \widetilde{H}_*(E)$ of $[\widehat{p}]$ under the homomorphism ψ.

10.7.5 **Theorem**. *Let $p : E \longrightarrow X$ be an n-fold ramified cover with multiplicity function $\mu : E \longrightarrow \mathbb{N}$. Then $p_\sharp = t_p : \widetilde{H}_*(X;\mathbb{Z}) \longrightarrow \widetilde{H}_*(E;\mathbb{Z})$, where t_p is the transfer in reduced homology.*

Proof: Consider the following commutative diagram.

$$\begin{tikzcd}
{[E,F_(E;\mathbb{N})]_*} \arrow[r] \arrow[d, "\tau^p"'] & {[E,F(E;\mathbb{N})]_*} \arrow[r, "\cong"] \arrow[d, "\tau^p"'] & \widetilde{H}^*(E,\widetilde{H}_*(E)) \arrow[r] \arrow[d, "\tau^p"'] & \mathrm{Hom}(\widetilde{H}_*(E),\widetilde{H}_*(E)) \arrow[d, "\mathrm{Hom}(t_p,1)"] \\
{[X,F_n(E;\mathbb{N})]_*} \arrow[r] & {[X,F(E;\mathbb{N})]_*} \arrow[r, "\cong"] & \widetilde{H}^*(X,\widetilde{H}_*(E)) \arrow[r] & \mathrm{Hom}(\widetilde{H}_*(X),\widetilde{H}_*(E))
\end{tikzcd}$$

The two squares on the left-hand side, where τ^p represents the cohomology transfer, commute obviously. The one on the right-hand side commutes by Corollary 10.6.2. Take $[i] \in [E,F_n(E;\mathbb{N})]_*$, where $i : E \hookrightarrow F_n(E;\mathbb{N})$ is the canonical inclusion. Chasing $[i]$ down and then right on the bottom of the diagram, we obtain $p_\sharp$, while chasing it to the right on the top of the diagram and then down, we obtain t_p. This is true, because the image of $[i]$ along the top row of the diagram is the identity homomorphism $1 \in \mathrm{Hom}(\widetilde{H}_*(E),\widetilde{H}_*(E))$. This follows from the naturality of the Kronecker product, since by Corollary 10.7.2, we have an explicit description of the weak homotopy equivalence that defines the isomorphism in the middle arrow.

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