Fiber bundles

Marcelo A. Aguilar* & Carlos Prieto*

* Instituto de Matemáticas, UNAM

2010

Date of version: May 2, 2012

Contents

Preface

The purpose of this book is the study of fiber bundles. The concept of a fiber bundle is one of those ubiquitous concepts in mathematics. Its first appearance was probably in the thirties of the last century within the study of the topology and geometry of manifolds. However it was not until the publication of Norman Steenrod's book [15] in 1951 that a systematic treatment of the concept was given. In the meantime some other books –not many– on the subject have appeared. Worthy to be mentioned is Sir Michael Atiyah's book [**?**] on K-Theory, where special fiber bundles are studied, namely the vector bundles, which constitute the basis for defining K-theory.

We start this book in a very general setup, where we define as *fibration* just a continuous map $p : E \longrightarrow B$. Thereon we begin to put some requirements to p and according to those, we put an adjective like *Serre fibration* for those maps p which have the homotopy lifting property for cubes, or *Hurewicz fibration* if the maps p have the homotopy lifting property for all spaces. We also have the *locally trivial fibrations*, which are always Serre fibrations. They are even Hurewicz fibrations whenever the base space B is paracompact. A special case are the covering maps, which are locally trivial fibrations whose fibers are discrete spaces.

This book was inspired in the notes of a course given by Dieter Puppe in Heidelberg some time in the seventies to whom we are deeply grateful. The influence of Albrecht Dold is also present.

Mexico City, Mexico — The authors[1]
Winter 2011-12

[1]M. Aguilar and C. Prieto were supported by PAPIIT-UNAM grants IN 101909 and IN 108712.

Chapter 1

Homotopy Theory of Fibrations

1.1 Introduction

If one wishes to study topological spaces, one way of doing it is the following. One may take a cell decomposition (or using cells one constructs a new space) and one tries to reduce its topological properties to algebraic or combinatorial relationships between the boundaries of the cells, for instance, one can construct simplicial complexes or apply a homology theory.

A second possibility can be illustrated by the following algebraic situation. One may take an exact sequence

$$O \longrightarrow F \xrightarrow{\iota} E \longrightarrow B \longrightarrow O$$

(of groups, say) and ask what possible values of E one can take for given F and B (for example, $E = F \times B$ is always possible).

It is a useful idea to compare this question with the following topological situations. The general setup will be as follows. Let $p : E \longrightarrow B$ be any continuous map. The inverse images $p^{-1}(b)$ of points b in B constitute a "decomposition" of E into "fibers" $p^{-1}(b)$. We get closer to the algebraic situation described above if all fibers were homeomorphic to each other as it will be the case in the following examples. The maps $p : E \longrightarrow B$, that we shall be dealing with will be generically called *fibrations*, without any conditions. Later on, according to their particular (lifting) properties, they will be qualified with a special name, such as *trivial fibration*, *Serre fibration*, *Hurewicz fibration*, *locally trivial fibration*, and so on.

1.1.1 Examples. The following should be *fibrations*.

(a) The *topological product* defined as follows. Let B and F be topological spaces and take the projection

$$p = \mathrm{proj}_1 : E = B \times F \longrightarrow B\,.$$

This should be a fibration for any definition, namely, the *trivial fibration* or the *product fibration*.

(b) The *Moebius strip* defined as follows. Let E be obtained from the square $I \times I$ by identifying for every $t \in I$ the pair $(0, t)$ with $(1, 1-t)$. B is obtained from I by identifying the end points of the interval. B is thus homeomorphic to $\mathbb{S}^1$. The mapping $(s, t) \mapsto s$ determines a continuous map $p : E \longrightarrow B$. Then $p^{-1}(s) \approx I$ for every $s \in B$ (see Figure 1.1).

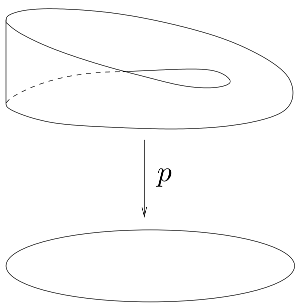


Figure 1.1

The space E is not homeomorphic to $\mathbb{S}^1 \times I$ since the boundary of $\mathbb{S}^1 \times I$ consists of two circles, i.e., it is not connected, but the boundary of E is a circle, i.e., it is connected. By means of

$$\mathrm{Bd}(M) = \{x \in M \mid H_2(M, M - x) = 0\}$$

one can define the boundary of $M = \mathbb{S}^1 \times I$, resp. $M = E$ in a topologically invariant way.

(c) The *Klein bottle* defined as follows. Let E be obtained from $I \times I$ by identifying for every $t \in I$ the pair $(0, t)$ with $(1, 1-t)$ and for every $s \in I$ the pair $(s, 0)$ with $(s, 1)$. Let $B = \mathbb{S}^1$ be obtained again as in (b) and $p : E \longrightarrow B$ be induced again by $(s, t) \mapsto s$, then $p^{-1}(s) \approx \mathbb{S}^1$ for every $s \in B$ (see Figure 1.2).

The space E is not homeomorphic to the *torus* $\mathbb{S}^1 \times \mathbb{S}^1$. As a proof of this fact we compute the homology of E using the cell decomposition shown in Figure 1.3.

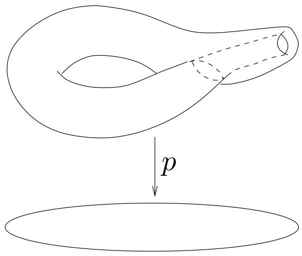


Figure 1.2

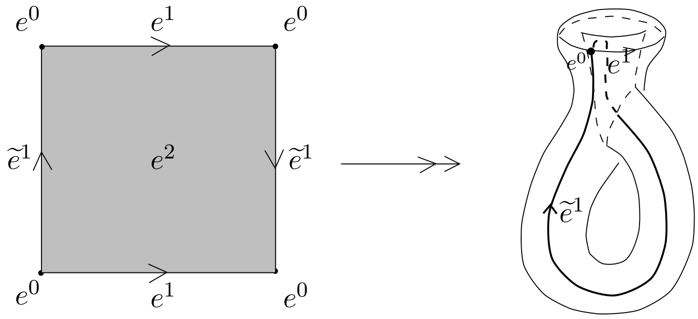


Figure 1.3

It consists of one 0-cell e^0, two 1-cells e^1 and $\widetilde{e}^1$ and one 2-cell e^2. In the cellular chain complex one has the following:

$$\begin{aligned}
\partial e^2 &= 2\widetilde{e}^1 \\
\partial e^1 &= \partial\widetilde{e}^1 \; = \; 0 \\
\partial e^0 &= 0\,,
\end{aligned}$$

from which we obtain

$$H_2(E) = 0\,, \qquad H_1(E) \cong \mathbb{Z} \oplus \mathbb{Z}_2\,.$$

Similarly, one obtains for the torus $\mathbb{S}^1 \times \mathbb{S}^1$

$$H_2(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{Z}\,, \qquad H_1(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}\,,$$

(see [1, 7.3.12]). Since the first and second homology groups of both spaces are different, they cannot be homeomorphic.

(d) The *covering maps*, of which a particularly important example will be the following. Take

$$\begin{aligned}
p : \mathbb{R} &\longrightarrow \mathbb{S}^1 \subset \mathbb{C}\,, \\
x &\longmapsto e^{2\pi \mathrm{i} x}\,,
\end{aligned}$$

(cf. Section 1.2). The fibers $p^{-1}(s)$, $s \in \mathbb{S}^1$, are homeomorphic to $\mathbb{Z}$ (as a set with the discrete topology). One has $\mathbb{R} \not\approx \mathbb{S}^1 \times \mathbb{Z}$ since $\mathbb{R}$ is connected while $\mathbb{S}^1 \times \mathbb{Z}$ has infinitely many components (see Figure 1.4).

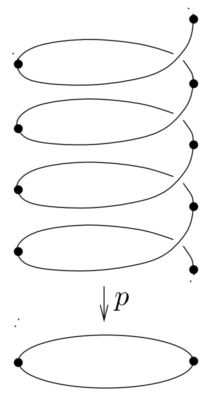


Figure 1.4

(e) The *tangent bundle of a smooth manifold*, of which a concrete example is the tangent bundle $T(\mathbb{S}^n)$ of the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ (cf. 1.6.6 (e)). Let

$$T(\mathbb{S}^n) = \big\{(x, v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} \mid x \perp v\big\}$$

be furnished with the relative topology, and take

$$\begin{aligned} p : T(\mathbb{S}^n) &\longrightarrow \mathbb{S}^n\,, \\ (x, v) &\longmapsto x\,. \end{aligned}$$

Consider the restriction of p to

$$T'(\mathbb{S}^n) = \{(x, v) \in T(\mathbb{S}^n) \mid v \neq 0\}\ .$$

The following is an interesting question: Does there exist a continuous map $s : \mathbb{S}^n \longrightarrow T'(\mathbb{S}^n)$ such that $p \circ s = \mathrm{id}_{\mathbb{S}^n}$? One such s is called a *section* of p. Geometrically, s can be described as a nonvanishing *continuous vector field* on $\mathbb{S}^n$.

1.1.2 Exercise. Prove that the fibrations of (b) and (c) have a section and that, on the contrary, that of (d) does not.

All examples introduced in 1.1.1 are going to be fibrations in a sense that we still have to state precisely. On the contrary, the following will not be one, even though all of its fibers are homeomorphic.

1.1.3 Example. Consider the space $E = I \times I \cup \{0\} \times [0,2] \subset \mathbb{R}^2$, the space $B = I \subset \mathbb{R}$, and $p : E \longrightarrow B$ such that $p : (s,t) \mapsto s$, as depicted in Figure 1.5.

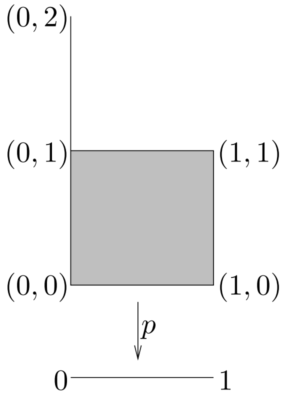


Figure 1.5

The map p has the following property: Not for every path $\omega : I \longrightarrow B$ and for every point $x_0 \in E$, such that $\omega(0) = p(x_0)$, there exists another path $\widetilde{\omega} : I \longrightarrow E$ such that $\widetilde{\omega}(0) = x_0$ and $p \circ \widetilde{\omega} = \omega$; i.e., not for every path in B, there exists a "lifting" to E with a given origin. For instance, if $x_0 = (0,2)$, there does not exist $\widetilde{\omega}$ unless ω is constant in a neighborhood of 0. (It is an *exercise* to prove this fact.) See Section 1.4 for a general treatment of this question.

1.2 General Definitions

In this section we present the general set up on which the rest of this book is supported.

1.2.1 Definition. (For the time being) we shall call *fibration* any continuous map $p : E \longrightarrow B$. E will be called the *total space* and B the *base space* of the fibration. Moreover, $p^{-1}(b)$ will be called the *fiber* over b, $(b \in B)$.

1.2.2 Definition. Let p and p' be fibrations. A pair of maps $(f, \overline{f})$ is called a *fiber map* from p to p' if the diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "\overline{f}"'] & B' \end{tikzcd}$$

is commutative. We denote this by $(f,\overline{f}) : p \longrightarrow p'$. In case that $B = B'$ and $\overline{f} = \mathrm{id}_B$ we call f a *fiber map over* B.

The commutativity of the diagram means that f maps the fiber over b into the fiber over $\overline{f}(b)$. If now f has the property of mapping fibers into fibers, then there is a function $\overline{f} : B \longrightarrow B'$ that makes the diagram commutative. If p is surjective then the function $\overline{f}$ is well defined by f. If, moreover, p is an identification, then $\overline{f}$ is continuous.

From definition 1.2.2 one may conclude the following.

(1) $(\mathrm{id}_E, \mathrm{id}_B) : p \longrightarrow p$ is a fiber map.

(2) If $(f,\overline{f}) : p \longrightarrow p'$ and $(g,\overline{g}) : p' \longrightarrow p''$ are fiber maps, then

$$(g,\overline{g}) \circ (f,\overline{f}) = (g \circ f, \overline{g} \circ \overline{f}) : p \longrightarrow p''$$

is also one.

This means that *there is a category, whose objects are fibrations, whose morphisms are fiber maps, and the identity morphisms and the compositions are given by* (1) *and* (2).

1.2.3 DEFINITION. $(g,\overline{g}) : p' \longrightarrow p$ is an *inverse* of $(f,\overline{f}) : p \longrightarrow p'$ if

$$\begin{aligned}(g,\overline{g}) \circ (f,\overline{f}) &= \mathrm{id}_p = (\mathrm{id}_E, \mathrm{id}_B)\\ (f,\overline{f}) \circ (g,\overline{g}) &= \mathrm{id}_{p'} = (\mathrm{id}_{E'}, \mathrm{id}_{B'})\,.\end{aligned}$$

$(f,\overline{f})$ is a *fiber equivalence* if it has an inverse. Two fibrations p, p' are said to be *equivalent* if there is a fiber equivalence between them. They are called *equivalent over* B if there is an equivalence of the form $(f, \mathrm{id}_B) : p \longrightarrow p'$.

If $(f,\overline{f})$ is a fiber map and f and $\overline{f}$ are homeomorphisms, then $(f,\overline{f})$ is an equivalence with inverse $(f^{-1}, \overline{f}^{-1})$.

1.2.4 DEFINITION. A fibration p is said to be *trivial* if it is equivalent to a product fibration (see 1.1.1(a)), that is, if we have a commutative diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & B' \times F \arrow[d, "\mathrm{proj}_{B'}"] \\ B \arrow[r, "\overline{f}"'] & B' \end{tikzcd}$$

In this case p is equivalent over B to the product fibration $\mathrm{proj}_B : B \times F \longrightarrow B$, namely, by means of the fiber equivalence

$$\left(\left(\overline{f}^{-1} \times \mathrm{id}_F\right) \circ f, \mathrm{id}_B\right) : p \longrightarrow \mathrm{proj}_B \,.$$

On the other hand, one cannot say in general that two equivalent fibrations with the same base space B are equivalent over B. For instance, the fibrations illustrated in Figure 1.6 are equivalent, but they are not equivalent over $B = \{0, 1\}$.

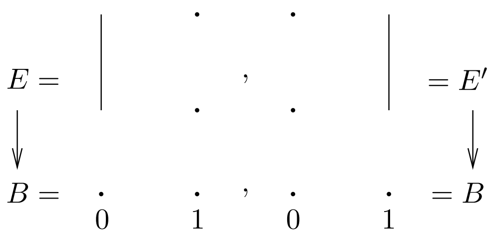


Figure 1.6

1.2.5 Definition. Let $p : B \longrightarrow E$ be a fibration and $A \subset B$. Then

$$p_A = p|_{p^{-1}(A)} : E_A = p^{-1}(A) \longrightarrow A$$

is called the *restriction* of the fibration p to A.

1.2.6 Exercise. Prove that if $p : E \longrightarrow B$ is trivial, then also $p_A : E_A \longrightarrow A$ is trivial.

1.2.7 Definition. A fibration p is *locally trivial* if every point $b \in B$ has a neighborhood U such that p_U es trivial.

1.2.8 **Theorem**. *Let $p : E \longrightarrow B$ be a locally trivial fibration. If B is connected, then all fibers of p are homeomorphic.*

Proof: In a trivial fibration, clarly all fibers are homeomorphic. Let $b_0 \in B$ be any point. Then the set

$$B_0 = \left\{b \in B \mid p^{-1}(b) \approx p^{-1}(b_0)\right\}$$

is open in E, namely let $b \in B_0$ and U be a neighborhood of b in B such that p is trivial over U. Then all fibers over U are homeomorphic and so $U \subset B_0$. Similarly one proves that $B - B_0$ is open in B. Since $B_0 \neq \emptyset$ and B is connected, then $B = B_0$. □

1.2.9 EXAMPLES. The following are locally trivial fibrations.

(a) The *Moebius strip fibration* $p : E \longrightarrow \mathbb{S}^1$ is not trivial, but it is locally trivial. If it were trivial, then there would be a space F and a homeomorphism f such that the diagram

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{S}^1 \times F \arrow[dl, "\mathrm{proj}_{\mathbb{S}^1}"] \\ & \mathbb{S}^1 & \end{tikzcd}$$

would be commutative. This implies that f induces a homeomorphism of the fibers, and so $F \approx I$; but we already saw that $E \not\approx \mathbb{S}^1 \times I$.

The fibration is locally trivial since $\mathbb{S}^1$ can be obtained from I by identifying the end points. We shall denote the points of $\mathbb{S}^1$, resp. E, by their inverse images in I, resp. $I \times I$. The sets $U = \mathbb{S}^1 - \{0\}$ and $V = \mathbb{S}^1 - \{\frac{1}{2}\}$ are open in $\mathbb{S}^1$ and the maps

$$\begin{aligned} \varphi : U \times I &\longrightarrow p^{-1}(U)\,, \\ (u,t) &\longmapsto (u,t)\,, \\ \psi : V \times I &\longrightarrow p^{-1}(V)\,, \\ (v,t) &\longmapsto \begin{cases} (v,t) & \text{if } v < \frac{1}{2}, \\ (v,1-t) & \text{if } v > \frac{1}{2}, \end{cases} \end{aligned}$$

are well defined and describe the local triaviality of p (see Section 1.2.7 and Figure 1.7).

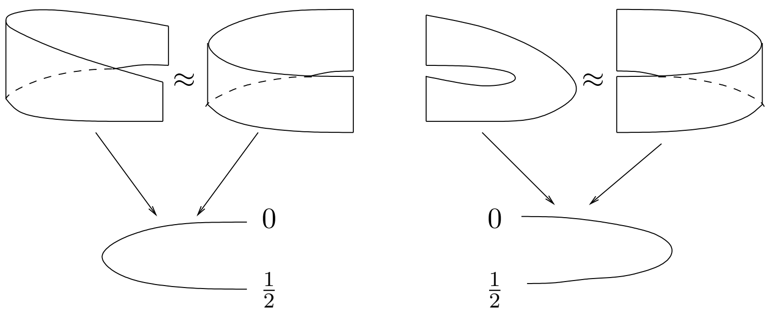


Figure 1.7

(b) In a similarly simple way one can see that the *Klein bottle fibration* and the *exponential fibration* $\mathbb{R} \longrightarrow \mathbb{S}^1$ are locally trivial but not trivial.

(c) On the contrary, example 1.1.3 is not locally trivial (*exercise*).

1.3 FURTHER EXAMPLES

There are important examples of fibrations, some of which we present in this section.

1.3.1 EXAMPLES. Further examples are the following.

(a) The *fibrations over projective spaces* defined as follows. Let $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ [1] and let $d = 1$, 2 or 4. Take the following fibrations:

$$\begin{tikzcd}
\mathbb{F}^{n+1} - \{0\} \arrow[dr, "p"'] & & \mathbb{S}^{d(n+1)-1} \arrow[ll, hook'] \arrow[dl, "p'"] \\
& \mathbb{F}\mathbb{P}^n\,, &
\end{tikzcd}$$

where p is the identification with respect to the equivalence relation $(x_0, x_1, \ldots, x_n) \sim (x_0', x_1', \ldots, x_n')$ (in $\mathbb{F}^{n+1} - \{0\}$) if and only if there exists $\lambda \in \mathbb{F}$ such that $(x_0, x_1, \ldots, x_n) = \lambda(x_0', x_1', \ldots, x_n')$. p' is the restriction to

$$\mathbb{S}^{d(n+1)-1} = \left\{x \in \mathbb{F}^{n+1} - \{0\} \mid \|x\|^2 = x_0^2 + \cdots + x_n^2 = 1\right\}\,.$$

$\mathbb{F}\mathbb{P}^n$ is the *real, complex* or *quaternionic projective space* of dimension n. One can prove that $\mathbb{F}\mathbb{P}^n$ is a dn-dimensional manifold.

The fibrations p and p' are locally trivial. Namely, let $V_i = \{(x_0, x_1, \ldots, x_n) \in \mathbb{F}^{n+1} - \{0\} \mid x_i \neq 0\}$ and let $U_i = pV_i$. One has that $V_i = p^{-1}(U_i)$, that is, the sets U_i constitute an open cover of $\mathbb{F}\mathbb{P}^n$. We shall prove that p and p' are trivial over U_i. To see this, we have to define homeomorphisms h_i and k_i that make the following diagrams commute.

$$\begin{tikzcd}
V_i \arrow[rr, "h_i"] \arrow[dr, "p_{U_i}"'] & & U_i \times (\mathbb{F} - \{0\})\,, \arrow[dl, "\mathrm{proj}_{U_i}"] \\
& U_i &
\end{tikzcd}
\qquad
\begin{tikzcd}
V_i \cap \mathbb{S}^{d(n+1)-1} \arrow[rr, "k_i"] \arrow[dr, "p'_{U_i}"'] & & U_i \times \mathbb{S}^{d-1} \arrow[dl, "\mathrm{proj}_{U_i}"] \\
& U_i &
\end{tikzcd}$$

Define

$$h_i(x_0, x_1, \ldots, x_n) = \big(p(x), |x_i|^{-1}\|x\|x_i\big)\,,$$

where $x = (x_0, x_1, \ldots, x_n)$, and define

$$g_i : U_i \times (\mathbb{F} - \{0\}) \longrightarrow V_i$$

by

$$g_i(p(x), \lambda) = \|x\|^{-1}|x_i|\lambda x_i^{-1}(x_0, x_1, \ldots, x_n)\,.$$

[1]The fields of real, complex or quaternionic numbers considered as topological spaces

It is easy to verify that g_i is well defined for each i and that h_i and g_i are continuous and inverse to each other. Thus h_i is a homeomorphism. $k_i = h_i|_{V_i \cap \mathbb{S}^{d(n+1)-1}}$ and $g_i|_{U_i \times \mathbb{S}^{d-1}}$ are inverse of each other (and have the desired images), and the diagrams obviously commute with h_i and k_i. □

(b) The *Hopf fibration* of the 3-sphere is the special case $\mathbb{F} = \mathbb{C}$, $n = 1$, $d = 2$, of the previous example, that we now study in more detail. Consider the diagram

$$\begin{tikzcd}
(z_0, z_1) \arrow[dd, mapsto] & \in & \mathbb{C}^2 - \{0\} \arrow[d] & \mathbb{S}^3 \arrow[l, hook'] \arrow[dd] \\
& & \mathbb{CP}^1 \arrow[d, "\approx"] & \\
\frac{z_0}{z_1} & \in & \mathbb{C} \cup \{\infty\} = & \mathbb{S}^2
\end{tikzcd}$$

Here we have a homeomorphism between $\mathbb{CP}^1$ and the Riemann sphere given by the map $p : (z_0, z_1) \mapsto \frac{z_0}{z_1}$, that is an identification. This is due to the fact that $p|_{\mathbb{S}^3}$ is a continuous surjective map from a compact space to a Hausdorff space. Recall that

$$\mathbb{S}^3 = \Big\{ (z_0, z_1) \mid |z_0|^2 + |z_1|^2 = 1 \Big\} \ .$$

We write $z_\nu = r_\nu s_\nu$ with $r_\nu \geq 0$ and $|s_\nu| = 1$, $(\nu = 0, 1)$. Then $r_1 = (1 - r_0^2)^{\frac{1}{2}}$ and thus every point in $\mathbb{S}^3$ is characterized by the numbers s_0, s_1 and $r = r_0$. Let

$$\begin{aligned} q : \mathbb{S}^1 \times \mathbb{S}^1 \times I &\longrightarrow \mathbb{S}^3 \\ (s_0, s_1, r) &\longmapsto \Big(r s_0, (1 - r^2)^{\frac{1}{2}} s_1 \Big) \ . \end{aligned}$$

q is an identification. For $r \neq 0, 1$, each $(r s_0, (1 - r^2)^{\frac{1}{2}} s_1)$ has only one inverse image. For $r = 0$, q identifies

$$(s_0, s_1, 0) \quad \text{with} \quad (s_0', s_1, 0) \, ,$$

and for $r = 1$ it identifies

$$(s_0, s_1, 1) \quad \text{with} \quad (s_0, s_1', 1) \, .$$

Given any two topological spaces X_0, X_1, the quotient space of $X_0 \times X_1 \times I$ with respect to such an identification (i.e., $(x_0, x_1, 0) \sim (x_0', x_1, 0)$, and $(x_0, x_1, 1) \sim (x_0, x_1', 1)$ for all $x_0, x_0' \in X_0$, $x_1, x_1' \in X_1$) is called the *join* of X_1 and X_2 and is usually denoted by $X_0 * X_1$.

What we proved above is then that one has a homeomorphism

$$\mathbb{S}^1 * \mathbb{S}^1 \approx \mathbb{S}^3 .$$

From this version of the 3-sphere $\mathbb{S}^3$ we can obtain the following: The points for which r is fixed and $r \neq 0, 1$ determine a torus; namely a space homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$. On the contrary for $r = 0$ or 1, they determine a 1-sphere. Further, the points such that $r \geq \frac{1}{2}$, resp. $r \leq \frac{1}{2}$ constitute a (space homeomorphic to a) solid torus. Thus we have that the 3-sphere $\mathbb{S}^3$ can be obtained from two solid tori by identifying their boundaries in such a way that the meridians of one of them corespond to the parallels of the other (see Figure 1.8). More precisely, we have

$$\mathbb{S}^3 \approx \mathbb{S}^1 \times \mathbb{B}^2 \sqcup \mathbb{B}^2 \times \mathbb{S}^1 / \sim , \qquad (s,t) \sim (s,t) \in \mathbb{S}^1 \times \mathbb{S}^1 .$$

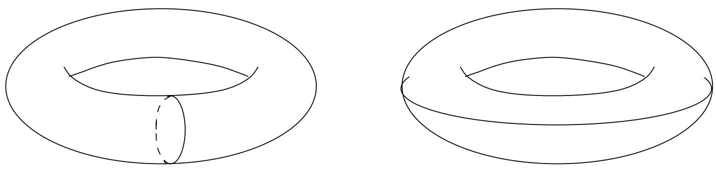


Figure 1.8

Now we can describe $p : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ by mapping

$$\left(r s_0, (1-r^2)^{\frac{1}{2}} s_1 \right) \longmapsto \left(r(1-r^2)^{-\frac{1}{2}} \right) \left(\frac{s_0}{s_1} \right) \in \mathbb{C} \cup \{\infty\} .$$

The inverse images of a point in $\mathbb{S}^2$ correspond to a fixed value of r. They constitute a circle that lies on the torus given by the equation $r =$ constant, if $r \neq 0, 1$. If $r = 0$ or 1, then they determine full circles. Each of these circles intersects each parallel and each meridian of the torus exactly once. Every two circles that are inverse images of a point are knotted. For this, one might analyze the case $p^{-1}(0)$ and $p^{-1}(z)$ ($z \neq 0, \infty$), or for two of those circles that lie on the same torus $r =$ constant.

One might try to study the general map

$$\begin{aligned} \mathbb{S}^1 * \mathbb{S}^1 &\longrightarrow \mathbb{S}^2 \\ [s_0, s_1, r] &\longmapsto \left(r(1-r^2)^{-\frac{1}{2}} \right) \left(\frac{s_0^m}{s_1^n} \right) \end{aligned}$$

where m and n are natural numbers. In general one does not obtain a locally trivial fibration, since the local triviality fails on the points

$r = 0, 1$. The inverse images of a point in $\mathbb{S}^2$ are again circles that lie on tori $r =$ constant, but they are multiply knotted. The reader can think about the case $n = 3$, $m = 2$, for which the circles (if $r \neq 0, 1$) are always knotted and build a *trefoil knot* (see Figure 1.9).

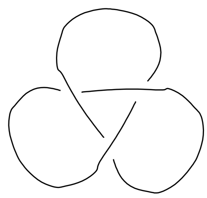


Figure 1.9

The relative position of two of these inverse image tori can be visualized as follows.

One stretches a (self-intersecting) surface along the first trefoil knot and chooses one side of it to be the front (i.e., one takes an orientation of the surface). After traveling along the second trefoil knot in the adequate sense, then one crosses the surface $2 \cdot 3 = 6$ times from the front to the back.

1.4 Homotopy Lifting

Let I be the unit interval $[0, 1]$ and $p : E \longrightarrow B$ be a fibration. We are interested in the following situation.

(1.4.1)
$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widetilde{h}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & B \end{tikzcd}$$

where the square is commutative. When does $\widetilde{h}$ exist that makes both triangles commutative?

1.4.2 Definition. We say that p has the *homotopy lifting property* or the *HLP* for the space X if given a pair of maps $(h, \widetilde{h}_0)$ as in (1.4.1), then there exists $\widetilde{h}$ such that (1.4.1) commutes.

We then say that $\widetilde{h}$ is a *lifting* of the homotopy h that *starts* with $\widetilde{h}_0$. Or we say that h *lifts* to $\widetilde{h}_0$.

1.4.3 **Theorem**. *A trivial fibration has the HLP for every space.*

Proof: A trivial fibration is equivalent to the product fibration. Therefore, we can restrict ourselves to the problem

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widetilde{h}_0"] \arrow[d, hook] & B \times F \arrow[d, "\mathrm{proj}_B"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & B\,. \end{tikzcd}$$

Define $\widetilde{h}$ by $\widetilde{h}(x,t) = (h(x,t), \mathrm{proj}_B \widetilde{h}_0(x,0))$. □

1.4.4 EXAMPLES.

(a) The fibration of example 1.1.3 does not have the HLP for any nonempty space X, since for instance the homotopy $h(x,t) = t$ cannot be lifted starting with $\widetilde{h}_0(x,0) = (0,2)$.

(b) There are fibrations that have the HLP for a one-point space $X = \{*\}$ but not for $X = \{*\} \times I \approx I$. An example of this is the double covering of the plane branched at the origin given, say, by

$$p : \mathbb{C} \longrightarrow \mathbb{C}\,, \quad \begin{aligned} z &\longmapsto \frac{z^2}{|z|} \\ 0 &\longmapsto 0\,, \end{aligned}$$

We have to prove that to each path $\omega : I \longrightarrow \mathbb{C}$ there exists a path $\widetilde{\omega} : I \longrightarrow \mathbb{C}$ such that $p \circ \widetilde{\omega} = \omega$ and such that $\widetilde{\omega}(0) \in p^{-1}(\omega(0))$ is preassigned. Now, since $I - \omega^{-1}(0)$ is an open set, it is an at most countable union of intervals I_n open in I. Since $p|_{\mathbb{C}-\{0\}}$ is a covering map (see Section 1.8 below), $\omega|_{I_n}$ can be lifted. Let $\widetilde{\omega}_n$ be a lifting. If $0 \in I_n$ (and $\widetilde{\omega}(0) \neq 0$) let $\widetilde{\omega}_n$ be such $\widetilde{\omega}_n(0) = \widetilde{\omega}(0)$. If we define

$$\widetilde{\omega}(t) = \begin{cases} \widetilde{\omega}_n(t) & \text{if } t \in I_n, \\ 0 & \text{if } t \in \omega^{-1}(0), \end{cases}$$

then $\widetilde{\omega}$ is such that $\widetilde{\omega}(0)$ is as we wanted, and $p \circ \widetilde{\omega} = \omega$. Moreover, $\widetilde{\omega}$ is continuous, since for $t \in I_n$, this is clear, and if $t_0 \in \omega^{-1}(0)$, then the continuity of $\widetilde{\omega}$ at t_0 follows from the fact that $|\widetilde{\omega}(t)| = |\omega(t)|$, that is, $|\widetilde{\omega}(t) - \widetilde{\omega}(t_0)| = |\omega(t) - \omega(t_0)|$ and from the continuity of ω.

Now, if $X = I$ and $h : X \times I \longrightarrow \mathbb{C}$ is given by $h(s,t) = (s - \frac{1}{2}, t - \frac{1}{2})$ there does not exist $\widetilde{h}$ for any $\widetilde{h}_0$, since p restricted to $p^{-1}(\partial h(I \times I))$ (that is, the inverse image of the boundary of $h(I \times I)$ (see Figure 1.10) is a twofold-covering map, and $\widetilde{h}$ would induce a section of it, fact that is not true (cf. Section 1.8).

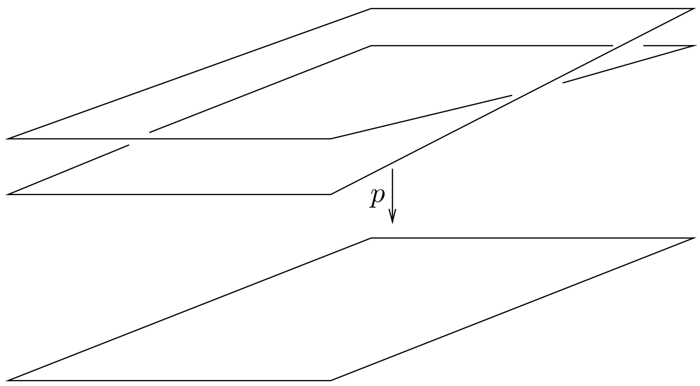


Figure 1.10

1.4.5 Exercise. Prove that if the group $\mathbb{Z}_2$ acts on $\mathbb{C}$ antipodally, then one has aan isomorphism $\mathbb{C}/\mathbb{Z}_2 \approx \mathbb{C}$ such that there is a commutative diagram

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathbb{C}/\mathbb{Z}_2 \arrow[rr, "\approx"'] & & \mathbb{C}\,. \end{tikzcd}$$

1.4.6 Definition. Let X be a topological space and $A \subset X$. We say that $p : E \longrightarrow B$ has the *relative homotopy lifting property* or the *relative HLP* for the pair (X, A) if every commutative square (given by h and $\widetilde{h}_0$)

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "\widetilde{h}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"'] & B \end{tikzcd}$$

admits a map $\widetilde{h}$ that makes both triangles commutative.

Even a trivial fibration might not always have the relative HLP as one can easily see in the case $B = \{*\}$, since in this case, the existence of $\widetilde{h}$ such that the upper triangle commutes implies an extension problem, and this problem is usually nontrivial (note, however, that the commutativity of the lower triangle is in this case always trivial).

1.4.7 **Theorem**. *The following statements are equivalent:*

(a) *p has the HLP for the closed unit ball $\mathbb{B}^n$, $n = 0, 1, 2, \dots$ ($\mathbb{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$).*

(b) *p has the relative HLP for the pair $(\mathbb{B}^n, \mathbb{S}^{n-1})$, $n = 0, 1, 2, \ldots$.*

(c) *p has the relative HLP for a* CW*-pair (X, A).*

(d) *p has the HLP for every* CW*-complex X.*

Proof: (a) $\Rightarrow$ (b) Let

$$k : \big(\mathbb{B}^n \times I, \mathbb{B}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I\big) \longrightarrow (\mathbb{B}^n \times I, \mathbb{B}^n \times \{0\})$$

be given by

$$k(x,t) = \begin{cases} \left(\frac{1+t}{2-t}x, t\right) & \text{if } |x| \le \frac{1}{2}(2-t), \\ \left(\frac{1}{2}(1+t)\frac{x}{|x|}, 2(1-|x|)\right) & \text{if } |x| \ge \frac{1}{2}(2-t). \end{cases}$$

k is a homeomorphism of pairs that converts the relative homotopy lifting problem for the pair $(\mathbb{B}^n, \mathbb{S}^{n-1})$ into a homotopy lifting problem for $\mathbb{B}^n$. See Figure 1.11.

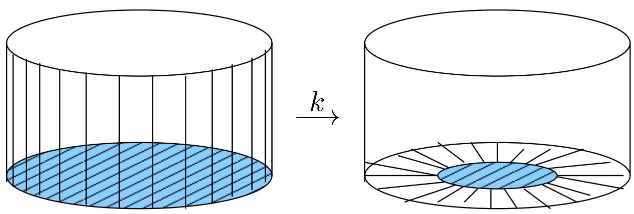


Figure 1.11

(c) $\Rightarrow$ (d) Just take $A = \emptyset$.

(d) $\Rightarrow$ (a) Just observe that $\mathbb{B}^n$ is a CW-complex.

(b) $\Rightarrow$ (c) Let X^n be the n-skeleton of X and let $X_n = X^n \cup A$. We shall inductively construct maps

$$\widetilde{h}_{n+1} : (X \times \{0\} \cup X_n \times I) \longrightarrow E$$

such that $\widetilde{h}_{n+1}|_{X \times \{0\} \cup X_{n-1} \times I} = \widetilde{h}_n$ and such that the composite $p \circ \widetilde{h}_n = h$, wherever it is defined.

$$\widetilde{h}_0 : (X \times \{0\} \cup A \times I) \longrightarrow E$$

is already given. Assume that $\widetilde{h}_n$ has already been constructed. Recall that $X \times I$ is a CW-complex with cells of the form

$$e^k \times (0,1)\,, \quad e^k \times \{0\}\,, \quad e^k \times \{1\}$$

where e^k represents any cell of X (see [1] or [8]).

Let e^n_j be an n-cell of $X - A$ and $\varphi_j : \mathbb{B}^n \longrightarrow X$ be its characteristic map. Consider

$$\begin{tikzcd}
\mathbb{B}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I \arrow[r] \arrow[d, hook] & X \times \{0\} \cup X_{n-1} \times I \arrow[r, "\widetilde{h}_n"] \arrow[d, hook] & E \arrow[d, "p"] \\
\mathbb{B}^n \times I \arrow[r, "\varphi_j \times \mathrm{id}"'] \arrow[urr, dashed, "\widetilde{g}_j"] & X \times I \arrow[r, "h"'] & B
\end{tikzcd}$$

By the hypothesis (b), there exists $\widetilde{g}_j$. Define $\widetilde{h}_{n+1}$ by

$$\widetilde{h}_{n+1}(x,t) = \begin{cases} \widetilde{h}_n(x,t) & \text{if } (x,t) \in X \times \{0\} \cup X_{n-1} \times I, \\ \widetilde{g}_j(\varphi_j^{-1}(x), t) & \text{if } x \in \overline{e}_j. \end{cases}$$

$\widetilde{h}_{n+1}$ is well defined, extends $\widetilde{h}_n$ and lifts $h|_{X \times \{0\} \cup X_n \times I}$.

Moreover, $\widetilde{h}_{n+1}$ is continuous. This follows from the fact that $\varphi_j \times \mathrm{id}$ is an identification and $\widetilde{h}_{n+1}(\varphi_j \times \mathrm{id}) = \widetilde{g}_j$. Therefore, $\widetilde{h}_{n+1}$ is continuous on each closed cell of $X \times \{0\} \cup X_n \times I$ (because $\varphi_j \times \mathrm{id}$ is the characteristic map of the cell $e^n_j \times (0,1)$).

To finish, define $\widetilde{h} : X \times I \longrightarrow E$ by $\widetilde{h}(x,t) = \widetilde{h}_n(x,t)$ if $(x,t) \in X_n \times I$. $\square$

1.4.8 DEFINITION. A fibration $p : E \longrightarrow B$ is said to be a *Serre fibration* if one (and hence all) of the statements (a) through (d) in the previous theorem holds. Moreover, we say that p is a *Hurewicz fibration* if p has the HLP for every space.

1.4.9 **Theorem**. *Let $p : E \longrightarrow B$ be a fibration and $\mathcal{U} = \{U\}$ be an open cover of the base space B such that for each $U \in \mathcal{U}$, the restriction p_U is a Serre fibration. Then p is also one. (This means that the property of being a Serre fibration is local with respect to the base space).*

Observe that the inverse is clear, as follows from the following exercise.

1.4.10 EXERCISE. Prove that if p has the HLP for a space X, then any restriction p_A, $A \subset B$, has it too.

1.4.11 **Corollary**. *Every locally trivial fibration is a Serre fibration.* $\square$

Proof of the theorem: We shall use condition (b) of 1.4.7 for each p_U and prove (a) for p. For technical reasons, we substitute the ball $\mathbb{B}^n$ with the homeomorphic cube I^n.

Subdivide $I^n \times I$ by successively halving the sides until each subcube is mapped by h into some $U \in \mathcal{U}$. Thus we obtain a decomposition of I^n, whose k-dimensional subcubes (*faces* if $k < n$) will be denoted by V_i^k, as well as a decomposition of I

$$0 < t_1 < t_2 < t_3 < \cdots < 1\,.$$

We shall extend $\widetilde{h}_0$ step by step along the "layers" $I^n \times [t_j, t_{j+1}]$ to finally obtain a lifting of h. To that end, let $V^k = \bigcup_i V_i^k$.

We shall successively solve the problem

$$\begin{tikzcd}
I^n \times \{0\} \cup V^{k-1} \times [0,t_1] \arrow[r, "\widetilde{h}^{k-1}"] \arrow[d, hook] & E \arrow[d, "p"] \\
I^n \times \{0\} \cup V^k \times [0,t_1] \arrow[r, "h|\cdots"'] \arrow[ur, dashed, "\widetilde{h}^k"'] & B\,,
\end{tikzcd}$$

$k = 0, 1, \ldots, n$, where $\widetilde{h}^0 = \widetilde{h}_0$.

Assume that $\widetilde{h}^{k-1}$ has already been constructed. Then we can solve the problem

$$\begin{tikzcd}
V_i^k \times \{0\} \cup \partial V_i^k \times [0,t_1] \arrow[r, "\widetilde{h}^{k-1}|\cdots"] \arrow[d, hook] & p^{-1}U \arrow[r, hook] \arrow[d, "p_u"] & E \arrow[d, "p"] \\
V_i^k \times [0,t_1] \arrow[r, "h|\cdots"'] \arrow[ur, dashed, "\widetilde{h}_i^k"'] & U \arrow[r, hook] & B\,,
\end{tikzcd}$$

since p_U is a Serre fibration. (Our subdivision of I^n into subcubes was fine enough to guarantee the existence of U such that

$$h(V_i^k \times [0,t_1]) \subset U\,,$$

$\partial V_i^k \subset V^{k-1}$ denotes the boundary of the subcube V_i^k.) The maps $\widetilde{h}_i^k$ can now be put together to produce a continuous map $\widetilde{h}^k : V^k \times [0,t_1] \longrightarrow E$ that extends $\widetilde{h}^{k-1}$ and lifts $h|\cdots$. We define $\widetilde{h}$ by means of $\widetilde{h}|_{I^n\times[0,t_1]} = \widetilde{h}^n$ on the first layer. The next layers are dealt with in a similar manner. □

An analogous statement to the previous theorem holds also for Hurewicz fibrations. To state it we need some preparation. We start by recalling the next definition.

1.4.12 Definition. Let X be a topological space. A *partition of unity* is a family of continuous functions $\{t_j : X \longrightarrow I\}_{j\in J}$ such that for each $x \in X$,

$t_j(x) \neq 0$ only for finitely many $j \in J$, and $\sum_{j\in J} t_j(x) = 1$. A partition of unity is called *locally finite* if every $x \in X$ has a neighborhood U with $t_j|_U \neq 0$ only for finitely many $j \in J$.

The family $\{V_j = {t_j}^{-1}(0,1]\}_{j\in J}$ is called the *associated open cover* of X for the given partition of unity. An open cover $\mathcal{U} = \{U_j\}_{j\in J}$ of X is called *numerable* if there exists a locally finite partition of unity $\{t_j : X \longrightarrow I\}_{j\in J}$ such that

$$ {t_j}^{-1}(0,1] \subset U_j. $$

In this case we say that the partion of unity is *subordinate* to the cover.

1.4.13 DEFINITION. A topological space X is said to be *paracompact* if every open cover of X is numerable.

The previous definition is usually presented as a theorem (cf. [13, 7.5.23]). The following theorem is due to Albrecht Dold [3].

1.4.14 **Theorem**. *Let $p : E \longrightarrow B$ be a fibration and $\mathcal{U} = \{U_j\}_{j\in J}$ be an open cover of B such that p_{U_j} is a Hurewicz fibration. Then*

(a) *if $\mathcal{U}$ is numerable, then p is a Hurewicz fibration;*

(b) *if $\mathcal{U}$ is open, then p has the HLP for every paracompact space.* □

We omit the proof, since it is quite intrincate and would pull us apart from the topics we are dealing with. See [3, Thm. 48] for a proof.

Since every CW-complex is paracompact (see [12] or [8]), we have the following.

1.4.15 **Corollary**. *Let $p : E \longrightarrow B$ be a fibration and $\mathcal{U} = \{U_j\}_{j\in J}$ be an open cover of B such that p_{U_j} is a Hurewicz fibration. Then*

(a) *if B is a* CW-*complex then p is a Hurewicz fibration;*

(b) *p has the HLP for every* CW-*complex.* □

Consider the *path space* $X^I = \{\omega : I \longrightarrow X\}$ furnished with the compact-open topology. Given a fibration $p : E \longrightarrow B$, take the *fibered product*

$$ E \times_B B^I = \{(e,\omega) \in E \times B^I \mid p(e) = \omega(0)\}. $$

1.4.16 DEFINITION. A continuous map

$$\Gamma : E \times_B B^I \longrightarrow E^I$$

is called *path-lifting map* (PLM) if the following hold:

(a) $\Gamma(e, \omega)(0) = e$, where $(e, \omega) \in E \times_B B^I$.

(b) $p\Gamma(e, \omega)(t) = \omega(t)$, where $(e, \omega) \in E \times_B B^I$ and $t \in I$.

1.4.17 **Theorem**. *A fibration $p : E \longrightarrow B$ is a Hurewicz fibration if and only if it has a PLM $\Gamma : E \times_B B^I \longrightarrow E^I$.*

Proof: Assume first that $p : E \longrightarrow B$ is a Hurewicz fibration and consider the lifting problem depicted in the following diagram:

$$\begin{tikzcd} E \times_B B^I \arrow[r, "\mathrm{proj}_1"] \arrow[d, hook, "i_0"'] & E \arrow[d, "p"] \\ E \times_B B^I \times I \arrow[r, "\varepsilon"'] \arrow[ur, dashed, "\widehat{\Gamma}"] & B \, , \end{tikzcd}$$

where i_0 is the inclusion into the bottom of the cylinder ($i_0(e, \omega) = (e, \omega, 0)$) and $\varepsilon(e, \omega, t) = \omega(t)$. Since the square is obviously commutative, and the fibration has the HLP for every space, there exists $\widehat{\Gamma} : E \times_B B^I \times I \longrightarrow I$, such that both triangloes commute. Defining $\Gamma : E \times_B B^I \longrightarrow E^I$ by

$$\Gamma(e, \omega)(t) = \widehat{\Gamma}(e, \omega, t)\, ,$$

we have the desired PLM.

Conversely, assume that there is a PLM $\Gamma : E \times_B B^I \longrightarrow E^I$ for $p : E \longrightarrow B$ and assume a general homotopy lifting problem

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & B \, . \end{tikzcd}$$

Define $\overline{h} : X \longrightarrow B^I$ by $\overline{h}(x)(t) = h(x, t)$, and consider the composite

$$\begin{aligned} h' : X &\longrightarrow E \times_B B^I \xrightarrow{\quad \Gamma \quad} E^I \\ x &\longmapsto (f(x), \overline{h}(x)) \longmapsto \Gamma(f(x), \overline{h}(x))\, . \end{aligned}$$

Then $\widetilde{h} : X \times I \longrightarrow E$ given by $\widetilde{h}(x, t) = h'(x)(t)$ is the desired lifting. □

1.4.18 EXERCISE. Proving the existence of PLMs show that the following are Hurewicz fibrations:

(a) The map $B^I \longrightarrow B$, given by $\omega \mapsto \omega(1)$.

(b) The map $P(B) = \{\omega \in B^I \mid \omega(0) = b_0\} \longrightarrow B$, given by $\omega \mapsto \omega(1)$. This is the so-called *path fibration* of B (see 3.4.7).

(c) Given $f : X \longrightarrow B$, the map $E_f = \{(x, \omega) \in X \times B^I \mid f(x) = \omega(1)\} \longrightarrow B$, given by $(x, \omega) \mapsto \omega(0)$. The space E_f is the so-called *mapping path space*, the fibration is the *mapping path fibration*, and its fiber over (a base point) $b_0 \in B$, $P_f = \{(x, \omega) \in X \times B^I \mid \omega(0) = b_0,\ f(x) = \omega(1)\}$, is the so-called *homotopy fiber* of f.

The following result states that every map factors as a homotopy equivalence followed by a Hurewicz fibration (i.e., every map can by replaced by a Hurewicz fibration, up to a homotopy equivalence). It is an easy *exercise* to prove it.

1.4.19 **Proposition**. *Given any continuous map $f : X \longrightarrow B$, the map $\varphi : X \longrightarrow E_f$ given by $x \mapsto (x, e_x)$, where $e_x : I \longrightarrow B$ is the constant path with value $f(x)$, is a homotopy equivalence. Moreover, there is a commutative triangle*

$$\begin{tikzcd} & E_f \arrow[d, "\widetilde{f}"] \\ X \arrow[ur, "\varphi"] \arrow[r, "f"'] & B , \end{tikzcd}$$

where $\widetilde{f}$ is the Hurewicz fibration of 1.4.18 (c). □

The following result will be important later. Given a fibration $p : E \longrightarrow B$ and a subspace $A \subset B$, recall its restriction $p_A : E_A \longrightarrow A$ (see Definition 1.2.5). We have the following result.

1.4.20 **Theorem**. *Assume that $p : E \longrightarrow B$ is a Hurewicz fibration and that both spaces E and B are normal. If $i : A \hookrightarrow B$ is a closed cofibration, then also $\widetilde{i} : E_A \hookrightarrow E$ is a closed cofibration.*

Before passing to the proof, we recall Theorem [1, 4.1.16], which reads as follows.

1.4.21 **Lemma**. *Let B be a normal space. Then an inclusion $A \hookrightarrow B$ is a closed cofibration if and only if there exist maps $u : B \longrightarrow I$ and $h : B \times I \longrightarrow B$ such that*

(i) $A \subset u^{-1}(0)$.

(ii) $h(b, 0) = b$ *for all* $b \in B$.

(iii) $h(a, t) = a$ *for all* $a \in A$ *and all* $t \in I$.

(iv) $h(b, t) \in A$ *for all* $b \in B$ *if* $t > u(b)$. □

Proof of 1.4.20: We shall apply Lemma 1.4.21. Assume that u and h are as in that lemma. Since $p : E \longrightarrow B$ is a Hurewicz fibration, the lifting problem

$$\begin{tikzcd} E \arrow[r, "\mathrm{id}"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ E \times I \arrow[r, "h\circ(p\times \mathrm{id})"'] \arrow[ur, dashed, "H'"] & B \end{tikzcd}$$

has a solution H'. Define $U : E \longrightarrow I$ by $U(e) = up(e)$, and $H : E \times I \longrightarrow E$ by

$$H(e,t) = \begin{cases} H'(e,t) & \text{if } t \leq U(e), \\ H'(e, U(e)) & \text{if } t \geq U(e). \end{cases}$$

Then obviously $E_A \subset U^{-1}(0)$, $H(e, 0) = e$ for all $e \in E$, and $H(e, t) = E$ for all $t \in I$ if $e \in E_A$. Thus the first three conditions in 1.4.21 hold. To verify (iv), assume $t > U(e)$. Then

$$pH(e,t) = pH'(e, U(e)) = h(p(e), up(e))\,. \tag{1.4.21}$$

But we have that if $s > up(e)$, then $h(p(e), s) \in A$. Since $A \subset B$ is closed, by the continuity of h, $h(p(e), up(e)) \in A$. Hence, from (1.4.21), $H(e, t) \in E_A$.

We have shown that U and H satisfy conditions (i)–(iv), thus $E_A \hookrightarrow E$ is a cofibration. □

The following definition generalizes the construction of the restricted fibration $p_A : E_A \longrightarrow A$.

1.4.22 Definition. Let $p : E \longrightarrow B$ be a fibration and $\alpha : A \longrightarrow B$ a continuous map. We define a new fibration $\alpha^*(p) : \widetilde{E} \longrightarrow B$ and a fiber map $(\beta, \alpha) : \alpha^*(p) \longrightarrow p$ as follows. Take

$$\widetilde{E} = \{(a, z) \in A \times E \mid \alpha(a) = p(z)\}$$

with the relative topology as subspace of $A \times E$.

$$\begin{aligned} \alpha^*(p)(a,z) &= a \\ \beta(a,z) &= z\,. \end{aligned}$$

$\alpha^*(p)$ is called the fibration *induced* by p through α. Thus one has a commutative diagram

$$\begin{tikzcd} \widetilde{E} \arrow[r, "\beta"] \arrow[d, "\alpha^*(p)"'] & E \arrow[d, "p"] \\ A \arrow[r, "\alpha"'] & B\,. \end{tikzcd}$$

In case that $\alpha : A \hookrightarrow B$ is an inclusion, the induced fibration $\alpha^*(p)$ is equivalent to the restriction $p_A : E_A \longrightarrow A$ (cf. 1.2.5).

1.4.23 EXERCISE. Prove that through a constant map, a trivial fibration is induced.

1.4.24 EXERCISE. Let $p : E \longrightarrow B$ be a fibration and $\alpha : A \longrightarrow B$ be continuous. Verify the following properties:

(a) If p is injective (resp. surjective), then so is $\alpha^*(p)$.

(b) If p is the product fibration, then $\alpha^*(p)$ is a trivial fibration (see 1.2.4).

(c) If p is locally trivial, then so is $\alpha^*(p)$.

(d) The map α admits a lifting $\widetilde{\alpha} : A \longrightarrow E$ (namely, a map such that $p \circ \widetilde{\alpha} = \alpha$) if and only if $\alpha^*(p)$ admits a section $s : A \longrightarrow \widetilde{E}$ (namely, a map such that $\alpha^*(p) \circ s = \mathrm{id}_A$).

The following is an important result.

1.4.25 **Proposition**. *Let $\alpha : A \longrightarrow B$ be continuous. If a fibration $p : E \longrightarrow B$ has the HLP for a space X, then so does $\alpha^*(p)$.*

Proof: We have to show that the homotopy lifting problem

$$\begin{tikzcd} X \arrow[r, "\widetilde{h}_0"] \arrow[d, hook, "i_0"'] & \widetilde{E} \arrow[d, "\alpha^*(p)"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & A \end{tikzcd}$$

has a solution. Consider the following homotopy lifting problem

$$\begin{tikzcd} X \arrow[r, "\widetilde{h}_0^1"] \arrow[d, hook, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "\alpha\circ h"'] \arrow[ur, dashed, "\widetilde{k}"] & B\,, \end{tikzcd}$$

where $\widetilde{h}_0(x) = (\widetilde{h}_0^1(x), \widetilde{h}_0^2(x)) \in \widetilde{E} \subset A \times E$. Since p has the HLP for X, this problem has a solution and thus $\widetilde{k} : X \times I \longrightarrow E$ exists making both triangles commutative. Define $\widetilde{h}$ by

$$\widetilde{h}(x,t) = (h(x,t), \widetilde{k}(x,t)) \in A \times E\,.$$

Obviously, this map is such that $\widetilde{h} : X \times I \longrightarrow \widetilde{E}$ and obviously is a solution of the initial problem. $\square$

1.4.26 **Corollary**. *Given a fibration $p : E \longrightarrow B$ and a map $\alpha : A \longrightarrow E$, the following hold.*

(a) *If p is a Serre fibration, then so is $\alpha^*(p)$.*

(b) *If p is a Hurewicz fibration, then so is $\alpha^*(p)$.* $\square$

1.4.27 Definition. Two fibrations $p_0 : E_0 \longrightarrow B$ and $p_1 : E_1 \longrightarrow B$ are called *fiber homotopy equivalent* (or to have the *same fiber homotopy type*) if there exist *fiber-preserving* maps, or maps *over* B, $\varphi : E_0 \longrightarrow E_1$ and $\psi : E_1 \longrightarrow E_0$, that is, maps such that the triangles

$$\begin{tikzcd} E_0 \arrow[rr, "\varphi"] \arrow[dr, "p_0"'] & & E_1 \arrow[dl, "p_1"] \\ & B & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E_1 \arrow[rr, "\psi"] \arrow[dr, "p_1"'] & & E_0 \arrow[dl, "p_0"] \\ & B & \end{tikzcd}$$

commute, and these maps are such that $\psi\circ\varphi \simeq_B \mathrm{id}_{E_0}$ and $\varphi\circ\psi \simeq_B \mathrm{id}_{E_1}$, that is, these composites are *fiber homotopic* to the identities in the sense that they are homotopic through homotopies H and K such that the triangles

$$\begin{tikzcd} E_0 \times I \arrow[rr, "H"] \arrow[dr, "p_0\circ\mathrm{proj}_1"'] & & E_0 \arrow[dl, "p_0"] \\ & B & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E_1 \times I \arrow[rr, "K"] \arrow[dr, "p_1\circ\mathrm{proj}_1"'] & & E_1 \arrow[dl, "p_1"] \\ & B & \end{tikzcd}$$

are commutative.

1.4.28 **Theorem**. *Let $p : E \longrightarrow B$ be a Hurewicz fibration and let $G_0, G_1 : X \times I \longrightarrow E$ be homotopies. Given other homotopies $H : p \circ G_0 \simeq p \circ G_1$ and $K : G_0 \circ i_0 \simeq G_1 \circ i_0$, where $i_0 : X \hookrightarrow X \times I$ is given by $i_0(x) = (x, 0)$, such that*

$$H(x, 0, t) = pK(x, t)\,,$$

there is a lifting $\widetilde{H} : X \times I \times I \longrightarrow E$ of H which is a homotopy from G_0 to G_1 and is an extension of K.

Proof: There is a homeomorphism of pairs

$$\lambda : (I \times I, I \times \{0\}) \longrightarrow (I \times I, I \times \partial I \cup \{0\} \times I)$$

as illustrated in Figure 1.12.

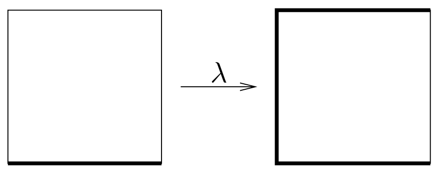


Figure 1.12

It is an *exercise* for the reader to figure out λ explicitly.

Define

$$f : X \times (I \times \partial I \cup \{0\} \times I) \longrightarrow E$$

by

$$\begin{aligned} f(x, s, 0) &= G_0(x, s)\,, \\ f(x, 0, t) &= K(x, t)\,, \\ f(x, s, 1) &= G_1(x, s)\,. \end{aligned}$$

Then the diagram

$$\begin{tikzcd}
X \times I \times \{0\} \arrow[r, "\mathrm{id}_X \times \lambda|", "\approx"'] \arrow[d, hook] \arrow[rr, bend left, "\widetilde{h}_0"] & X \times (I \times \partial I \cup \{0\} \times I) \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \times I \arrow[r, "\approx", "\mathrm{id}_X \times \lambda"'] \arrow[urr, dashed, "\widetilde{h}"] \arrow[rr, bend right, "h"'] & X \times I \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"'] & B
\end{tikzcd}$$

commutes and both the exterior square as well as the right square pose lifting problems. Since p is a Hurewicz fibration, there exists $\widetilde{h}$ (that solves the exterior problem). Then $\widetilde{H} = \widetilde{h} \circ (\mathrm{id}_X \times \lambda)^{-1}$ solves the problem on the right. This is the desired homotopy. $\square$

The following result combines the concept of homotopic maps with that of equivalent Hurewicz fibrations.

1.4.29 **Theorem**. *Let $p : E \longrightarrow B$ be a Hurewicz fibration and let $\alpha_0, \alpha_1 : A \longrightarrow B$ be homotopic. Then the fibrations $\alpha_0^*(p) : \widetilde{E}_0 \longrightarrow A$ and $\alpha_1^*(p) : \widetilde{E}_1 \longrightarrow A$ induced by p through α_0 and α_1, respectively, are fiber homotopy equivalent.*

Proof: Let $p_0 = \alpha_0^*(p)$ and $p_1 = \alpha_0^*(p)$ be the induced fibrations, and let $\beta_0 : \widetilde{E}_0 \longrightarrow E$ and $\beta_1 : \widetilde{E}_1 \longrightarrow E$ be the corresponding projection maps such that $p \circ \beta_0 = \alpha_0 \circ p_0$ and $p \circ \beta_1 = \alpha_1 \circ p_1$. Given a homotopy $F : A \times I \longrightarrow B$ from α_0 to α_1, there are maps $G_0 : \widetilde{E}_0 \times I \longrightarrow E$ and $G_1 : \widetilde{E}_1 \times I \longrightarrow E$ that solve the lifting problems

$$\begin{tikzcd} \widetilde{E}_0 \arrow[rr, "\beta_0"] \arrow[d, hook, "i_0"'] & & E \arrow[d, "p"] \\ \widetilde{E}_0 \times I \arrow[r, "p_0 \times \mathrm{id}_I"'] \arrow[urr, dashed, "G_0"] & A \times I \arrow[r, "F"'] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{E}_1 \arrow[rr, "\beta_1"] \arrow[d, hook, "i_1"'] & & E \arrow[d, "p"] \\ \widetilde{E}_1 \times I \arrow[r, "p_1 \times \mathrm{id}_I"'] \arrow[urr, dashed, "G_1"] & A \times I \arrow[r, "F"'] & B \end{tikzcd},$$

where i_0 and i_1 are the inclusions into the bottom and into the top of the corresponding cylinders, respectively. Let $\widetilde{\beta}_0 : \widetilde{E}_0 \longrightarrow \widetilde{E}_1$ be given by

$$\widetilde{\beta}_0(a, e) = (a, G_0(a, e, 1))\,,$$

and $\widetilde{\beta}_1 : \widetilde{E}_1 \longrightarrow \widetilde{E}_0$ be given by

$$\widetilde{\beta}_1(a, e) = (a, G_1(a, e, 0))\,.$$

Then

$$p \circ (G_0 \circ (\widetilde{\beta}_1 \times \mathrm{id}_I)) = F \circ (p_0 \times \mathrm{id}_I) \circ (\widetilde{\beta}_1 \times \mathrm{id}_I) = F \circ (p_1 \times \mathrm{id}_I) = p \circ G_1$$

and

$$G_0 \circ (\widetilde{\beta}_1 \times \mathrm{id}_I) \circ i_0 = G_1 \circ i_0\,.$$

Hence, from Theorem 1.4.28, it follows that $G_1 \simeq_B G_0 \circ (\widetilde{\beta}_1 \times \mathrm{id}_I)$. Similarly, $G_0 \simeq_B G_1 \circ (\widetilde{\beta}_0 \times \mathrm{id}_I)$. Thus the mappings

$$\widetilde{E}_1 \ni (a, e) \longmapsto (a, G_0(\widetilde{\beta}_1(a, e), 1)) = \widetilde{\beta}_0 \widetilde{\beta}_1(a, e) \in \widetilde{E}_1$$

$$\widetilde{E}_1 \ni (a,e) \longmapsto (a, G_1(a,e,1)) = (a,e) \in \widetilde{E}_1\,,$$

are homotopic (over A), since $G_1(a,e,1) = e$; similarly, the mappings

$$\widetilde{E}_0 \ni (a,e) \longmapsto (a, G_1(\widetilde{\beta}_0(a,e),0)) = \widetilde{\beta}_1\widetilde{\beta}_0(a,e) \in \widetilde{E}_0$$

$$\widetilde{E}_0 \ni (a,e) \longmapsto (a, G_0(a,e,0)) = (a,e) \in \widetilde{E}_0\,,$$

are homotopic (over A), that is

$$\widetilde{\beta}_0 \circ \widetilde{\beta}_1 \simeq_A \mathrm{id}_{\widetilde{E}_1} \quad \text{and} \quad \widetilde{\beta}_1 \circ \widetilde{\beta}_0 \simeq_A \mathrm{id}_{\widetilde{E}_0}\,. \qquad \square$$

1.4.30 **Corollary**. *If $p : E \longrightarrow B$ is a Hurewicz fibration and B is contractible, then p is fiber homotopy equivalent to the trivial fibration $B \times p^{-1}(b) \longrightarrow B$ for any $b \in B$.*

Proof: If B is contractible, then $\mathrm{id}_B \simeq c_b$, where $c_b : B \longrightarrow B$ is the constant map with value b. Obviously $\mathrm{id}_B^*(p)$ is equivalent to p and by 1.4.23 the induced fibration $c_b^*(p)$ is trivial. Hence, by 1.4.29, p is fiber homotopy equivalent to a trivial fibration. $\square$

1.5 Translation of the Fiber

Given a fibration $p : E \longrightarrow B$, a map $f_0 : X \longrightarrow F_0 = p^{-1}(b_0)$ from a space X to the fiber over a point $b_0 \in B$, and a path $\omega : I \longrightarrow B$ in the base space such that $\omega(0) = b_0$ and $\omega(1) = b_1$, we wish to translate f_0 homotopically in such a way that at the time t we have a map into the fiber F_t over $\omega(t)$. We have the following.

1.5.1 Definition. Under *translation of the fiber* we understand the following. Consider the problem

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widetilde{h}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & B\,, \end{tikzcd}$$

where

$$\widetilde{h}_0(x,0) = f_0(x) \quad \text{and} \quad h(x,t) = \omega(t)\,.$$

We assume further that p has the HLP for X and $X \times I$ (this is not always the case, as seen in 1.4.4 (b)), then we can solve the problem and there exists such a map $\widetilde{h}$. Since $\widetilde{h}(x,1) \in F_1 = p^{-1}(b_1)$ we may define $f_1 : X \longrightarrow F_1$ by $f_1(x) = \widetilde{h}(x,1)$ and say that f_1 *is obtained from f_0 by translation along ω.*

1.5.2 **Theorem**. *Let $f_0, f_0' : X \longrightarrow F_0$ be homotopic maps and let $\omega, \omega' : I \longrightarrow B$ be homotopic paths relative to the end points, such that $\omega(0) = \omega'(0) = b_0$, $\omega(1) = \omega'(1) = b_1$. Assume that f_1, resp. f_1', is obtained from f_0 by translation along ω, resp. from f_0' by translation along ω', then f_1 and f_1' are homotopic.*

Proof: Let $\widetilde{h}$, resp. $\widetilde{h}'$ be a lifting of h, resp. h', such that $\widetilde{h}(x,0) = f_0(x)$, resp. $\widetilde{h}'(x,0) = f_0'(x)$, where h and h' are given by $h(x,t) = \omega(t)$ and $h'(x,t) = \omega'(t)$, respectively. Then we define f_1 and f_1' by $f_1(x) = \widetilde{h}(x,1)$ and $f_1'(x) = \widetilde{h}'(x,1)$.

Let now $g : X \times I \longrightarrow F_0$ be a homotopy such that $g(x,0) = f_0(x)$ and $g(x,1) = f_0'(x)$, and let $\lambda : I \times I \longrightarrow B$ be a homotopy such that $\lambda(s,0) = \omega(s)$ and $\lambda(s,1) = \omega'(s)$, $\lambda(0,t) = b_0$ and $\lambda(1,t) = b_1$ for all s, t.

Consider the problem

$$\begin{tikzcd} X \times (I \times \partial I \cup \{0\} \times I) \arrow[r, "\widetilde{H}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"'] & B\,, \end{tikzcd}$$

where

$$\begin{aligned} H(x,s,t) &= \lambda(s,t) \\ \widetilde{H}_0(x,s,0) &= \widetilde{h}(x,s) \\ \widetilde{H}_0(x,s,1) &= \widetilde{h}'(x,s) \\ \widetilde{H}_0(x,0,t) &= g(x,t)\,. \end{aligned}$$

Since the pair $(X \times I \times I, X \times (I \times \partial I \cup \{0\} \times I))$ is homeomorphic to the pair $(X \times I \times I, X \times \{0\} \times I)$ (see Figure 1.13, and compare with the proof of 1.4.7), and p has the HLP for $X \times I$, the solution of the problem exists.

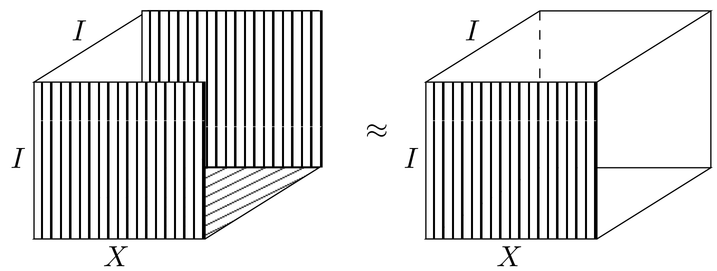


Figure 1.13

Since $p\widetilde{H}(x,1,t) = H(x,1,t) = \lambda(1,t) = b_1$, $F(x,t) = \widetilde{H}(x,1,t)$ defines a homotopy $F : X \times I \longrightarrow F_1$ from f_1 to f_1'. $\square$

1.5.3 **Theorem**. *Let ω_1, resp. ω_2, be a path from b_0 to b_1, resp. from b_1 to b_2, and assume that f_1, resp. f_2, is obtained from f_0, resp. f_1, by translation along ω_1, resp. along ω_2. Then f_2 is obtained from f_0 by translation along the product path $\omega_1\omega_2$.*

Proof: Let $\widetilde{h}_1$ be the lifting that determines f_1 and $\widetilde{h}_2$ the one that determines f_2. Then the homotopy

$$\widetilde{h}(x,t) = \begin{cases} \widetilde{h}_1(x,2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \widetilde{h}_2(x,2t-1) & \text{if } \frac{1}{2} \le t \le 1\,. \end{cases}$$

is such that $p\widetilde{h}(x,t) = (\omega_1\omega_2)(t)$, $\widetilde{h}(x,0) = \widetilde{h}_1(x,0) = f_0(x)$ and $\widetilde{h}(x.1) = \widetilde{h}_2(x,1) = f_2(x)$. □

1.5.4 Definition. Let $[X,Y]$ denote the set of homotopy classes of maps $X \longrightarrow Y$. For each path $\omega : b_0 \simeq b_1$, there is a function $\Phi_\omega : [X,F_0] \longrightarrow [X,F_1]$ that sends the homotopy class of any map $f_0 : X \longrightarrow F_0$ to that of the map $f_1 : X \longrightarrow F_1$ obtained from f_0 by translation along ω.

Theorem 1.5.2 guarantees that the function Φ_ω is well defined, and Theorem 1.5.3 shows that

$$\Phi_{\omega_2} \circ \Phi_{\omega_1} = \Phi_{\omega_1\omega_2}\,.$$

Let $e_{b_\nu} : I \longrightarrow B$ be the constant path with value $e_{b_\nu}(t) = b_\nu$, $\nu = 0,1$. Then

$$\Phi_{e_{b_\nu}} = \mathrm{id}_{[X,F_\nu]}\,,$$

as one can easily verify. Moreover, if $\overline{\omega}$ is the inverse path of ω, then by Theorem 1.5.3 and the previous remark,

$$\Phi_{\overline{\omega}} \circ \Phi_\omega = \mathrm{id}_{[X,F_0]} \quad \text{and} \quad \Phi_\omega \circ \Phi_{\overline{\omega}} = \mathrm{id}_{[X,F_1]}\,.$$

This shows, in particular, that Φ_ω is always bijective. Since by 1.5.2 Φ_ω depends only on the homotopy class of ω, we can summarize all previous remarks in the following theorem. Before stating it we have a definition.

1.5.5 Definition. For a topological space B we define its *fundamental groupoid* $\Pi_1(B)$ as the category whose objects are the points in B, whose morphisms $b_0 \longrightarrow b_1$ are the homotopy classes of paths $\omega : b_0 \simeq b_1$, the identity morphism of each b is $\mathrm{id}_b = [e_b]$, where e_b is the constant path with value b, and the composition is given by the product of paths $[\omega_1] \circ [\omega_0] = [\omega_0\omega_1]$.

We then have the following.

1.5.6 Theorem. *Given a topological space X and a fibration $p : E \longrightarrow B$ that has the HLP for X and $X \times I$, there is a contravariant functor $\Phi : \Pi_1(B) \longrightarrow \mathcal{S}et$ given in the objects by $\Phi(b) = [X, F]$, $F = p^{-1}(b)$, and in the morphisms by $\Phi([\omega]) = \Phi_\omega : [X, F_0] \longrightarrow [X, F_1]$, $F_\nu = p^{-1}(\omega(\nu))$, $\nu = 0, 1$.* □

The fundamental groupoid $\Pi_1(B)$ is a *small category*, that is, its objects constitute a set (the underlying set of the space B). Given a map $f : B \longrightarrow B'$, there is a covariant functor $\widehat{f} : \Pi_1(B) \longrightarrow \Pi_1(B')$ that coincides with f in the objects and is such that for a path $\omega : b_0 \simeq b_1$, one has $\widehat{f}([\omega]) = [f \circ \omega]$. Obviously, the functor $\widehat{f}$ depends only on the homotopy class of f. We have the following.

1.5.7 Proposition. *The assignment $B \mapsto \Pi_1(B)$ is a functor from the homotopy category $\mathcal{T}op^h$ of topological spaces and homotopy classes of maps, to the category $\mathcal{C}at$ of small categories and functors between them.* □

1.5.8 Theorem. *Let X and Y be topological spaces and let $p : E \longrightarrow B$ have the HLP for X, Y, $X \times I$ and $Y \times I$. If ω is a path in B from b_0 to b_1 and $\beta \in [Y, X]$, then the following diagram commutes*

$$\begin{tikzcd} {[X, F_0]} \arrow[r, "\beta^*"] \arrow[d, "\Phi_\omega"'] & {[Y, F_0]} \arrow[d, "\Phi_\omega"] \\ {[X, F_1]} \arrow[r, "\beta^*"'] & {[Y, F_1]}\,, \end{tikzcd}$$

where $F_\nu = p^{-1}(b_\nu)$, $\nu = 0, 1$. In other words, if $\alpha \in [X, F_0]$, then $(\Phi_\omega(\alpha)) \circ \beta = \Phi_\omega(\alpha \circ \beta)$, since by definition, $\beta^*(\alpha) = \alpha \circ \beta$.

Proof: Let $f_0 : X \longrightarrow F_0$ represent the homotopy class α and $g : Y \longrightarrow X$ represent β. Let moreover $\widetilde{h} : X \times I \longrightarrow E$ be a lifting that determines the translation of f_0. So the homotopy $\widetilde{h}' = \widetilde{h} \circ (g \times \mathrm{id}) : Y \times I \longrightarrow E$ determines the translation of $f \circ g$, as one can see in the diagram

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "g"] \arrow[d, hook] & X \times \{0\} \arrow[r, "f_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "g \times \mathrm{id}"'] \arrow[urr, dashed, "\widetilde{h}'"] & X \times I \arrow[r] \arrow[ur, dashed, "\widetilde{h}"'] & B\,, \\ (y, t) \arrow[r, mapsto] & (g(y), t) \arrow[r, mapsto] & \omega(t)\,. \end{tikzcd}$$

The map $y \mapsto \widetilde{h}'(y, 1)$ provides a representative of $\Phi_\omega \beta^*(\alpha)$. On the other hand, $f_1 \circ g$ represents $\beta^* \Phi_\omega(\alpha)$, and since $\widetilde{h}'(y, 1) = f_1 g(y)$, one gets the assertion of the theorem. □

From now on we adopt the following hypothesis: For each fiber $F_b = p^{-1}(b)$ of a fibration $p : E \longrightarrow B$, $b \in B$, there exists a space X_b with the same homotopy type of F_b such that p has the HLP for X_b and $X_b \times I$.

Let $\alpha_0 \in [X_0, F_0]$ be represented by a homotopy equivalence. We define

$$\varphi_\omega = \Phi_\omega(\alpha_0) \circ \alpha_0^{-1} \in [F_0, F_1]\,.$$

If $\beta_0 \in [X_0, F_0]$ is represented by another homotopy equivalence, then by Theorem 1.5.8 we have

$$\begin{aligned}
\Phi_\omega(\alpha_0) \circ \alpha_0^{-1} &= \Phi_\omega(\beta_0 \circ \beta_0^{-1} \circ \alpha_0) \circ \alpha_0^{-1} \circ \beta_0 \circ \beta_0^{-1} \\
&= \Phi_\omega(\beta_0) \circ \beta_0^{-1} \circ \alpha_0 \circ \alpha_0^{-1} \circ \beta_0 \circ \beta_0^{-1} \\
&= \Phi_\omega(\beta_0) \circ \beta_0^{-1}\,.
\end{aligned}$$

So, Φ_ω is independent of the chosen homotopy equivalence α. Let now $\alpha_1 \in [X_1, F_1]$ be a homotopy equivalence. From 1.5.6 and 1.5.8, one has that

$$\begin{aligned}
\varphi_{\omega_1\omega_2} &= \Phi_{\omega_1\omega_2}(\alpha_0) \circ \alpha_0^{-1} = \Phi_{\omega_2}(\Phi_{\omega_1}(\alpha_0)) \circ \alpha_0^{-1} \\
&= \Phi_{\omega_2}(\alpha_1 \circ \alpha_1^{-1} \circ \Phi_{\omega_1}(\alpha_0)) \circ \alpha_0^{-1} \\
&= (\Phi_{\omega_2}(\alpha_1) \circ \alpha_1^{-1} \circ \Phi_{\omega_1}(\alpha_0)) \circ \alpha_0^{-1} \\
&= \varphi_{\omega_2} \circ \varphi_{\omega_1}
\end{aligned}$$

and

$$\varphi_{e_0} = \Phi_{e_0}(\alpha_0) \circ \alpha_0^{-1} = \alpha_0 \circ \alpha_0^{-1} = [\mathrm{id}] \in [F_0, F_0]\,.$$

Thus φ is a functor from the fundamental groupoid of B, $\Pi_1(B)$, to the homotopy category $\mathcal{T}op^h$. In particular we have the following.

1.5.9 **Theorem**. *Let $p : E \longrightarrow B$ be either*

(a) *a Hurewicz fibration, or*

(b) *a Serre fibration such that each of its fibers has the homotopy type of a* CW*-complex.*

Then there is a functor

$$\begin{aligned}
\varphi : \Pi_1(B) &\longrightarrow \mathcal{T}op^h \\
B \ni b &\longmapsto p^{-1}(b) \\
(\omega : b_1 \simeq b_2) &\longmapsto \varphi_\omega \in [p^{-1}(b_1), p^{-1}(b_2)]\,.
\end{aligned}$$

□

There are some consequences of the previous theorem. Since every morphism in the fundamental groupoid is an isomorphism we have the following.

1.5.10 **Corollary**. *φ_ω is a homotopy equivalence for every ω.* □

Another is the following.

1.5.11 **Corollary**. *If B is path connected (0-connected), then all the fibers of p have the same homotopy type.* □

1.6 Homotopy Sets and Homotopy Groups

In this section we analyze sets of homotopy classes of pointed maps between two pointed spaces. We study when these sets have a group structure, and as special cases, we shall obtain the homotopy groups of a space, and particularly, its fundamental group.

1.6.1 Definition. Under a *pointed topological space* we shall understand a pair $(X, *)$ consisting of a topological space X and a *base point* $* \in X$. A *pointed map* between pointed spaces is a continuous map $f : X \longrightarrow Y$ such that $f(*) = *$. A *pointed homotopy* is a homotopy $h : X \times I \longrightarrow Y$ such that $h(*, t) = *$ for every $t \in I$.

Pointed spaces and pointed maps build a category, $\mathcal{T}op_*$ that will be the one we shall work with in this section. Therefore, we shall frequently omit the adjective "pointed" in the sequel.

1.6.2 Definition. Let X and Y be pointed spaces. We shall denote by $\pi(X, Y)$ the set of pointed homotopy classes of pointed maps $a : X \longrightarrow Y$. By $k : X \longrightarrow Y$, given by $k(x) = *$, $x \in X$, we denote the constant map whose homotopy class $[k] \in \pi(X, Y)$ represents a special element in $\pi(X, Y)$ that will be denoted by $0 = [k]$. Let $f : X' \longrightarrow X$, $g : Y \longrightarrow Y'$ be (pointed) maps. We define a function

$$\begin{aligned} \pi(f, g) : \pi(X, Y) &\longrightarrow \pi(X', Y') \\ [a] \quad &\longmapsto [g \circ a \circ f] \end{aligned}$$

that does not depend on the choice of the representative $a \in [a]$. The following rules are easily verified.

$$\begin{aligned} \pi(f', g') \circ \pi(f, g) &= \pi(f \circ f', g' \circ g)\,, \\ \pi(\mathrm{id}_X, \mathrm{id}_Y) &= \mathrm{id}_{\pi(X,Y)}\,, \\ \pi(f, g)(0) &= 0\,. \end{aligned}$$

We thus have the following.

1.6.3 **Theorem**. *π is a two-variable functor (contravariant in the first variable and covariant in the second) from the category $\mathcal{T}op_*$ of pointed spaces and pointed maps to the category $\mathcal{S}et_*$ of pointed sets and pointed functions.* □

We shall use the following notatation

$$g_* = \pi(\mathrm{id}, g)\,, \quad f^* = \pi(f, \mathrm{id})\,.$$

1.6.4 DEFINITION. Let X and Y be pointed spaces. We define their *smash product* $X \wedge Y$ as the quotient space

$$X \wedge Y = X \times Y / X \vee Y\,,$$

where their *wedge sum*, or simply *wedge*, $X \vee Y$ is defined by $X \vee Y = X \times \{*\} \cup \{*\} \times Y \subset X \times Y$. The base point of $X \wedge Y$ is the image of $X \vee Y$ (or of $(*, *)$) under the quotient map $q : X \times Y \longrightarrow X \wedge Y$. The point $q(x, y) \in X \wedge Y$ will be denoted by $x \wedge y$. One has that $x \wedge * = * \wedge y = *$.

1.6.5 **Theorem**. *There are natural pointed homeomorphisms*

(1) $X \wedge Y \approx Y \wedge X$,

(2) $(X \wedge Y) \wedge Z \approx X \wedge (Y \wedge Z)$ *if X and Z are locally compact, or if X and Y are compact, or if all involved spaces are compactly generated and one takes the compactly generated product instead (see [1, 4.3.22], [13, 6.7] or [16]).*

Proof: The homeomorphism in (1) is induced by the homeomorphism $T : X \times Y \longrightarrow Y \times X$ given by $T(x, y) = (y, x)$.

For (2) we have the following diagram

$$\begin{tikzcd}
 & X \times Y \times Z \arrow[dl, "\pi \times \mathrm{id}_Z"'] \arrow[dr, "\mathrm{id}_X \times \pi'"] & \\
(X \wedge Y) \times Z \arrow[d] & & X \times (Y \wedge Z) \arrow[d] \\
(X \wedge Y) \wedge Z \arrow[rr, dashed, "f"'] & & X \wedge (Y \wedge Z)\,,
\end{tikzcd}$$

where π, π' as well as the two vertical maps are identifications. f defines a bijection such that $f((x \wedge y) \wedge z) = x \wedge (y \wedge z)$. f will be a homeomorphism when the maps $\pi \times \mathrm{id}_Z$ and $\mathrm{id}_X \times \pi'$ are identifications. This is the case under the given hypotheses. □

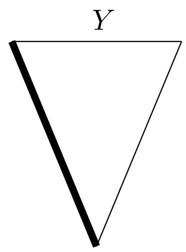
1.6.6 Examples.

(a) $* \wedge Y = *$.

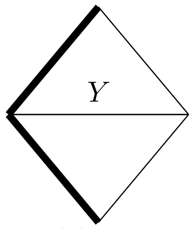
(b) $\mathbb{S}^0 \wedge Y \approx Y$.

(c) $I \wedge Y = CX$ with $0 \in I$ as the base point is the (*reduced*) *cone* of Y. (See Figure 1.14 (c), where the thick line represents the base point.)

(d) $\mathbb{S}^1 \wedge Y = \Sigma Y$ is the (*reduced*) *suspension* of Y. (See Figure 1.14 (d), where the thick line represents the base point.)



(c) CY



(d) ΣY

Figure 1.14

(e) Let $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ be the unit n-sphere with $* = (1, 0, 0, ..., 0)$ as the base point.

There is a pointed homeomorphism

$$\varphi : \Sigma\mathbb{S}^n = \mathbb{S}^1 \wedge \mathbb{S}^n \approx \mathbb{S}^{n+1}$$

given as follows. If we describe the points of $\mathbb{S}^1$ by

$$(\cos 2t, \sin 2t)\,, \quad t \in [0, \pi]\,,$$

then φ is given by

$$\varphi((\cos 2t, \sin 2t) \wedge (x_0, \ldots, x_n)) =$$
$$= \left(\cos^2 t + x_0 \sin^2 t, x_1 \sin^2 t, \ldots, x_n \sin^2 t, \sqrt{\tfrac{1-x_0}{2}} \sin 2t\right) \in \mathbb{S}^{n+1}\,.$$

1.6.7 Definition. Let $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$ be pointed maps. We define $f \wedge g : X \wedge Y \longrightarrow X' \wedge Y'$ by

$$(f \wedge g)\ (x \wedge y) = f(x) \wedge f(y).$$

$f \wedge g$ is continuous, since in the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "f\times g"] \arrow[d, "q"'] & X' \times Y' \arrow[d] \\ (X \wedge Y) \arrow[r, "f\wedge g"'] & X' \wedge Y', \end{tikzcd}$$

q is an identification.

1.6.8 **Theorem**.

(1) $\wedge$ *is a two-variable covariant functor.*

(2) $\wedge$ *is compatible with the homotopy relation, i.e., if* $f_0 \simeq f_1$ *and* $g_0 \simeq g_1$*, then* $f_0 \wedge g_0 \simeq f_1 \wedge g_1$.

Proof: (1) follows immediately.

(2) is obtained as follows: Let $h : I \times X \longrightarrow X'$ be a homotopy between f_0 and f_1, and let $q : X \times Y \longrightarrow X \wedge Y$, $q' : X' \times Y' \longrightarrow X' \wedge Y'$ be the respective identifications. Then in the diagram

$$\begin{tikzcd} I \times X \times Y \arrow[r, "h\times g"] \arrow[d, "\mathrm{id}\times q"'] & X' \times Y' \arrow[d, "q'"] \\ I \times (X \wedge Y) \arrow[r] & X' \wedge Y' \end{tikzcd}$$

the map $\mathrm{id}\times q$ is again an identification and therefore the arrow at the bottom describes a homotopy $f_0 \wedge g \simeq f_1 \wedge g$. To prove $f \wedge g_0 \simeq f \wedge g_1$ one proceeds similarly; the general case follows combining the two previous cases. □

1.6.9 DEFINITION. Let $\mathbb{S}^1 = I/\{0,1\}$, where we denote its points simply by their inverse images in I. Let $0 \in \mathbb{S}^1$ be the base point. Let moreover $f, g : \Sigma X \longrightarrow Y$ be pointed maps. We define $f + g : \Sigma X \longrightarrow Y$ by

$$(f+g)(t \wedge x) = \begin{cases} f(2t \wedge x) & \text{if } 0 \le t \le \frac{1}{2}, \\ g((2t-1) \wedge x) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

$f + g$ is well defined and is continuous. If f_t and g_t are homotopies, then also $f_t + g_t$ is one, so that $[f] + [g] = [f + g]$ defines an operation "+" in $\pi(\Sigma X, Y)$.

1.6.10 **Theorem**. $(\pi(\Sigma X, Y); +)$ *is a group with the selected element* 0 *as neutral element.*

Proof: Observe that, as we did above, one can write a homotopy as a family

$$h_t : X \longrightarrow Y\,, \qquad h_t(x) = h(x,t)\,,$$

of pointed maps. On the other hand, a path $f : I \longrightarrow X$ such that $f(0) = f(1)$ induces a continuous map $f : \mathbb{S}^1 \longrightarrow X$ (we denote it by the same symbol).

Associativity: The map

$$\varphi_t(s) = \begin{cases} \frac{1}{2}s(2-t) & \text{if } 1 \le s \le \frac{1}{2}, \\ s - \frac{1}{4}t & \text{if } \frac{1}{2} \le s \le \frac{3}{4}, \\ s(1+t) & \text{if } \frac{3}{4} \le s \le 1, \end{cases}$$

describes a pointed homotopy $\varphi_t : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$. By 1.6.8

$$((f+g)+h) \circ (\varphi_t \wedge x) : \mathbb{S}^1 \wedge X \longrightarrow Y$$

is a homotopy. From the fact that $\varphi_0 = \mathrm{id}_{\mathbb{S}^1}$ and that $((f+g)+h) \circ (\varphi_1 \wedge x) = f + (g+h)$ the associativity is obtained.

Neutral element: The map

$$\psi_t(s) = \begin{cases} s(1+t) & \text{if } 0 \le s \le \frac{1}{2}, \\ t + (1-t)s & \text{if } \frac{1}{2} \le s \le 1, \end{cases}$$

gives a homotopy $\psi_t : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$. If k is the constant map, we have that $g_t = f \circ (\psi_t \wedge \mathrm{id}_X) : \mathbb{S}^1 \wedge X \longrightarrow Y$ is a homotopy between $g_0 = f$ and $g_1 = f + k$.

Existence of the inverse: By

$$\overline{f}(t \wedge x) = f((1-t) \wedge x)$$

a continuous map $\overline{f} : \Sigma X \longrightarrow Y$ is defined. The homotopy $\chi_t : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ given by

$$\chi_t(s) = \begin{cases} 2st & \text{if } 0 \le s \le \frac{1}{2}, \\ 2t(1-s) & \text{if } \frac{1}{2} \le s \le 1, \end{cases}$$

is such that $f \circ (\chi_0 \wedge \mathrm{id}_X) = k$ and $f \circ (\chi_1 \wedge \mathrm{id}_X) = f + \overline{f}$. □

1.6.11 Definition. For the special case $X = \mathbb{S}^{n-1}$, we define

$$\pi_n(Y) = \pi(\Sigma\mathbb{S}^{n-1}, Y)\,, \qquad n \ge 1$$

and call it the *nth homotopy group* of Y. In particular, for $n = 1$ we call it the *fundamental group* of Y. This last group is not necessarily abelian.

1.6.12 EXERCISE. Prove that the fundamental group $\pi_1(X)$ is the group of isomorphisms of the base point to itself in the fundamental grupoid $\Pi_1(X)$ defined above in 1.5.5.

1.6.13 **Theorem**. *Let $f : Y \longrightarrow Y'$, $g : X' \longrightarrow X$ be pointed maps. Then*

$$f_* : \pi(\Sigma X, Y) \longrightarrow \pi(\Sigma X, Y')$$

and

$$(\Sigma g)^* : \pi(\Sigma X, Y) \longrightarrow \pi(\Sigma X', Y')$$

are homomorphisms, where $\Sigma g = \mathrm{id}_{\mathbb{S}^1} \wedge g$.

Proof: Let $a, b : \Sigma X \longrightarrow Y$ represent two elements in $\pi(\Sigma X, Y)$. One has

$$\begin{aligned}(f \circ (a+b))\,(t \wedge x) &= \begin{cases} f \circ a\ (2t \wedge x) & \text{if } 0 \le t \le \frac{1}{2}, \\ g \circ b((2t-1) \wedge x) & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \\ &= ((f \circ a) + (f \circ b))\,(t \wedge x)\,;\end{aligned}$$

hence $f_*([a] + [b]) = f_*[a] + f_*[b]$, and so f_* is a homomorphism.

On the other hand, the equalities

$$\begin{aligned}((a+b) \circ (\mathrm{id} \wedge g))\,(t \wedge x) &= (a+b)(t \wedge g(x)) \\ &= \begin{cases} a(2t \wedge g(x)) & \text{if } 0 \le t \le \frac{1}{2}, \\ b((2t-1) \wedge g(x)) & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \\ &= \begin{cases} a \circ (\mathrm{id} \wedge g)\ (2t \wedge x) & \text{if } 0 \le t \le \frac{1}{2}, \\ b \circ (\mathrm{id} \wedge g)\ ((2t-1) \wedge x) & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \\ &= (a \circ \Sigma g + b \circ \Sigma g)(t \wedge x)\,,\end{aligned}$$

imply that $(\Sigma g)^*([a] + [b]) = (\Sigma g)^*[a] + (\Sigma g)^*[b]$; therefore, $(\Sigma g)^*$ is a homomorphism. □

1.6.14 REMARK. Not every map $\Sigma X' \longrightarrow \Sigma X$ induces a homomorphism $\pi(\Sigma X, Y) \longrightarrow \pi(\Sigma X', Y)$. For example, take $X = \mathbb{S}^1$, $X' = \mathbb{S}^2$, $Y = \mathbb{S}^2$, and let $h : \Sigma X' = \mathbb{S}^3 \longrightarrow \mathbb{S}^2 = \Sigma X$ be the Hopf fibration 1.3.1 (b). Then $\pi(\mathbb{S}^2, \mathbb{S}^2) = \pi_2(\mathbb{S}^2) \cong \mathbb{Z}$, $\pi(\mathbb{S}^3, \mathbb{S}^2) = \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ (cf. Subsection **1.8.2**), and h^* is given by $h^*(n) = n^2$, therefore, h^* is not a homomorphism.

To prove the last assertion, the argument is as follows: h has Hopf invariant 1 (see [1, 10.6]) and if $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ has degree n ($[f] = n$), then $f \circ h$ has Hopf invariant $n^2 \cdot 1$. The assignment $(g : \mathbb{S}^3 \longrightarrow \mathbb{S}^2) \mapsto$ (Hopf invariant of g) induces an isomorphism $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ (cf. [**?**, **?**]).

1.6.15 Note. By 1.6.5, $\mathbb{S}^1 \wedge (\mathbb{S}^1 \wedge X) = \Sigma(\Sigma X)$ is homeomorphic to $(\mathbb{S}^1 \wedge \mathbb{S}^1) \wedge X$ through the map $s \wedge (t \wedge x) \mapsto (s \wedge t) \wedge x$. Thus $\Sigma\Sigma X = \Sigma^2 X$ can be considered as

$$I \times I \times X/(\partial I)^2 \times X \cup I^2 \times *$$

and we can denote the image of (s,t,x) under the identification simply as $s \wedge t \wedge x$.

We may define another operation $+'$ between two maps $f, g : \Sigma^2 X \longrightarrow Y$ as follows

$$(f +' g)\ (s \wedge t \wedge x) = \begin{cases} f(s \wedge 2t \wedge x) & \text{if } 0 \le t \le \frac{1}{2}, \\ g(s \wedge (2t-1) \wedge x) & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

and analogously to 1.6.9 one can show the compatibility of $+'$ with the homotopy relation (i.e., $+'$ induces a well-defined operation in $\pi(\Sigma^2 X, Y)$ with k as two-sided neutral element, namely, $k +' f \cong f +' k$).

1.6.16 **Theorem**. *$+$ and $+'$ induce the same group operation in $\pi(\Sigma^2 X, Y)$ and this group is abelian.*

Proof: Take $f, f', g, g' : \Sigma^2 X \longrightarrow Y$. One has

$$\begin{aligned} ((f+g) +' (f'+g'))\,(s \wedge t \wedge x) \ &= \\ &= \begin{cases} (f+g)(s \wedge 2t \wedge x) & \text{if } 0 \le t \le \frac{1}{2}, \\ (f'+g')(s \wedge (2t-1) \wedge x) & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \\ &= \begin{cases} f(2s \wedge 2t \wedge x) & \text{if } 0 \le s \le \frac{1}{2},\ 0 \le t \le \frac{1}{2}, \\ g((2s-1) \wedge 2t \wedge x) & \text{if } \frac{1}{2} \le s \le 1,\ 0 \le t \le \frac{1}{2}, \\ f'(2s \wedge (2t-1) \wedge x) & \text{if } 0 \le s \le \frac{1}{2},\ \frac{1}{2} \le t \le 1, \\ g'((2s-1) \wedge (2t-1) \wedge x) & \text{if } \frac{1}{2} \le s \le 1,\ \frac{1}{2} \le t \le 1, \end{cases} \\ &= \begin{cases} (f +' f')(2s \wedge t \wedge x) & \text{if } 0 \le s \le \frac{1}{2}, \\ (g +' g')((2s-1) \wedge t \wedge x) & \text{if } \frac{1}{2} \le s \le 1, \end{cases} \\ &= \ ((f +' f') + (g +' g'))(s \wedge t \wedge x)\,. \end{aligned}$$

from there one obtains by taking special values for the maps

$$\begin{aligned} & f + g \simeq (f +' k) + (k +' g) = (f + k) +' (k + g) \ \simeq \ f +' g\,; \\ & f + g \simeq (k +' f) + (g +' k) = (k + g) +' (f + k) \simeq g +' f \simeq g + f \\ & \qquad \simeq g + f\,. \end{aligned}$$

The first of these equation shows that $[f] + [g] = [f] +' [g]$, and the second, that $[f] + [g] = [g] + [f]$. $\square$

1.6.17 DEFINITION. The *suspension function*

$$\Sigma : \pi(X, Y) \longrightarrow \pi(\Sigma X, \Sigma Y)$$

is given by $\Sigma[f] = [\Sigma f]$ if $[f] \in \pi(X, Y)$ is the homotopy class of a map $f : X \longrightarrow Y$ and $\Sigma f = \mathrm{id}_{\mathbb{S}^1} \wedge f$.

1.6.18 **Theorem**. *The suspension function*

$$\Sigma : \pi(\Sigma X, Y) \longrightarrow \pi(\Sigma^2 X, \Sigma Y)$$

is a group homomorphism. We shall call it henceforth the *suspension homomorphism.*

Proof: Just observe that one has

$$\begin{aligned}
\Sigma(f+g)\ (s \wedge t \wedge x) &= s \wedge (f+g)\ (t \wedge x) \\
&= \begin{cases} s \wedge f(2t \wedge x) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ s \wedge g((2t-1) \wedge x) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases} \\
&= \begin{cases} \Sigma f(s \wedge 2t \wedge x) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \Sigma g(s \wedge (2t-1) \wedge x) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases} \\
&= (\Sigma f +' \Sigma g)\ \ (s \wedge t \wedge x)\,.
\end{aligned}$$

Hence, $\Sigma([f]+[g]) = \Sigma[f] +' \Sigma[g]$ and by 1.6.16 one gets the assertion. □

1.6.19 REMARK. (**Freudenthal suspension theorem**) Under adequate hypotheses on X and Y, the function $\Sigma : \pi(X, Y) \longrightarrow \pi(\Sigma X, \Sigma Y)$ is a bijection; for example, if $\pi_i(Y) = 0$ for $i < n$, and X is a CW-complex such that $\dim X < 2n - 1$.

In particular, if $X = \mathbb{S}^m$, $Y = \mathbb{S}^n$, $m < 2n - 1$, then

$$\Sigma : \pi_m(\mathbb{S}^n) \longrightarrow \pi_{m+1}(\mathbb{S}^{n+1})$$

is an isomorphism (cf. [1, 6.2.4]).

1.7 THE EXACT HOMOTOPY SEQUENCE OF A FIBRATION

One of the most useful algebraic tools is that of an exact sequence. In this section we show how the homotopy sets and groups introduced in the previous section fit together to yield a long exact sequence.

We shall work here under the following assumptions: All spaces, maps and homotopies, as well as all constructions made, will be pointed. It will usually be easy to distinguish in the new constructed spaces, which is the base point. The base point will be generically denoted by $*$, as we do for the one-point space. A map $f : X \longrightarrow Y$ will be called *nullhomotopic* if it is homotopic to the constant map; this fact will be denoted by $f \simeq 0$. The fibration $p : E \longrightarrow B$ will always be a Serre fibration, and $i : F = p^{-1}(*) \hookrightarrow E$ will denote the inclusion of the fiber (F and E have the same base point). X will be a CW-complex and $* \in X$ will be a 0-cell of some adequate CW-decomposition.

1.7.1 **Lemma**. *The sequence*

$$\pi(X, F) \xrightarrow{i_*} \pi(X, E) \xrightarrow{p_*} \pi(X, B)$$

is exact as a sequence of pointed sets. That is, the image of i_*, $\operatorname{im}(i_*) = i_*(\pi(X, F))$, *is equal to the kernel of* p_*, $\ker(p_*) = p_*^{-1}(0)$.

This exactness concept is consistent with the usual exactness concept for sequences of groups, provided that one takes the neutral elements 0 (that are the homotopy classes of the constant maps) of the groups as base points.

Proof: $\operatorname{im}(i_*) \subset \ker(p_*)$, since $pi(F) = \{*\}$.

$\ker(p_*) \subset \operatorname{im}(i_*)$: If $[f] \in \ker(p_*)$, then $p \circ f \simeq 0$. Let $h_t : X \longrightarrow B$ be a homotopy such that $h_0 = p \circ f$, $h_1 = k$, k the constant map. We apply the HLP for the pair $(X, *)$ (cf. 1.4.7) in the diagram

$$\begin{tikzcd} X \times \{0\} \cup \{*\} \times I \arrow[r, "\widetilde{h}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & B \end{tikzcd}$$

to obtain a lifting $\widetilde{h}$ of h, if we define $\widetilde{h}_0$ by $\widetilde{h}_0(x, 0) = f(x)$ and $\widetilde{h}_0(*, t) = *$.

Since $p\widetilde{h}(x, 1) = h(x, 1) = *$, $h(x, 1) \in F$ and it determines a map $g : X \longrightarrow F$ by setting $g(x) = h(x, 1)$. Then $\widetilde{h} : f \simeq i \circ g$, so that one has $i_*[g] = [i \circ g] = [f]$. Thus $[f] \in \operatorname{im}(i_*)$. □

In what follows, we shall define the *connecting homomorphism* $\Delta : \pi(\Sigma X, B) \longrightarrow \pi(X, F)$.

Let $q : I \times X \longrightarrow \Sigma X = I \times X/\partial I \times X \cup I \times \{*\}$ be the natural identification. For each $f : \Sigma X \longrightarrow B$, $f \circ q$ is a homotopy. The problem

$$\begin{tikzcd}
\{0\} \times X \cup I \times \{*\} \arrow[rr, "\text{const}"] \arrow[d, hook] & & E \arrow[d, "p"] \\
I \times X \arrow[r, "q"'] \arrow[urr, dashed, "\widetilde{h}"] & \Sigma X \arrow[r, "f"'] & B
\end{tikzcd}$$

has a solution under our general assumptions.

The equality $p\widetilde{h}(1, x) = fq(1, x) = f(*) = *$ means that there exists a (unique) map $g : X \longrightarrow F$ such that $g(x) = \widetilde{h}(1, x)$.

If g is obtained from f as shown above, we shall briefly say that g *corresponds* to f (*through* $\widetilde{h}$).

1.7.2 **Lemma**. *If $f_0, f_1 : \Sigma X \longrightarrow B$ are homotopic, and g_0 and g_1 correspond to f_0 and f_1, respectively, then g_0 and g_1 are homotopic.*

Proof: Let $\widetilde{h}_\nu : I \times X \longrightarrow E$ be the homotopy through which g_ν is constructed starting from f_ν ($\nu = 0, 1$), and let f_t be a homotopy between f_0 and f_1.

Consider the problem

$$\begin{tikzcd}
\{0\} \times (I \times X) \cup I \times (\partial I \times X \cup I \times \{*\}) \arrow[r, "\widetilde{H}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\
I \times I \times X \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B
\end{tikzcd}$$

where $\widetilde{H}_0$ and H are defined by

$$\begin{aligned}
H(s, t, x) &= f_t(s \wedge x)\,, \\
\widetilde{H}_0(0, t, x) &= *\,, \\
\widetilde{H}_0(s, t, *) &= *\,, \\
\widetilde{H}_0(s, 0, x) &= \widetilde{h}_0(s, x)\,, \\
\widetilde{H}_0(s, 1, x) &= \widetilde{h}_1(s, x)\,.
\end{aligned}$$

The HLP for the pair $(I \times X, \partial I \times X \cup I \times \{*\})$ provides us with the existence of $\widetilde{H}$. Since $p\widetilde{H}(1, t, x) = H(1, t, x) = *$, $\widetilde{H}$ determines a homotopy $g'_t : X \longrightarrow F$ through $g'_t(x) = \widetilde{H}(1, t, x)$, where $g'_0 = g_0$ and $g'_1 = g_1$; g'_t is thus the desired homotopy. $\square$

1.7.3 Definition. Define Δ by

$$\Delta[f] = [g]\,,$$

where $[f] \in \pi(\Sigma X, B)$ and $g : X \longrightarrow F$ corresponds to f.

1.7.4 **Lemma**. *The sequence*

$$\pi(\Sigma X, E) \xrightarrow{p_*} \pi(\Sigma X, B) \xrightarrow{\Delta} \pi(X, F) \xrightarrow{i_*} \pi(X, E)$$

is exact.

Proof: $\operatorname{im}(\Delta) \subset \ker(i_*)$: If $g : X \longrightarrow F$ corresponds to $f : \Sigma X \longrightarrow B$, then let $\widetilde{h} : I \times X \longrightarrow E$ be the homotopy through which g is defined (cf. 1.7.3). $\widetilde{h}$ is a homotopy that starts with k (the constant map) and ends with $i \circ g$. Thus $i_*\Delta[f] = i_*[g] = [i \circ g] = [k] = 0$.

$\operatorname{im}(\Delta) \supset \ker(i_*)$: Take $g : X \longrightarrow F$ and let $\widetilde{h} : I \times X \longrightarrow E$ be a (null)homotopy such that $\widetilde{h}(0, x) = *$ and $\widetilde{h}(1, x) = ig(x) = g(x) \in F$. Let moreover $h = p \circ \widetilde{h}$. Since $h(\partial I \times X \cup I \times \{*\}) = \{*\}$, the map h is compatible with the identification q and so it determines a continuous map $f : \Sigma X \longrightarrow B$ such that $f \circ q = h$. In a diagram

$$\begin{tikzcd} I \times X \arrow[rr, "q"] \arrow[dr, "h"'] & & \Sigma X \arrow[dl, dashed, "f"] \\ & B\,. & \end{tikzcd}$$

Clearly, g corresponds to f through $\widetilde{h}$, that is, $\Delta[f] = [g]$. Thus $[g] \in \operatorname{im}(\Delta)$.

$\operatorname{im}(p_*) \subset \ker(\Delta)$: Take $\widetilde{f} : \Sigma X \longrightarrow E$ and consider the commutative diagram

$$\begin{tikzcd} \{0\} \times X \cup I \times \{*\} \arrow[rr, "\mathrm{const}"] \arrow[d, hook] & & E \arrow[d, "p"] \\ I \times X \arrow[r, "q"'] \arrow[urr, "\widetilde{h}"] & \Sigma X \arrow[r, "p\circ\widetilde{f}"'] \arrow[ur, "\widetilde{f}"'] & B\,, \end{tikzcd}$$

where $\widetilde{h}(t, x) = \widetilde{f}(t \wedge x)$. Thus $g : x \mapsto \widetilde{h}(1, x) = \widetilde{f}(1 \wedge x) = *$ corresponds to $f = p \circ \widetilde{f}$. In other words, $\Delta p^*[\widetilde{f}] = \Delta[f] = [g] = [k] = 0$.

$\operatorname{im}(p^*) \supset \ker(\Delta)$: If $g : X \longrightarrow F$ corresponds to $f : \Sigma X \longrightarrow Y$ through $\widetilde{h} : I \times X \longrightarrow E$, and is such that $[g] = *$, then let $g_t : X \longrightarrow F$ be a nullhomotopy such that $g_0 = g$ and $g_1 = k$, and define $\widetilde{h}' : I \times X \longrightarrow E$ by

$$\widetilde{h}'(t, x) = \begin{cases} \widetilde{h}(2t, x) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ g_{2t-1}(x) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The map $\widetilde{h}'$ is compatible with the identification $q : I \times X \longrightarrow \Sigma X$ and therefore it defines a map $\widetilde{f} : \Sigma X \longrightarrow E$ such that $q \circ \widetilde{f} = \widetilde{h}'$. Now, $p \circ \widetilde{f} = f + k \simeq f$. Thus $p_*[\widetilde{f}] = [f]$, i.e., $[f] \in \operatorname{im}(p_*)$. $\square$

1.7.5 **Lemma**. $\Delta : \pi(\Sigma^2 X, B) \longrightarrow \pi(\Sigma X, F)$ *is a homomorphism.*

Proof: Take $f_\nu : \Sigma^2 X \longrightarrow B$, $\nu = 1, 2$, and let g_ν correspond to f_ν through $\widetilde{h}_\nu$. Add f_1 and f_2 with respect to the second coordinate (see 1.6.15, 1.6.16). Defining $\widetilde{h} : I \times \Sigma X \longrightarrow E$ by

$$\widetilde{h}(t, s \wedge x) = \begin{cases} \widetilde{h}_1(t, 2s \wedge x) & \text{if } 0 \le s \le \frac{1}{2}, \\ \widetilde{h}_2(t, (2s-1) \wedge x) & \text{if } \frac{1}{2} \le s \le 1, \end{cases}$$

one obtains the following commutative diagram.

$$\begin{tikzcd}
\{0\} \times \Sigma X \cup I \times \{*\} \arrow[rr, "\text{const}"] \arrow[d, hook] & & E \arrow[d, "p"] \\
I \times \Sigma X \arrow[r, "q"'] \arrow[urr, "\widetilde{h}"] & \Sigma^2 X \arrow[r, "f_1 +' f_2"'] & B\,.
\end{tikzcd}$$

Thus the map g given by

$$\begin{aligned} g(s \wedge x) = \widetilde{h}(1, s \wedge x) &= \begin{cases} g_1(2s \wedge x) & \text{if } 0 \le s \le \frac{1}{2}, \\ g_2((2s-1) \wedge x) & \text{if } \frac{1}{2} \le s \le 1, \end{cases} \\ &= (g_1 + g_2)(s \wedge x)\,, \end{aligned}$$

corresponds to $f_1 +' f_2$ through $\widetilde{h}$, i.e., $\Delta([f_1] + [f_2]) = \Delta[f_1 +' f_2] = [g] = [g_1 + g_2] = [g_1] + [g_2] = \Delta[f_1] + \Delta[f_2]$. □

In what follows we set $\pi_n(X, Y) = \pi(\Sigma^n X, Y)$, $n = 0, 1, 2, \ldots$.

1.7.6 **Theorem**. *Given a Serre fibration $p : E \longrightarrow B$ and a pointed* CW-*complex X, the long sequence*

$$\cdots \xrightarrow{\Delta} \pi_n(X, F) \xrightarrow{i_*} \pi_n(X, E) \xrightarrow{p_*} \pi_n(X, B) \xrightarrow{\Delta} \pi_{n-1}(X, F) \xrightarrow{i_*}$$
$$\longrightarrow \cdots \xrightarrow{\Delta} \pi_0(X, F) \xrightarrow{i_*} \pi_0(X, E) \xrightarrow{p_*} \pi_0(X, B)$$

is exact, and all arrows (maybe excepting the last three) represent group homomorphisms.

The *proof* combines 1.7.1, 1.7.4 and 1.7.5. □

1.7.7 EXERCISE. Let $f : Y \longrightarrow X$ be a map between CW-complexes. Prove that the diagram

$$\begin{tikzcd}
\pi_n(X, F) \arrow[r, "i_*"] \arrow[d, "(\Sigma^n f)^*"] & \pi_n(X, E) \arrow[r, "p_*"] \arrow[d, "(\Sigma^n f)^*"] & \pi_n(X, B) \arrow[r, "\Delta"] \arrow[d, "(\Sigma^n f)^*"] & \pi_{n-1}(X, F) \arrow[d, "(\Sigma^{n-1} f)^*"] \\
\pi_n(Y, F) \arrow[r, "i_*"] & \pi_n(Y, E) \arrow[r, "p_*"] & \pi_n(Y, B) \arrow[r, "\Delta"] & \pi_{n-1}(Y, F)
\end{tikzcd}$$

commutes.

In general, $\pi(X, F)$ does not have a group structure; Δ is not a homomorphism in those cases. However, it is true that Δ sends the right cosets of $\mathrm{im}(p_*)$ exactly onto one element. We have the following.

1.7.8 **Theorem**. *$\Delta\alpha_0 = \Delta\alpha_1$ if and only if $\alpha_0 - \alpha_1 \in \mathrm{im}(p_*)$.*

Proof: Assume that $\Delta\alpha_0 = \Delta\alpha_1$. Let f_ν represent α_ν, and let g_ν correspond to f_ν through a homotopy $\widetilde{h}_\nu$, $\nu = 0, 1$. By assumption, $g_0 \simeq g_1$, say via the homotopy g_t. If we define a map $l : \Sigma X \longrightarrow E$ by

$$l(s \wedge x) = \begin{cases} \widetilde{h}_0(4s, x) & \text{if } 0 \leq s \leq \frac{1}{4}, \\ g_{4s-1}(x) & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2}, \\ \widetilde{h}_1(2 - 2s, x) & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

then we have that $p \circ l = (f_0 + k) + \overline{f}_1$ ($\overline{f}_1$ is the inverse of f_1), since by definition of $\widetilde{h}_\nu$, one has that $p\widetilde{h}_\nu(s, x) = f_\nu(s \wedge x)$. Passing to homotopy classes we have

$$p_*[l] = [p \circ l] = [f_0] + [k] + [\overline{f}_1] = [f_0] - [f_1] = \alpha_0 - \alpha_1 \in \mathrm{im}(p_*)\,.$$

Conversely, let us suppose that $\alpha_0 - \alpha_1 \in \mathrm{im}(p_*)$. More specifically, $\alpha_0 = p_*(\beta) + \alpha_1$. Choose representatives f_1 of α_1 and l of β and take $f_0 = p \circ l + f_1$ as a representative of α_0.

If g_1 corresponds to f_1 through $\widetilde{h}_1$, then define $\widetilde{h}_0$ by

$$\widetilde{h}_0(s, x) = \begin{cases} l(2s \wedge x) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \widetilde{h}_1(2s - 1, x) & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

to obtain that $p \circ \widetilde{h}_0 = (p \circ l + f_1) \circ q = f_0 \circ q$, so that $\widetilde{h}_0$ lifts $f_0 \circ q$. Since $\widetilde{h}_0(1, x) = \widetilde{h}_1(1, x) = g_1(x)$, then g_1 also corresponds to f_0, that is,

$$\Delta\alpha_1 = \Delta[f_1] = [g_0] = \Delta[f_0] = \Delta\alpha_0\,.$$

□

1.7.9 **Theorem**. *If $i : F \hookrightarrow E$ is nullhomotopic, then*

$$\Delta : \pi_n(X, B) \longrightarrow \pi_{n-1}(X, F)$$

has a right inverse homomorphism (if $n \geq 2$).

Proof: Assume first that $n = 2$ and let g be a pointed nullhomotopy of i, that is, $g : I \times F \longrightarrow E$ and $g(1, y) = y = i(y)$, $g(0, y) = *$, $g(t, *) = *$. For each $f : \Sigma X \longrightarrow F$ that represents $[f] \in \pi_1(X, F)$, via the diagram

$$\begin{tikzcd} I \times F \arrow[rr, "g"] & & E \arrow[d, "p"] \\ I \times \Sigma X \arrow[u, "\mathrm{id}\times f"] \arrow[r, "q"'] & \Sigma^2 X \arrow[r, dashed, "Gf"'] & B \end{tikzcd}$$

one defines a map Gf. If $f_0 \simeq f_1$ through a homotopy f_t, then $Gf_0 \simeq Gf_1$ via Gf_t, since the homotopy $(Gf_t) \circ q$ is compatible with the identification q, i.e., if $(Gf_t) \circ q$ is a homotopy, then (Gf_t) is a homotopy. Thus G induces a function

$$\Gamma : \pi_1(X, F) \longrightarrow \pi_2(X, B)\,.$$

Γ is a homomorphism, since one easily shows that $G(f_1 + f_2) = Gf_1 +' Gf_2$ (cf. 1.6.15).

To see that $\Delta\Gamma = \mathrm{id}_{\pi_1(X,F)}$, we have to construct a map that corresponds to Gf using the diagram

$$\begin{tikzcd} \{0\} \times \Sigma X \cup I \times \{*\} \arrow[rr, "\mathrm{const}"] \arrow[d, hook] & & E \arrow[d, "p"] \\ I \times \Sigma X \arrow[urr, dashed, "\widetilde{h}"] \arrow[r, "q"'] & \Sigma^2 X \arrow[r, "Gf"'] & B\,. \end{tikzcd}$$

But setting $\widetilde{h} = g \circ (\mathrm{id} \times f)$ (cf. 1.7.9) one gets $\widetilde{h}(1, z) = g(1, f(z)) = f(z)$. Thus f corresponds to Gf through $\widetilde{h}$, and so $\Delta\Gamma[f] = \Delta[Gf] = [f]$.

For any $n > 2$, just replace X in the previous case with $\Sigma^{n-1}X$. $\square$

1.8 Applications

In this section, we explain some particular instances of (locally trivial) fibrations that have special interest in algebraic topology.

1.8.1 Covering Maps

One of the most useful tools of algebraic topology for computing the fundamental group of a space is the concept of a covering map, that we analyze succintly in what follows. See [13] or [1] for a thorough treatment.

1.8.1 Definition. A *covering map* is a locally trivial fibration such that each fiber is discrete.

1.8.2 **Theorem**. *In a covering map, the path lifting is unique. That is, if $p : E \longrightarrow B$ is a covering map, $\omega : I \longrightarrow B$ is a path and $x_0 \in E$ is a point such that $p(x_0) = \omega(0)$, then there exists a unique path $\widetilde{\omega} : I \longrightarrow E$ such that $p \circ \widetilde{\omega} = \omega$ and $\widetilde{\omega}(0) = x_0$.*

Proof: Let $\widetilde{\omega}_0$ and $\widetilde{\omega}_1$ be liftings of ω. We apply the HLP to the pair $(I, \partial I)$ to obtain in the diagram

$$\begin{tikzcd}
\{0\} \times I \cup (I \times \partial I) \arrow[r, "\widetilde{h}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\
I \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & B
\end{tikzcd}$$

a map $\widetilde{h}$, where $h(t,s) = \omega(t)$, $\widetilde{h}_0(0,s) = x_0$, $\widetilde{h}_0(t,0) = \widetilde{\omega}_0(t)$, $\widetilde{h}_0(t,1) = \widetilde{\omega}_1(t)$. For fixed t, the mapping $s \mapsto \widetilde{h}(t,s)$ defines a continuous map into the fiber over $\omega(t)$, and is thus constant, since the fiber is discrete. Hence $\widetilde{\omega}_0(t) = \widetilde{h}(t,0) = \widetilde{h}(t,1) = \widetilde{\omega}_1(t)$. □

Of course, the previous theorem and its proof are still valid if p is a Serre fibration and each fiber admits only constant paths.

1.8.3 **Corollary**.

(a) *For a covering map, the homotopy lifting is unique.* (This follows since a homotopy is nothing else but a family of paths.)

(b) *For a covering map, the translation of the fiber along a path in B is unique.* (This follows from the fact that in order to translate a fiber one has to lift a particular homotopy.) □

In what follows we shall consider again pointed spaces, pointed maps, pointed homotopies, etc.

1.8.4 **Lemma**. *If X is connected and Y is discrete, then $\pi(X,Y) = 0$. In particular, for Y discrete and any X, $\pi_n(X,Y) = 0$ for $n \geq 1$, since ΣX is 0-connected (path connected).* □

Let $p : E \longrightarrow B$ be a covering map. From the long homotopy exact sequence

$$\cdots \longrightarrow \pi_n(X,F) \longrightarrow \pi_n(X,E) \longrightarrow \pi_n(X,B) \longrightarrow \pi_{n-1}(X,F) \longrightarrow \cdots$$

and the fact that by 1.8.4, $\pi_n(X,F) = 0$ if $n \geq 2$, one gets the following.

1.8.5 **Theorem**. *For a covering map $p : E \longrightarrow B$,*

$$p_* : \pi_n(X,E) \longrightarrow \pi_n(X,B)$$

is an isomorphism if $n \geq 2$ and a monomorphism if $n = 1$. □

If we apply the previous theorem to the covering map $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ (cf. 1.1.1(d)), since $\pi(X,\mathbb{R}) = 0$ because $\mathbb{R}$ is contractible, then we obtain the following.

1.8.6 **Theorem**. $\pi_n(\mathbb{S}^1) = 0$ *for* $n \geq 2$. □

For a locally trivial fibration $p : E \longrightarrow B$ one has the following.

1.8.7 **Proposition**. *If B is connected, then p is surjective (see 1.2.8).* □

Suppose that in the covering map $p : E \longrightarrow B$, the total space E is 0-connected (i.e., $\pi_0(E) = 0$), and that B is connected. Then B es 0-connected. The exact sequence

$$\pi_1(E) \xrightarrow{p_*} \pi_1(B) \xrightarrow{\Delta} \pi_0(F) \xrightarrow{i_*} \pi_0(E) = 0\,,$$

together with 1.7.8, gives the following.

1.8.8 **Theorem**. *Let $p : E \longrightarrow B$ be a covering map. Then $\pi_1(B)/\mathrm{im}(p^*) \cong \pi_0(F)$ (as sets, since $\mathrm{im}(p^*)$ does not have to be a normal subgroup of $\pi_1(B)$ (cf. for instance [7, III.17.1]).* □

Since F is discrete, $\pi_0(F) = F$. Therefore, F has at most as many elements as $\pi_1(B)$; in particular, we have the following.

1.8.9 **Corollary**. *Let $p : E \longrightarrow B$ be a covering map. If B is 1-connected (simply connected), that is, if $\pi_1(B) = 0$, then p is a homeomorphism.*

Proof: p is bijective, since $\pi_0(F) = 0$, and p^{-1} is continuous, since the projection p of a locally trivial fibration is an open map. $\square$

In particular, $B = \mathbb{S}^n$ does not admit nontrivial covering maps with path-connected total space if $n \geq 2$.

If E is simply connected, we have an isomorphism of sets

$$\pi_1(B) \cong \pi_0(F) \cong F\,.$$

Considering the special case of the covering map

$$p : \mathbb{R} \longrightarrow \mathbb{S}^1$$

(cf. 1.1.1(d)), we obtain the following.

1.8.10 **Theorem**. *There is a group isomorphism $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.*

Proof: Let $\omega_{n,m} : I \longrightarrow \mathbb{R}$ be the path $t \mapsto m + nt$ from m to $m+n$, $m, n \in \mathbb{Z}$. Any other path ω' from m to $m+n$ is homotopic to $\omega_{m,n}$, since $h_s : I \longrightarrow \mathbb{R}$, given by

$$t \longmapsto (1-s)\omega_{m,n}(t) + s\omega'(t)$$

is a homotopy from $\omega_{m,n}$ to ω' relative to the end points.

In particular,

$$\omega_{0,m} + \omega_{m,n} \simeq \omega_{0,m+n}\,.$$

Each path $\omega : I \longrightarrow \mathbb{S}^1$ with $\omega(0) = \omega(1)$ can be lifted to $\widetilde{\omega} : I \longrightarrow \mathbb{R}$, so that $\widetilde{\omega}(0) = 0$ and $\widetilde{\omega}(1) = k$ (for some $k \in \mathbb{Z}$). One has

$$\omega = p \circ \widetilde{\omega} \simeq p \circ \omega_{0,k} = p \circ \omega_{n,n+k}\,.$$

Take $\lambda_k = p \circ \omega_{0,k}$. Since

$$\begin{aligned}
[\lambda_k] + [\lambda_l] &= [p \circ \omega_{0,k}] + [p \circ \omega_{k,k+l}] \\
&= [p \circ \omega_{0,k+l}] \\
&= [\lambda_{k+l}]\,,
\end{aligned}$$

we have that $[\lambda_1]$ generates $\pi_1(\mathbb{S}^1)$. Therefore, $\pi_1(\mathbb{S}^1)$ is cyclic and infinite as a set. Hence it is free. $\square$

1.8.2 Spherical Fibrations

There are cases in which special fibers, base spaces, or even total spaces of a given fibration make the long homotopy exact sequence collapse. One obtains short exact sequences or even isomorphisms that provide us with valuable information. In what follows, we shall analyze cases in which one or more of those spaces are spheres.

We assume well known that

$$\pi_i(\mathbb{S}^n) = 0 \quad \text{if} \quad i < n, \quad \text{and} \quad \pi_n(\mathbb{S}^n) = \mathbb{Z} \quad \text{if} \quad n \geq 1$$

(see [1, 5.1.22] or [7, IV.2]; see also 1.6.19).

Take $n \geq 1$ and consider the fibrations 1.3.1

$$p : \mathbb{S}^{d(n+1)-1} \longrightarrow \mathbb{FP}^n$$

with fiber embedding

$$i : \mathbb{S}^{d-1} \hookrightarrow \mathbb{S}^{d(n+1)-1} .$$

The map i is nullhomotopic. Thus, from the homotopy exact sequence of p, we obtain the short exact sequences

$$0 \longrightarrow \pi_j(\mathbb{S}^{d(n+1)-1}) \xrightarrow{p_*} \pi_j(\mathbb{FP}^n) \xrightarrow{\Delta} \pi_{j-1}(\mathbb{S}^{d-1}) \longrightarrow 0 .$$

If $j \geq 2$, then by Theorem 1.7.9 we know that this sequence splits. Let us consider individual cases.

1.8.11 Examples. The following special cases are interesting:

1. $n = d = j = 1$.

   Then $\mathbb{RP}^1 \approx \mathbb{S}^1$. Thus p_* is multiplication by 2. The sequence does not split in this case. The exact sequence is then isomorphic to

   $$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0 .$$

2. $d = j = 1$, $n > 1$. Then $\pi_1(\mathbb{S}^{1(n+1)-1}) = \pi_1(\mathbb{S}^n) = 0$, thus

   $$\pi_1(\mathbb{RP}^n) \cong \pi_0(\mathbb{S}^0) \cong \mathbb{Z}_2 ,$$

   $\cong$ first as sets, but also as groups, since there is only one group with two elements.

3. $d = 1$, $j \geq 2$, $n \geq 2$.

   Since $\pi_j(\mathbb{S}^0) = 0$, we obtain

$$\pi_j(\mathbb{S}^n) \cong \pi_j(\mathbb{RP}^n)\,.$$

4. $d = 4$, $j \geq 2$.

   From Theorem 1.7.9 one has

$$\pi_j(\mathbb{HP}^n) \cong \pi_j(\mathbb{S}^{4n+3}) \oplus \pi_{j-1}(\mathbb{S}^3)\,.$$

   For $n = 1$, in particular, one has

$$\pi_j(\mathbb{S}^4) \cong \pi_j(\mathbb{S}^7) \oplus \pi_{j-i}(\mathbb{S}^3)\,.$$

1.8.12 NOTE. The homeomorphism $\mathbb{FP}^1 \approx \mathbb{S}^d$ can be given similarly to the case $\mathbb{F} = \mathbb{C}$ (cf. 1.3.1 (b)). Namely, via

$$\mathbb{F}^2 - \{0\} \supset \mathbb{S}^{d(n+1)-1} \ni (w_0, w_1) \longmapsto w_0 w_1^{-1} \in \mathbb{F} \cup \{\infty\} \cong \mathbb{S}^d\,.$$

(*Exercise*).

5. $d = 2$, $n \geq 1$.

   One has

$$\pi_j(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \text{if } j = 2, \\ \pi_j(\mathbb{S}^{2n+1}) & \text{if } n = 1, \end{cases}$$

   since $\pi_{j-1}(\mathbb{S}^1) = 0$ if $j \neq 2$ (cf. 1.8.6). In particular, one has (for $n = 1$) that

$$\pi_j(\mathbb{S}^2) \cong \pi_j(\mathbb{S}^3) \quad \text{if} \quad j > 2\,.$$

1.8.13 REMARK. With the help of the Cayley numbers (octonians), one can construct an analogous fibration to the previous ones

$$\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \longrightarrow \mathbb{S}^3$$

(cf. [15, 20.6]) and conclude from it that

$$\pi_j(\mathbb{S}^8) \cong \pi_j(\mathbb{S}^{15}) \oplus \pi_{j-1}(\mathbb{S}^7) \quad \text{if} \quad j \geq 1\,,$$

(it is nontrivial if $j \geq 8$).

1.8.3 Fibrations with a Section

Sections play an important role in many aspects of the theory and applications of the fibrations. We analyze here some implications of the existence of a section for a given fibration.

1.8.14 Definition. Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration. A map $s : B \longrightarrow E$ is called a *section* of p if $p \circ s = \mathrm{id}_B$.

Given a section $s : B \longrightarrow E$ of a fibration $p : E \longrightarrow B$, s_* is in the exact sequence

$$\begin{tikzcd} \pi_j(F) \arrow[r, "i_*"] & \pi_j(E) \arrow[r, shift left, "p_*"] & \pi_j(B) \arrow[l, shift left, "s_*"] \end{tikzcd}$$

a right inverse of p_*. Thus p_* is surjective, s_* is injective and the sequence splits. Hence, for $j \geq 2$ one has

$$\pi_j(E) \cong \pi_j(F) \oplus \pi_j(B)\,.$$

This last equation is valid, in particular, for the product fibration $p : E = F \times B \longrightarrow B$ and in this case one may easily check it directly. In this sense, a fibration with section behaves as a product with respect to the homotopy groups (cf. Sections 1.1 and 1.2).

If n is odd, the *unitary tangent bundle* $p : ST(\mathbb{S}^n) \longrightarrow \mathbb{S}^n$ of the unit tangent vectors to the sphere $\mathbb{S}^n$ has a section, namely, the map $s : \mathbb{S}^n \longrightarrow ST(\mathbb{S}^n) = \{(x,y) \in \mathbb{S}^n \times \mathbb{S}^n | x \perp y\}$ given by

$$s(x) = s(x_0, \ldots, x_n) = (x, (-x_1, x_0, -x_3, x_2, ..., -x_n, x_{n-1}))\,.$$

Thus we have the following result.

1.8.15 **Proposition**. *There is an isomorphism*

$$\pi_j(ST(\mathbb{S}^n)) \cong \pi_j(\mathbb{S}^{n-1}) \oplus \pi_j(\mathbb{S}^n) \quad \textit{if} \quad j \geq 2\,.$$

□

Chapter 2

Fiber Bundles

2.1 Introduction

2.2 Topological Groups

2.2.1 Definition. A *topological group* G is a topological space G together with a group structure such that the function

$$\begin{aligned} \nu : G \times G &\longrightarrow G\,, \\ (g,h) &\longmapsto g^{-1} \cdot h\,, \end{aligned}$$

is continuous. We frequently write gh instead of $g \cdot h$ for the product of $g, h \in G$. Sometimes, when the group is additive, we write $g + h$. In the former case we write 1 or once in a while e for the neutral element of G; in the latter case we write 0 for it.

2.2.2 Exercise. Prove that the maps μ and ι given by

$$\begin{aligned} \mu : G \times G &\longrightarrow G\,, \\ (g,h) &\longmapsto g \cdot h\,, \end{aligned}$$

$$\begin{aligned} \iota : G &\longrightarrow G\,, \\ g &\longmapsto g^{-1}\,, \end{aligned}$$

are continuous if and only if the map $\nu : G \times G \longrightarrow G$ given above is continuous.

2.2.3 Examples.

1. Let $(\mathbb{R}^n, +)$, resp. $(\mathbb{C}^n, +)$, be the real, resp. complex, n-dimensional vector space with the usual topology and the usual sum of vectors. They both are topological groups for every n.

2. Let $\mathrm{GL}_n(\mathbb{R})$ be the set of real invertible $n \times n$ matrices with the group structure given by matrix multiplication and the topology given as follows. Fix an ordering of the entries of each matrix, so that it can be considered as an n^2-tuple of real numbers, i.e., as an element of $\mathbb{R}^{n^2}$. This way, $\mathrm{GL}_n(\mathbb{R})$ can be seen as an (open) subspace of $\mathbb{R}^{n^2}$ with the relative topology. In this case, ν is continuous, since the entries of the product matrix AB^{-1} are rational functions of the entries of A and B. Thus $\mathrm{GL}_n(\mathbb{R})$ is a topological group. In particular, the group $\mathrm{GL}_1(\mathbb{R})$ is the multiplicative group of the nonzero real numbers, also written as $\mathbb{R}^*$. The group $\mathrm{GL}_n(\mathbb{R})$ is called the *general linear group of real* $n \times n$ *matrices*.

3. Let $\mathrm{GL}_n(\mathbb{C})$ be the set of complex invertible $n \times n$ matrices with the group structure and topology analogous to the previous example. Similarly, $\mathrm{GL}_n(\mathbb{C})$ is a topological group. In particular, the group $\mathrm{GL}_1(\mathbb{C})$ is the multiplicative group of the nonzero complex numbers, also written as $\mathbb{C}^*$. The group $\mathrm{GL}_n(\mathbb{C})$ is called the *general linear group of complex* $n \times n$ *matrices*.

2.2.4 **Theorem**. *Every subgroup H of a topological group G with the relative topology is a topological group.*

Proof: Let μ' be the induced multiplication in H. Let $i : H \hookrightarrow G$ be the inclusion. $i\mu' = \mu|_{H \times H}$ is continuous, and since H has the relative topology, μ' is continuous. Similarly, one proves that the map sending an element in H to its inverse is continuous. $\square$

2.2.5 EXAMPLES.

1. The following are important subgroups of $\mathrm{GL}_n(\mathbb{R})$:

$$\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = 1\}$$

   is the *special linear group* of real $n \times n$ matrices.

$$\mathrm{O}_n = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid AA^* = 1\},$$

   where A^* is the transposed matrix of A and 1 is the unit matrix, is the *orthogonal group* of $n \times n$ matrices.

$$\mathrm{SO}_n = \mathrm{O}_n \cap \mathrm{SL}_n(\mathbb{R})$$

is the *special orthogonal group* of $n \times n$ matrices.

All these subgroups are closed in $\mathrm{GL}_n(\mathbb{R})$. $\mathrm{SL}_n(\mathbb{R})$ for being the inverse image of the closed set $\{1\} \subset \mathbb{R}$ under the continuous map $A \mapsto \det(A)$. That O_n is closed can be proved as follows. Let $A = (a_{ij}) \in \mathrm{O}_n$. Then the matrix AA^* has entries $\sum_{k=1}^n a_{ki}a_{kj}$. Therefore, O_n is the inverse image of the closed set $\{1\} \subset \mathrm{GL}_n(\mathbb{R})$ under the continuous mapping

$$A = (a_{ij}) \longmapsto \sum_{k=1}^{n} a_{ki}a_{kj} \,,$$

where 1 denotes the unit matrix, with ones in the diagonal and zeroes elsewhere. The subgroup SO_n is closed, since it is the intersection of two closed subgroups. Since $\mathrm{O}_n \subset \mathbb{R}^{n^2}$ is clearly bounded, the groups O_n and SO_n are even compact.

2. The following are special cases:

$$\begin{aligned}
\mathrm{O}_1 &= \{1,-1\} = \mathbb{Z}_2 = \mathbb{S}^0 \,, \\
\mathrm{SL}_1(\mathbb{R}) &= \{1\} \,, \\
\mathrm{SO}_2 &\cong \mathbb{S}^1 \,, \\
\mathrm{O}_n &\approx \mathrm{SO}_n \times \mathbb{Z}_2 \quad \text{(as topological spaces)} \,, \\
\mathrm{SO}_3 &\approx \mathbb{RP}^3 \quad \text{(as topological spaces).}
\end{aligned}$$

For the last of the previous statements, we sketch a proof. Each element in SO_3 is a rotation around some axis. Let $\mathbb{B}^3 \subset \mathbb{R}^3$ be the unit ball and let $f : \mathbb{B}^3 \longrightarrow \mathrm{SO}_3$ be the map that sends an element $x \in \mathbb{B}^3$ to the rotation around the axis determined by x by an angle $\pi|x|$. f is clearly surjective; that is, it is an identification (see Figure 2.1).

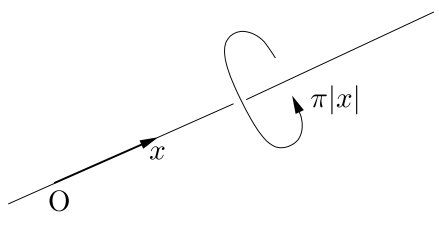


Figure 2.1

From $f(x) = f(y)$ it follows that either $x = y$ or $x = -y$ and $|x| = |y| = 1$. That is, f identifies antipodal points of $\mathbb{S}^2 \subset \mathbb{B}^3$, and thus f induces a homeomorphism

$$\mathbb{RP}^3 = \mathbb{B}^3/\!\sim \xrightarrow{\ \approx\ } \mathrm{SO}_3 \,,$$

where $x \sim y$ if either $x = y$ or $x = -y$ and $|x| = 1$. It is thus enough to prove that f is continuous, which is left as an *exercise* to the reader.

3. The following are subgroups of $\mathrm{GL}_n(\mathbb{C})$:

$$\begin{aligned}
\mathrm{SL}_n(\mathbb{C}) &= \{A \mid \det(A) = 1\}\,,\\
\mathrm{GL}_n(\mathbb{R}) &= \{A \mid A = \overline{A}\}\,,\\
\mathrm{O}_n(\mathbb{C}) &= \{A \mid AA^* = 1\}\,,\\
\mathrm{U}_n &= \{A \mid \overline{A}A^* = 1\}\,,\\
\mathrm{O}_n &= \{A \mid A = \overline{A} = (A^*)^{-1}\,,
\end{aligned}$$

where A^* is again the transposed matrix of $A \in \mathrm{GL}_n(\mathbb{C})$ and $\overline{A}$ is the complex conjugate matrix.

The group $\mathrm{SL}_n(\mathbb{C})$ is the *special linear group* of complex $n \times n$ matrices, and the group U_n is the *unitary group* of $n \times n$ matrices.

$$\mathrm{SU}_n = \mathrm{U}_n \cap \mathrm{SL}_n(\mathbb{C})$$

is the *special unitary group* of $n \times n$ matrices.

4. There is an embedding

$$r : \mathrm{GL}_n(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{R})\,,$$

as follows. Each $\mathbb{C}$-linear transformation of $\mathbb{C}^n$ is also an $\mathbb{R}$-linear transformation. If we consider $\mathbb{C}^n$ as a real vector space, then we obtain a vector space isomorphic to $\mathbb{R}^{2n}$. This isomorphism can be given by

$$z = (x_1 + \mathrm{i}y_1, \ldots, x_n + \mathrm{i}y_n) \longmapsto (x_1, \ldots, x_n, y_1, \ldots, y_n) = (x, y)\,,$$

from which we obtain that if $z' = zC$, $C \in \mathrm{GL}_n(\mathbb{C})$, then $C = A + \mathrm{i}B$, with A and B real matrices. Hence,

$$(x', y') = (x, y)\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\,;$$

thus we can define

$$r(C) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}\,.$$

r is a topological embedding (inclusion), since it is continuous, injective, and has an inverse given by

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longmapsto A + \mathrm{i}B$$

which is obviously continuous as well.

2.2.6 Definition. Let H be a subgroup of a topological group G. Let G/H be the set of left cosets xH, $x \in G$. We topologize G/H by requiring that the quotient map

$$p : G \longrightarrow G/H$$

be an identification. We call this topological space the *homogeneous space* of the group G modulo H.

2.2.7 **Theorem**. *p is an open map, that is, if $A \subset G$ is an open set, then its image pA is open. (Recall that there are identifications that are not open maps.)*

Proof: That pA is open means, by definition of an identification that $p^{-1}pA$ is open. But

$$p^{-1}pA = AH = \bigcup_{x \in H} Ax\,.$$

Now, if A is open, then also Ax is open, since the map $G \longrightarrow G$ given by $y \mapsto yx$ is a homeomorphism (the proof of this fact is an easy *exercise* for the reader). Thus $p^{-1}pA$ is a union of open sets, thus open. □

2.2.8 **Theorem**. *If H is a normal subgroup of G, then G/H is a topological group.*

Proof: By means of the commutativity of the diagrams

$$\begin{tikzcd} G \times G \arrow[r, "\mu"] \arrow[d, "p\times p"'] & G \arrow[d, "p"] \\ G/H \times G/H \arrow[r, dashed, "\overline{\mu}"'] & G/H\,, \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\iota"] \arrow[d, "p"'] & G \arrow[d, "p"] \\ G/H \arrow[r, dashed, "\overline{\iota}"'] & G/H\,, \end{tikzcd}$$

one may define maps $\overline{\mu}$, $\overline{\iota}$. $\overline{\mu}$ is the canonical multiplication in G/H, and $\overline{\iota}$ determines canonically the inverses in G/H. Since p is open, so is also $p \times p$, and this last being surjective makes it an identification too. Therefore, both $\overline{\mu}$ and $\overline{\iota}$ are continuous. □

2.2.9 Exercise. Prove the previous theorem using the maps ν and $\overline{\nu}$ instead of the maps μ, ι, $\overline{\mu}$, and $\overline{\iota}$.

2.2.10 **Theorem**. *The homogeneous space G/H is Hausdorff if and only if H is closed in G.*

Proof: If G/H is Hausdorff, then the point $p(1) \in G/H$ is closed ($1 \in G$ is the neutral element) and so $p^{-1}p(1) = H$ is closed.

Conversely, let H be closed. Consider the relation

$$R = \{(x, y) \mid x^{-1}y \in H\} \subset G \times G\,.$$

R is closed in $G \times G$, since it is the inverse image of H under the continuous map $\nu : G \times G \longrightarrow G$ given by $(x, y) \mapsto x^{-1}y$. Let x_1H and x_2H be different cosets in G/H. Then $(x_1, x_2) \notin R$, and since R is closed, there exist open neighborhoods U_ν of x_ν ($\nu = 1, 2$) such that $(U_1 \times U_2) \cap R = \emptyset$. Since p is an open map, pU_ν is a neighborhood of $p(x_\nu) = x_\nu H$. These neighborhoods pU_1 and pU_2 are disjoint, since if, on the contrary, there were elements $y_\nu \in U_\nu$ such that $p(y_1) = p(y_2)$, then one would have that $(y_1, y_2) \in R$. But this contradicts the choice of U_1 and U_2. $\square$

This theorem shows the importance of taking only closed subgroups of a given topological group.

2.2.11 **Corollary**. $\{1\}$ *is closed in* G *if and only if* G *is Hausdorff.* $\square$

2.2.12 DEFINITION. Let G be a topological group and X a topological space. We say that G *acts* on X *on the left* if there is a continuous map

$$\lambda : G \times X \longrightarrow X$$

such that, if we denote $\lambda(g, x)$ by gx, then the following hold:

(a) $(g_1g_2)x = g_1(g_2x)$,

(b) $1x = x$.

From $g^{-1}(gx) = (g^{-1}g)x = 1x = x$ and $g(g^{-1}x) = x$ it follows that the map

$$\begin{aligned} \widehat{g} : X &\longrightarrow X\,, \\ x &\longmapsto gx\,, \end{aligned}$$

is a homeomorphism of X for each $g \in G$. Condition (a) implies that the mapping $g \mapsto \widehat{g}$ is a homomorphism from G into the group $\mathrm{Homeo}(X)$ of homeomorphisms of X onto itself. If G acts on X we say that X is a *left* G*-space*.

2.2.13 Note. There is a corresponding notion of a group G *acting* on a space X *on the right*, if instead of the map λ one has a map $\rho : X \times G \longrightarrow X$, $(x, g) \mapsto xg$, that satisfies conditions corresponding to (a) and (b). In this case we speak of X as a *right G-space*.

2.2.14 Exercise. Give a precise formulation for (a) and (b) in the case of a right action of G on X. Then prove that there is a one-to-one correspondence between left actions and right actions of G on X given by the formula

$$xg = g^{-1}x\,.$$

2.2.15 Definition. *G acts effectively* on X if $gx = x$ for all elements $x \in X$, then $g = 1$. In this case, we may consider G as a subgroup of $\mathrm{Homeo}(X)$, through the embedding $g \mapsto \widehat{g}$.

2.2.16 Definition. *G acts transitively* on X if for any $x, y \in X$ there exists an element $g \in G$ such that $y = gx$. In this case, there is a continuous surjection from G onto X through the mapping $g \mapsto gx_0$ for some (any) fixed $x_0 \in X$ (see 2.2.20 below.)

2.2.17 Definition. *G acts freely* on X if $gx = x$ for some element $x \in X$, then $g = 1$.

2.2.18 Examples.

1. $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathbb{R}^n$ through $(A, x) \mapsto Ax$, for any invertible $n \times n$ matrix A and any vector x in $\mathbb{R}^n$ (written vertically). Conditions (a) and (b) in Definition 2.2.12 are obviously satisfied. This is an effective and transitive action.

2. Let H be a subgroup of a topological group G. G acts on the homogeneous space G/H as follows. By the commutativity of

$$\begin{tikzcd} G \times G \arrow[r, "\mu"] \arrow[d, "\mathrm{id}\times p"'] & G \arrow[d, "p"] \\ G \times G/H \arrow[r, dashed, "\lambda"'] & G/H \end{tikzcd}$$

a map λ is uniquely defined. The action λ is continuous, since by 2.2.7, the product of maps $\mathrm{id} \times p$ is an identification. The map λ is then given by

$$\begin{array}{ccc} (g_1, g_2H) & \longmapsto & (g_1g_2)H \\ \| & & \| \\ (g_1, p(g_2)) & \longmapsto & p(g_1g_2)\,. \end{array}$$

With this, it is routine to verify (a) and (b) in 2.2.12. The action λ is always transitive, but not necessarily effective. (For instance, if G is abelian and $H \neq \{1\}$, it is not effective. It is never free.).

2.2.19 Exercise. Prove the following:

(a) If G acts freely on X, then it also acts effectively.

(b) The orthogonal group O_n acts effectively on $\mathbb{R}^n$, but it does not act freely.

2.2.20 Remark. Many transitive actions can be reduced to the one of Example 2.2.18, 2.

Let $x_0 \in X$ be a fixed element. As we already noted, by $g \mapsto gx_0$ one defines a map $f : G \longrightarrow X$. This map f is surjective when G acts transitively on X. Take $H = \{g \in G \mid gx_0 = x_0\} = f^{-1}(x_0)$. Then H is a subgroup of G. It is called the *isotropy subgroup* of x_0 and is usually denoted by G_{x_0}. This subgroup is closed whenever the point x_0 is closed in X. Let us consider the problem

$$\begin{tikzcd} & G \arrow[dl, "p"'] \arrow[dr, "f"] & \\ G/H \arrow[rr, dashed, "\overline{f}"'] & & X \end{tikzcd}.$$

The map $\overline{f}$ exists. Namely, one has

$$\begin{aligned} p(g_1) = p(g_2) &\Leftrightarrow g_1^{-1}g_2 \in H \Leftrightarrow g_1^{-1}g_2x_0 = x_0 \\ &\Leftrightarrow g_2x_0 = g_1x_0 \Leftrightarrow f(g_1) = f(g_2)\,. \end{aligned}$$

Thus the map $\overline{f}$ is even bijective. $\overline{f}$ is continuous, since p is an identification. Under adequate assumptions, one can prove that $\overline{f}$ is a homeomorphism. For example, if the space X is Hausdorff and the quotient space G/H is compact. The map $\overline{f}$ is compatible with the actions of G on G/H and on X, in other words, it is *equivariant*. That is, the diagram

$$\begin{tikzcd} G \times G/H \arrow[r] \arrow[d, "\mathrm{id}\times\overline{f}"'] & G/H \arrow[d, "\overline{f}"] \\ G \times X \arrow[r] & X \end{tikzcd}$$

is commutative.

2.2.21 DEFINITION. Let G be a topological group. If X is a G-space and $x \in X$, then the subspace

$$Gx = \{gx \mid g \in G\} \subset X$$

is called the *orbit* of x under the action of G. The orbits decompose the space X in disjoint subspaces. Namely, assume that $gx = hy$ for some $g, h \in G$, $x, y \in X$, then for any $k \in G$, $kx = kg^{-1}gx = kg^{-1}hy \in Gy$; hence, $Gx \subset Gy$. Similarly, one proves under the same assumption that $Gy \subset Gx$. Thus the orbits of any two points are either equal or disjoint.

We denote by X/G the set of orbits of X under the action of G. Let $q : X \longrightarrow X/G$ denote the mapping $x \mapsto Gx$. We endow X/G with the quotient topology induced by q. We call this the *orbit space* of X (with respect to G).

2.2.22 EXERCISE. Assume that X is a Hausdorff G-space and that G is compact. Prove the following:

(a) X/G is Hausdorff.

(b) $q : X \longrightarrow X/G$ is a closed map.

(c) $q : X \longrightarrow X/G$ is a proper map, namely, for each compact set $K \subset X/G$, the inverse image $q^{-1}K \subset X$ is compact.

(d) X is compact if and only if X/G is compact.

(e) X is locally compact if and only if X/G is locally compact.

2.2.23 EXAMPLE. Let $\mathbb{F}$ be any of the fields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Then $\mathbb{F} - \{0\} \subset \mathbb{F}$ is a topological group with the relative topology and the multiplication given by the field multiplication. There is an action of this group on $\mathbb{F}^{n+1} - \{0\}$, given by $\lambda(x_0, x_1, \ldots, x_n) = (\lambda x_0, \lambda x_1, \ldots, \lambda x_n)$ for $\lambda \in \mathbb{F} - \{0\}$ and $(x_0, x_1, \ldots, x_n) \in \mathbb{F}^{n+1} - \{0\}$. The orbit space $\mathbb{F}^{n+1} - \{0\}/\mathbb{F} - \{0\}$ is the projective space $\mathbb{FP}^n$ defined in 1.3.1 (a).

2.2.24 EXERCISE. Let d be the (real) dimension of $\mathbb{F}$ (see 1.3.1 (a)) and let $\mathbb{S}^{d(n+1)-1} \subset \mathbb{F}^{n+1} - \{0\}$ be the unit sphere (see 1.6.6 (e)). Prove that $\mathbb{S}^{d-1} \subset \mathbb{F} - \{0\}$ is a closed subgroup. Moreover, prove that the restriction of the action given in 2.2.23 gives an action of $\mathbb{S}^{d-1}$ on $\mathbb{S}^{d(n+1)-1}$. Conclude that there is a canonical homeomorphism

$$\mathbb{S}^{d(n+1)-1}/\mathbb{S}^{d-1} \approx \mathbb{FP}^n .$$

2.2.25 Example. The group O_n acts on the sphere $\mathbb{S}^{n-1}$ through $(A,x) \mapsto Ax$ (cf. Example 2.2.18, 1). Take $x_0 \in \mathbb{S}^{n-1}$ to be the vector such that $x_0^* = (0,\dots,0,1)$. The equation $Ax_0 = x_0$ is equivalent to the matrix equation

$$A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}, \quad B \in \mathrm{O}_{n-1}\,.$$

By means of the embedding given by

$$B \longmapsto \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

we may consider the group O_{n-1} as a subgroup of O_n and by 2.2.20 we have a homeomorphism

$$\overline{f} : \mathrm{O}_n/\mathrm{O}_{n-1} \approx \mathbb{S}^{n-1}\,,$$

(since O_n is compact and $\mathbb{S}^{n-1}$ is Hausdorff).

2.2.26 Exercise. Similarly to the previous example, give a transitive action of the group U_n on the sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$. Conclude that there is a homeomorphism

$$\mathrm{U}_n/\mathrm{U}_{n-1} \approx \mathbb{S}^{2n-1}\,.$$

2.2.27 Note. See Subsection **2.5.1** for further examples similar to 2.2.25 and 2.2.26.

2.3 Fiber Bundles

In what follows, B and F will be topological spaces, and G a topological group acting effectively on F (see 2.2.15). F, G and the action will be the same along this section. We shall prepare the definition of a fiber bundle.[1]

2.3.1 Definition. A *set bundle* B with fiber F is a family

$$\mathcal{F} = \{\mathcal{F}_x \mid x \in B\}$$

of sets, that are equivalent (as sets) to F, that is, $\mathcal{F}_x \approx F$ for all x. A *local chart* for $\mathcal{F}$ is a family

$$\varphi = \{\varphi_x : F \longrightarrow \mathcal{F}_x \mid x \in U_\varphi\}$$

[1]The definition of a fiber bundle that we shall give below was proposed by A. Dold.

of maps, where U_φ is an open set in B, and each map φ_x is bijective. If we want to emphasize that a set bundle $\mathcal{F}$ is a bundle over B, we sometimes denote it by the pair $(\mathcal{F}, B)$.

An *atlas* for $\mathcal{F}$ with respect to the group G is a set $\mathcal{A}$ of local charts for $\mathcal{F}$ such that the following conditions are satisfied:

(B1) $\bigcup_{\varphi \in \mathcal{A}} U_\varphi = B$.

(B2) Given $\varphi, \psi \in \mathcal{A}$ and $x \in U_\varphi \cap U_\psi$, $g(x) = \psi_x^{-1}\varphi_x : F \longrightarrow F$ is an element of $G \subset \mathrm{Homeo}(F)$ (cf. 2.2.15).

(B3) The map

$$\begin{aligned} g : U_\varphi \cap U_\psi &\longrightarrow G \\ x &\longmapsto g(x) \end{aligned}$$

is continuous.

An atlas is said to be *trivial* if it consists of only one chart.

2.3.2 EXAMPLE. Let $p : E \longrightarrow B$ be a locally trivial fibration, all of whose fibers are homeomorphic to F. We obtain a set bundle $\mathcal{F} = \{\mathcal{F}_x\}$ by defining $\mathcal{F}_x = p^{-1}(x)$. The fact that p is locally trivial means that there is an open cover $\{U_j \mid j \in J\}$ of B and homeomorphisms Φ_j such that the diagram

$$\begin{tikzcd} U_j \times F \arrow[rr, "\Phi_j"] \arrow[dr, "\mathrm{proj}_1"'] & & p^{-1}U_j \arrow[dl, "p_{U_j}"] \\ & U_j & \end{tikzcd}$$

commutes. We give local charts as follows. For each $j \in J$, take

$$\varphi_j = \{\varphi_{j,x} : F \longrightarrow p^{-1}x = \mathcal{F}_x \mid x \in U_j\}$$

by defining $\varphi_{j,x}(y) = \Phi_j(x, y)$. The set $\mathcal{A} = \{\varphi_j \mid j \in J\}$ is an atlas. (B1) clearly holds. In order for (B2) and (B3) to hold, we need a topological group G with the following properties:

(a) $G \subset \mathrm{Homeo}(F)$ (as a subgroup).

(b) The homeomorphisms

$$g_{ij}(x) = \varphi_{i,x}^{-1}\varphi_{j,x} : F \longrightarrow F$$

are all elements of G.

(c) The group G acts on F, i.e., the obvious map $G \times F \longrightarrow F$ is continuous.

(d) The map $g_{ij} : U_i \cap U_j \longrightarrow G$ given by $x \mapsto g_{ij}(x)$ is continuous.

Endowing $\mathrm{Homeo}(F)$, for instance, with the compact-open topology, and taking $G = \mathrm{Homeo}(F)$, all conditions (a)–(d) are satisfied. The only remaining question is the following: Is G a topological group with this topology and does it act continuously on F? The answer is yes if, for example, F is compact and Hausdorff (cf. Steenrod [15, 5.4]).

Another possibility is to furnish $\mathrm{Homeo}(F)$ with the k-topology associated to the compact-open one. Thus, if F is also compactly generated, then taking $G = \mathrm{Homeo}(F)$, G is a topological group that acts on F and (a)–(d) are satisfied (for (c) see [16, 5.2 and 5.9]).

2.3.3 DEFINITION. Let $\mathcal{F}$ and $\mathcal{F}'$ be set bundles over B and B', respectively, both with the same fiber F. A *set bundle map* $(f, \overline{f}) : \mathcal{F} \longrightarrow \mathcal{F}'$ consists of a continuous map $\overline{f} : B \longrightarrow B'$, and a family $f = \{f_x \mid x \in B\}$ of bijections $f_x : \mathcal{F}_x \longrightarrow \mathcal{F}_{\overline{f}(x)}$.

Let $\mathcal{A}$ and $\mathcal{A}'$ be atlases for $\mathcal{F}$ and $\mathcal{F}'$ with respect to the group G (G acts on F always in a fixed manner). $(f, \overline{f})$ is said to be *compatible* with $\mathcal{A}$ and $\mathcal{A}'$ if the following conditions hold:

(C1) If $\varphi \in \mathcal{A}$, $\psi \in \mathcal{A}'$, and $x \in U_\varphi \cap \overline{f}^{-1} U'_\psi$, then the bijection

$$\psi_y^{-1} f_x \varphi_x : F \longrightarrow F\,,$$

where $y = \overline{f}(x)$, is an element $g(x) \in G$; in particular, it is a homeomorphism.

(C2) The map $g : U_\varphi \cap \overline{f}^{-1} U'_\psi \longrightarrow G$ given by $x \mapsto g(x)$ is continuous.

The next theorem shows that set bundles build a category.

2.3.4 **Theorem**.

(a) (e, id_B), *where* $e_x = \mathrm{id}_{\mathcal{F}_x}$, *is a set bundle map* $(\mathcal{F}, B) \longrightarrow (\mathcal{F}, B)$ *compatible with the atlases* $\mathcal{A}$ *and* $\mathcal{A}$. *We denote* (e, id_B) *by* $\mathrm{id}_{(\mathcal{F},B)}$ *or simply by* $\mathrm{id}_{\mathcal{F}}$.

(b) *If* $(f,\overline{f}):(\mathcal{F},B)\longrightarrow(\mathcal{F}',B')$ *is compatible with* $\mathcal{A}$ *and* $\mathcal{A}'$*, and* $(f',\overline{f}'):(\mathcal{F}',B')\longrightarrow(\mathcal{F}'',B'')$ *is compatible with* $\mathcal{A}'$ *and* $\mathcal{A}''$*, then* $(h,\overline{h}):(\mathcal{F},B)\longrightarrow(\mathcal{F}'',B'')$ *is compatible with* $\mathcal{A}$ *and* $\mathcal{A}''$*, where* $\overline{h}=\overline{f}'\circ\overline{f}$ *and* $h=\{h_x=f'_{\overline{f}(x)}\circ f_x:\mathcal{F}_x\longrightarrow\mathcal{F}''_{\overline{h}(x)}\}$*. We denote* $(h,\overline{h})$ *by* $(f',\overline{f}')\circ(f,\overline{f})$*.*

Proof: (a) is clear.

For (b), take $\varphi\in\mathcal{A}$, $\chi\in\mathcal{A}''$, and $x\in U_\varphi\cap\overline{h}^{-1}U_\chi$. We choose $\psi\in\mathcal{A}'$ such that $y=\overline{f}(x)\in U_\psi$. Then, for $z=\overline{f}'(y)$,

$$\begin{aligned}g''(x)&=\chi_z^{-1}\circ h_x\circ\varphi_x=\chi_z^{-1}\circ f'_y\circ f_x\circ\varphi_x\\&=(\chi_z^{-1}\circ f'_y\circ\psi_y)\circ(\psi_y^{-1}\circ f_x\circ\varphi_x)\\&=g'(y)g(x)\in G\,,\end{aligned}$$

since by assumption both $g'(y)$ and $g(x)$ lie in G. It still remains to prove that the mapping $x\mapsto g''(x)$ is a continuous map $g'':U_\varphi\cap\overline{h}^{-1}U''_\chi\longrightarrow G$. This is true because

$$\{U_\varphi\cap\overline{f}^{-1}U'_\psi\cap\overline{h}^{-1}U''_\chi\mid\psi\in\mathcal{A}'\}$$

is an open cover of $U_\varphi\cap\overline{h}^{-1}U''_\chi$, and g'' is continuous on each open set of the cover, since there

$$\begin{aligned}g''(x)&=\mu(g'(y),g(x))\\&=\mu\circ(g'\times g)\circ(\overline{f}\times\mathrm{id})\circ\Delta(x)\,,\end{aligned}$$

where $\Delta:X\longrightarrow X\times X$ is the diagonal map and $\mu:G\times G\longrightarrow G$ is the multiplication in G. □

2.3.5 **Theorem**. *Let* $\mathcal{F}$ *be a set bundle over* B *with two given atlases* $\mathcal{A}$ *and* $\mathcal{A}'$*. Then the set bundle map* $\mathrm{id}_{\mathcal{F}}=(e,\mathrm{id}_B)$ *is compatible with with* $\mathcal{A}$ *and* $\mathcal{A}'$ *if and only if* $\mathcal{A}\cup\mathcal{A}'$ *is an atlas.*

Proof: This follows immediately from Definition 2.3.3. □

2.3.6 Definition. Two atlases $\mathcal{A}$ and $\mathcal{A}'$ of a set bundle are *equivalent* if $\mathcal{A}\cup\mathcal{A}'$ is again an atlas. This is an equivalence relation.

2.3.7 **Theorem**. *Let* $\mathcal{A}$ *be an atlas for a set bundle* $\mathcal{F}$*. The following statements hold:*

(a) *The union* $\widehat{\mathcal{A}}$ *of all atlases equivalent to* $\mathcal{A}$ *is again an atlas.*

(b) *The atlas* $\widehat{\mathcal{A}}$ *is the largest that is equivalent to* $\mathcal{A}$.

(c) *The atlas* $\widehat{\mathcal{A}}$ *is maximal in the ordered set (with respect to inclusion) of all atlases for* $\mathcal{F}$.

Proof: (a) (B1) is clear. Take $\varphi, \psi \in \widehat{\mathcal{A}}$. There exists atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ that are equivalent to $\mathcal{A}$ such that $\varphi \in \mathcal{A}_1$ and $\psi \in \mathcal{A}_2$. $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent, and therefore, (B2) and (B3) hold for φ and ψ.

(b) $\widehat{\mathcal{A}}$ is equivalent to $\mathcal{A}$, since $\widehat{\mathcal{A}} \cup \mathcal{A} = \widehat{\mathcal{A}}$ is again an atlas (because $\mathcal{A} \subset \widehat{\mathcal{A}}$, 2.3.5).

(c) If $\widehat{\mathcal{A}} \subset \mathcal{B}$ one would have that $\mathcal{A} \subset \mathcal{B}$ and so $\mathcal{B}$ would be equivalent to $\mathcal{A}$, and therefore, $\mathcal{B} \subset \widehat{\mathcal{A}}$ (by definition of $\widehat{\mathcal{A}}$). Thus, $\widehat{\mathcal{A}} = \mathcal{B}$. □

2.3.8 DEFINITION. A set bundle $\mathcal{F}$ over B with fiber F, together with an action of G on F and a *maximal atlas* $\mathcal{A}$ with respect to the group G, is called *fiber bundle*. The group G is called the *structure group* of the fiber bundle.

Such a fiber bundle will be denoted by

$$\xi = (F, G, B; \mathcal{F}, \mathcal{A})\,.$$

A fiber bundle will be called *trivial* if its atlas is equivalent to the trivial one (see 2.3.1).

2.3.9 REMARK. We could have defined a fiber bundle as a set bundle together with an equivalence class of atlases, since by 2.3.7, maximal atlases and equivalence classes of atlases are in one-to-one correspondence; that is each equivalence class contains exactly one maximal atlas (namely, the union of all atlases in the class).

As it is frequent, we shall write instead of the equivalence class of an atlas for ξ simply $\mathcal{A}$, even though this atlas is not maximal. The concept of fiber bundle is introduced, since an atlas for a set bundle is nothing else but an auxiliary concept, which does not have to belong to the structure. This will be clearer when we determine a locally trivial fibration for this fiber bundle. A special atlas will describe then the local trivialization, while the fibration will only depend on the equivalence class of the atlases (cf. also 2.3.2).

2.3.10 **Theorem**. *Let* $(f, \overline{f}) : (\mathcal{F}, B) \longrightarrow (\mathcal{F}', B')$ *be a set bundle map. Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be equivalent atlases for* $\mathcal{F}$, *and* $\mathcal{A}'_1$ *and* $\mathcal{A}'_2$ *equivalent atlases for* $\mathcal{F}'$. *Then* $(f, \overline{f})$ *is compatible with* $\mathcal{A}_1$ *and* $\mathcal{A}'_1$ *if and only if it is compatible with* $\mathcal{A}_2$ *and* $\mathcal{A}'_2$.

Proof: Consider the following diagram of set bundle maps.

$$\begin{tikzcd}
(\mathcal{F},\mathcal{A}_1) \arrow[r, dashed, "(f,\overline{f})"] \arrow[d, "(e,\mathrm{id}_B)"'] & (\mathcal{F}',\mathcal{A}'_1) \\
(\mathcal{F},\mathcal{A}_2) \arrow[r, "(f,\overline{f})"'] & (\mathcal{F}',\mathcal{A}'_2)\,. \arrow[u, "(e',\mathrm{id}_{B'})"']
\end{tikzcd}$$

By assumption, (e, id_B) and $(e', \mathrm{id}_{B'})$ are compatible with the atlases (see 2.3.5). If the bundle map $(f, \overline{f})$ on the bottom is compatible with the atlases, then by 2.3.4 so is also the bundle map on the top. ☐

2.3.11 Definition. A (*fiber*) *bundle map* $\xi \longrightarrow \xi'$, where

$$\xi = (F, G, B; \mathcal{F}, \mathcal{A}) \quad \text{and} \quad \xi' = (F, G, B'; \mathcal{F}', \mathcal{A}')\,,$$

is a set bundle map $(f, \overline{f}) : \mathcal{F} \longrightarrow \mathcal{F}'$ that is compatible with the associated maximal atlases $\mathcal{A}$ and $\mathcal{A}'$.

A bundle map will be denoted again by $(f, \overline{f})$. By Theorem 2.3.10, a set bundle map that is compatible with some atlas is compatible with the corresponding maximal atlas. This is consequent with our convention (see 2.3.9) to allow in the notation for ξ also atlases that are not maximal.

2.3.12 Note. By 2.3.4 one has that fiber bundles, together with bundle maps constitute a category. As usual, a *bundle equivalence* is a bundle map with an inverse.

This is a good oportunity to get to know the different equivalence concepts. "Set bundles with an atlas" and "bundle maps compatible with an atlas" constitute a category. In this category, (e, id) is an equivalence. It provides us with atlas equivalence; this will be important in 2.3.14 below, (see also 2.3.21).

A bundle map (f, id_B) is called an *equivalence over* B (cf. 2.3.13 below). If we consider (f, id_B) as a map of bundles with atlas, we obtain an equivalence relation, which is stronger than the one given by (e, id_B). It is now permitted, for example, to replace the bundle fibers with equivalent (homeomorphic) fibers without leaving the equivalence class. This equivalence concept is important for the bundle classification (see Section 2.8; see also 2.4.5). An equivalence $(f, \overline{f})$ of general type is independent of the specific type of the space B, i.e., we may replace B with homeomorphic spaces.

2.3.13 **Theorem**. *If* $(f, \overline{f}) : \xi \longrightarrow \xi'$ *is a bundle map and* $\overline{f} : B \longrightarrow B'$ *is a homeomorphism, then* $(f, \overline{f})$ *is an bundle equivalence.*

Proof: We have to define a bundle map $(f', \overline{f}')$ that is an inverse of $(f, \overline{f})$. We do it as follows. Take $\overline{f}' = \overline{f}^{-1}$ and $f' = \{f'_y \mid y \in B'\}$ such that $f'_y = f_x^{-1}$ if $\overline{f}(x) = y$. Then $(f', \overline{f}')$ is compatible with the atlas (cf. 2.3.3). Namely, to prove (C1), take $\varphi \in \mathcal{A}$, $\psi \in \mathcal{A}'$, and $y \in U_\psi \cap \overline{f}^{-1} U_\varphi$. Then

$$g'(y) = \varphi_x^{-1} f'_y \psi_y = (\psi_y^{-1} f_x \varphi_x)^{-1} = g(x)^{-1}\,,$$

since $\overline{f}'(y) = x \in U_\varphi \cap \overline{f}' U_\psi = U_\varphi \cap \overline{f}^{-1} U_\psi$.

To prove (C2), we have that the mapping $y \mapsto g'(y) = g(\overline{f}'(y))^{-1}$ is continuous, since $\overline{f}'$, g, and the map ι (which sends a group element to its inverse) are continuous. $\square$

2.3.14 Construction. Let $\mathcal{F}$ be a set bundle over B with fiber F. We shall assign to $\mathcal{F}$ a locally trivial fibration over B. To do this, let us assume that $\mathcal{F}_x \cap \mathcal{F}_y = \emptyset$ if $x, y \in B$ are different points. If this assumption does not hold in $\mathcal{F}$ *a priori*, we replace the sets $\mathcal{F}_x$ with $\{x\} \times \mathcal{F}_x$.

Let $\mathcal{A} = \{\varphi_j \mid j \in J\}$ be an atlas for $\mathcal{F}$ with respect to the group G, where $\varphi_j = \{\varphi_{j,x} : F \longrightarrow \mathcal{F}_x \mid x \in U_j\}$, where we write U_j instead of φ_j.

We define $p : E \longrightarrow B$ as follows. Take

$$E = \bigcup_{x \in B} \mathcal{F}_x \qquad \text{and} \qquad p(\mathcal{F}_x) = \{x\}\,.$$

We now endow E with a topology. Using the map Φ_j given by $(x, y) \mapsto \varphi_{j,x}(y)$ we have the next commutative diagram.

(2.3.15)
$$\begin{tikzcd} U_j \times F \arrow[rr, "\Phi_j"] \arrow[dr, "\mathrm{proj}_1"'] & & p^{-1}U_j = \bigcup_{x \in U_j} \mathcal{F}_x \arrow[dl, "p_{U_j}"] \\ & U_j\,. & \end{tikzcd}$$

By requiring that Φ is an identification in the diagram

$$\begin{tikzcd} \bigcup_j U_j \times F \times \{j\} \arrow[d, "\Phi"'] \arrow[dr, "\mathrm{proj}_1"] & \\ E \arrow[r, "p"'] & B\,, \end{tikzcd}$$

where $\Phi(x, y, j) = \Phi_j(x, y)$, we endow E with a topology and with it p turns out to be continuous.

We call $p : E \longrightarrow B$ the fibration *determined* by the set bundle $\mathcal{F}$.

2.3.16 **Lemma**. *Take $X_i = U_i \times F \times \{i\}$ and $E_i = p^{-1}U_i$. Then the restricted map $\Phi_i = \Phi|_{X_i}$ is a homeomorphism $X_i \longrightarrow E_i$.*

Proof: The map

$$\begin{tikzcd} G_{ij} : \Phi_j^{-1}(E_i \cap E_j) \arrow[r, "\Phi_j"] & E_i \cap E_j \arrow[r, "\Phi_i^{-1}"] & \Phi_i^{-1}(E_i \cap E_j) \\ (x,y,j) \arrow[rr, mapsto] & & (x, \varphi_{i,x}^{-1}\varphi_{j,x}(y), i) \end{tikzcd}$$

is continuous, since $\varphi_{i,x}^{-1} \circ \varphi_{j,x} \in G$ and G acts continuously on F. The map G_{ji} is inverse to G_{ij}, and therefore, G_{ij} is a homeomorphism. Let A be open in X_i. We have to prove that $\Phi_i A$ is open in E_i, that is, by the very definition of an identification, that $\Phi^{-1}\Phi_i A$ is open in $\bigcup_{j\in J} X_j$. This is equivalent to saying that

$$\begin{aligned} X_i \cap \Phi^{-1}\Phi_i A &= \Phi_j^{-1}\Phi_i(A \cap \Phi_i^{-1}(E_i \cap E_j)) \\ &= G_{ij}(A \cap \Phi_i^{-1}(E_i \cap E_j)) \end{aligned}$$

is open in X_j. But this is open in $\Phi_j^{-1}(E_i \cap E_j)$, since G_{ji} is a homeomorphism and $\Phi_j^{-1}(E_i \cap E_j) = (U_i \cap U_j) \times F \times \{j\}$ is open in X_j, we have that $X_i \cap \Phi^{-1}\Phi_i A$ is open in X_i. $\square$

From Diagram (2.3.15) and the previous lemma, we obtain the following two consequences.

2.3.17 **Proposition**. *p is locally trivial.* $\square$

2.3.18 **Proposition**. *The identification Φ is an open map.* $\square$

In what follows, we see that not only a fiber bundle gives rise to a locally trivial fibration, but also that a bundle map induces a fiber map.

2.3.19 Construction. Let $\mathcal{F}$ and $\mathcal{F}'$ be set bundles with atlases $\mathcal{A}$ and $\mathcal{A}'$. Let $(f, \overline{f}) : \mathcal{F} \longrightarrow \mathcal{F}'$ be a bundle map that is compatible with the atlases. We now want to construct a fiber map $(\widehat{f}, \overline{f})$ between the locally trivial fibrations determined by the given set bundles (2.3.14), namely,

$$\begin{tikzcd} E \arrow[r, dashed, "\widehat{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "\overline{f}"'] & B' , \end{tikzcd}$$

Taking $\widehat{f}(z) = f_x(z)$ if $z \in \mathcal{F}_x$, the diagram is commutative.

2.3.20 **Theorem**. *The map* $\widehat{f}$ *is continuous, and thus* $(\widehat{f}, \overline{f})$ *is a fiber map.*

Proof: The topologies in E and E' are given through identifications Φ and Ψ. One has

$$\begin{tikzcd} \bigcup_{\varphi\in\mathcal{A}} U_\varphi \times F \times \{\varphi\} \arrow[d, "\Phi"'] & \bigcup_{\psi\in\mathcal{A}'} U'_\psi \times F \times \{\psi\} \arrow[d, "\Psi"] \\ E \arrow[r, "\widehat{f}"'] & E\,. \end{tikzcd}$$

It is then enough to prove that $\widehat{f} \circ \Phi$ is continuous. For that, since the sets $(U_\varphi \cap \overline{f}^{-1} U_\psi) \times F \times \{\varphi\}$ build an open cover of $\bigcup U_\varphi \times F \times \{\varphi\}$, we only check that $\widehat{f} \circ \Phi|_{(U_\varphi \cap \overline{f}^{-1} U_\psi)\times F \times\{\varphi\}}$ is continuous for all $\varphi \in \mathcal{A}$ and all $\psi \in \mathcal{A}'$. One has

$$\begin{aligned} \widehat{f}\Phi(x, v, \varphi) &= \widehat{f}\varphi_x(v) = f_x\varphi_x(v) \\ &= \psi_y(\psi_y^{-1} f_x \varphi_x)(v) \\ &= \psi_y g(x)(v) \\ &= \Psi(y, g(x)v, \psi)\,, \end{aligned}$$

where $y = \overline{f}(x)$. The last term clearly depends continuously on (x, v), thus we obtain the desired continuity. $\square$

Let $\mathcal{F}$ be a set bundle over B with two atlases $\mathcal{A}$ and $\mathcal{A}'$. The set map $p : E \longrightarrow B$ (as in 2.3.14) depends only on $\mathcal{F}$. However, there are two topologies $\mathcal{T}$ and $\mathcal{T}'$ in E.

2.3.21 **Theorem**. *If* $\mathcal{A}$ *and* $\mathcal{A}'$ *are equivalent atlases, then the topologies* $\mathcal{T}$ *and* $\mathcal{T}'$*, generated by* $\mathcal{A}$ *and* $\mathcal{A}'$ *on* E *are the same.*

Proof: By 2.3.5, the bundle map $(e, \mathrm{id}_B) : \mathcal{F} \longrightarrow \mathcal{F}$ is compatible with the atlases. By 2.3.19, we have that the identity map

$$\mathrm{id}_E = \widehat{e} : (E, \mathcal{T}) \longrightarrow (E, \mathcal{T}')$$

is continuous. Similarly, one may prove that the inverse map is also continuous. $\square$

By Theorem 2.3.21 we may assign to each fiber bundle ξ (see Definition 2.3.8) a fibration $p_\xi : E \longrightarrow B$ and to each bundle map $(f, \overline{f})$ a fiber map $(\widehat{f}, \overline{f})$. We call them the fibration *determined* by the fiber bundle ξ and the fiber map *determined* by the bundle map $(f, \overline{f})$. This assignment is compatible with the composition of maps, as can easily be verified; thus we have the following.

2.3.22 **Theorem**. *The assignments*

$$\xi \longmapsto (p_\xi : E \longrightarrow B)$$
$$(f, \overline{f}) \longmapsto ((\widehat{f}, \overline{f}) : p_\xi \longrightarrow p_{\xi'})$$

define a functor from the category of fiber bundles and bundle maps to the category of locally trivial fibrations and fiber maps. To a trivial bundle, a trivial fibration is assigned. □

2.3.1 Tangent Bundles

As an application of the previous concepts, we shall construct the bundle of tangent vectors of a differentiable manifold.

2.3.23 Definition. A *one-one relation* f is a triple of sets $f = (X, Y, F)$ such that $F \subset X \times Y$ and such that for each $x \in X$ there exists at most one $y \in Y$ with $(x, y) \in F$.

The set
$$\mathrm{Def}(f) = \{x \in X \mid \exists y \in Y \text{ with } (x, y) \in F\}$$
is called the *definition domain* of the relation. We write this relation as $f : X \longrightarrow Y$. If $x \in \mathrm{Def}(f)$ and $(x, y) \in F$, then we write $y = f(x)$.

Let M and N be differentiable manifolds. A one-one relation $f : M \longrightarrow N$ is *differentiable* if

(1) $\mathrm{Def}(f) \subset M$ is an open set.

(2) $f|_{\mathrm{Def}(f)}$ is a differentiable map.

The composition of two differentiable one-one relations is again a differentiable one-one relation.

Let M be an n-dimensional smooth (i.e., of class C^∞) manifold. Take $x \in M$ and let ϑ_x be the set of differentiable one-one relations

$$f : M \longrightarrow \mathbb{R} \quad \text{with} \quad x \in \mathrm{Def}(f)\,.$$

ϑ_x is a vector space; namely, if $f, g \in \vartheta_x$, then $f + g \in \vartheta_x$ is given by $\mathrm{Def}(f+g) = \mathrm{Def}(f) \cap \mathrm{Def}(g)$, and for all $x' \in \mathrm{Def}(f) \cap \mathrm{Def}(g)$, $(f+g)(x') = f(x') + g(x') \in \mathbb{R}$. Moreover, if $\alpha \in \mathbb{R}$ and $f \in \vartheta_x$, then $\alpha f \in \vartheta_x$ is given by $\mathrm{Def}(\alpha f) = \mathrm{Def}(f)$, and for all $x' \in \mathrm{Def}(f)$, $(\alpha f)(x') = \alpha(f(x')) \in \mathbb{R}$. In fact, ϑ_x has also a multiplication that makes it an *algebra* over $\mathbb{R}$. Namely, if $f, g \in \vartheta_x$, then $f \cdot g \in \vartheta_x$ is given by $\mathrm{Def}(f \cdot g) = \mathrm{Def}(f) \cap \mathrm{Def}(g)$, and for all $x' \in \mathrm{Def}(f) \cap \mathrm{Def}(g)$, $(f \cdot g)(x') = f(x')g(x') \in \mathbb{R}$.

2.3.24 DEFINITION. A *tangent vector* of M at x is a map

$$X : \vartheta_x \longrightarrow \mathbb{R}$$

with the following properties:

(1) If $f, g \in \vartheta_x$ and $f(x) = g(x)$ for all y in some neighborhood of x, then

$$Xf = Xg\,.$$

This means that Xf depends only on the *germ* of f around x.

(2) $X(\alpha f + \beta g) = \alpha(Xf) + \beta(Xg)$ for $\alpha, \beta \in \mathbb{R}$. This means that X is *linear*.

(3) $X(f \cdot g) = (Xf)g(x) + f(x)(Xg)$. This means that X is a *derivation*.

See [11, 2.2].

Let $T_x(M)$ be the set of all tangent vectors of M at x. By means of the usual function addition and multiplication by a scalar, $T_x(M)$ gets a vector space structure. This is the *tangent space* of M at x.

2.3.25 CONSTRUCTION. Let $h : M \longrightarrow N$ be differentiable one-one relation, and take $x \in \mathrm{Def}(h)$. Defining

$$[dh_x X]f = X(f \circ h)\,, \quad f \in \vartheta_{h(x)}(N)\,,$$

one has a linear transformation

$$dh_x : T_x(M) \longrightarrow T_{h(x)}(N)\,.$$

The linear transformation dh_x is called the *derivative* of h at x. For a composite $M \xrightarrow{h} N \xrightarrow{g} P$ one has

$$d(g \circ h)_x = dg_{h(x)} \circ dh_x\,.$$

This equation is the *chain rule* for the derivative of a composite. For the identity map $\mathrm{id} : M \longrightarrow M$ in a neighborhood of x in M, one has

$$d(\mathrm{id})_x = \mathrm{id}_{T_x(M)}\,.$$

This, together with the chain rule, lets one obtain that if h is a *local diffeomorphism* around x, that is if h and h^{-1} are differentiable one-one relations, then

$$(dh^{-1})_{h(x)} = (dh_x)^{-1}\,.$$

In particular, dh_x is a linear isomorphism.

2.3.26 EXERCISE. Consider the category of pointed differentiable manifolds (M, x) with maps $h : (M, x) \longrightarrow (N, y)$ given by one-one relations h such that $x \in \mathrm{Def}(h)$ and $y = h(x)$. Verify that this is, indeed, a category and prove that the assignments

$$\begin{aligned} (M, x) &\longrightarrow T_x(M)\,, \\ h &\longmapsto dh_x\,, \end{aligned}$$

determine a functor from the just defined category to the category of finite dimensional vector spaces and linear transformations.

Take $M = \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Let $D_j(x)$ be the tangent vector at x given by

$$D_j(x) f = \frac{\partial f}{\partial x_j}|_x\,,$$

$j = 1, 2, \ldots, n$.

2.3.27 **Lemma**. *The vectors $D_1(x), \ldots, D_n(x)$ build a basis of the tangent space $T_x(\mathbb{R}^n)$.*

For the *proof* see [11, 2.3]. □

Via the mapping $(\alpha_1, \ldots, \alpha_n) \mapsto \sum \alpha_j D_j(x)$ one obtains an isomorphism $\mathbb{R}^n \longrightarrow T_x(\mathbb{R}^n)$ through which we identify both spaces.

2.3.28 DEFINITION. Let M be a differentiable n-manifold and $h : \mathbb{R}^n \longrightarrow M$ a differentiable one-one relation. If h^{-1} is also a differentiable one-one relation, then they determine a *diffeomorphism* $\mathrm{Im}(h) \approx \mathrm{Def}(h)$, where $\mathrm{Im}(h)$ is the *image* of the one-one relation h, that will be called *local chart*. For $x \in \mathrm{Im}(h)$ and $y = h^{-1}(x)$ one has an isomorphism

$$dh_y : T_y(\mathbb{R}^n) = \mathbb{R}^n \longrightarrow T_x(M)$$

(which, in particular, is bijective).

Let M be a differentiable n-manifold. We have a set bundle over M with fiber $\mathbb{R}^n$ (2.3.28) given by

$$T(M) = \mathcal{T} = \{T_x(M) \mid x \in M\}\,.$$

If h is a local chart for the manifold M, one can give a local chart $\varphi = \{\varphi_x(M) \mid x \in \operatorname{Im}(h)\}$ for $\mathcal{T}$ by

$$\varphi_x = dh_y : \mathbb{R}^n \longrightarrow T_x(M)\,, \quad y = h^{-1}(x)\,.$$

If we start with an atlas of local charts for M, we obtain an atlas indexed by $\{\varphi\}$ for $\mathcal{T}$ with respect to the group $\mathrm{GL}_n(\mathbb{R})$. Namely, take another local chart $\widetilde{h} : \mathbb{R}^n \longrightarrow M$ and $\widetilde{\varphi}_x = (d\widetilde{h})_{\widetilde{y}}$, $\widetilde{y} = \widetilde{h}^{-1}(x)$. For $x \in \operatorname{Im}(h) \cap \operatorname{Im}(\widetilde{h})$ (cf. 2.3.25) one has

$$\varphi_x^{-1} \circ \widetilde{\varphi}_x = (dh_y)^{-1} \circ d\widetilde{h}_{\widetilde{y}} = d(h^{-1} \circ \widetilde{h})_{\widetilde{y}}\,.$$

Thus $g(x) = d(h^{-1} \circ \widetilde{h})_{\widetilde{y}}$, since it is a linear transformation, is an element of $\mathrm{GL}_n(\mathbb{R})$, because $h^{-1} \circ \widetilde{h}$ has a differentiable inverse.

We still have to prove that $x \mapsto g(x)$ is a continuous map on $\operatorname{Im}(h) \cap \operatorname{Im}(\widetilde{h})$. This follows from the next result.

2.3.29 **Lemma**. *If $k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and its inverse $k^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are differentiable one-one relations, then the map*

$$\begin{aligned} \operatorname{Def}(k) &\longrightarrow \mathrm{GL}_n(\mathbb{R}) \\ x &\longmapsto dk_x \end{aligned}$$

is continuous.

Proof: Applying the basis and the identification of 2.3.27 one gets dk_x expressed by the jacobian of k in x. Thus the map is continuous. $\square$

2.4 Coordinate Transformations

In this section we explain how a fiber bundle is assembled.

2.4.1 Definition. Let $\xi = (F, G, B; \mathcal{F}, \mathcal{A})$ be a fiber bundle with (a not necessarily maximal) atlas $\mathcal{A} = \{\varphi_j \mid j \in J\}$. We shall again briefly write U_i instead of U_{φ_i} and $g_{ij}(x)$ instead of $\varphi_{i,x}^{-1} \circ \varphi_{j,x}$ for $x \in U_i \cap U_j$. The so-defined maps $g_{ij} : U_i \cap U_j \longrightarrow G$ will be called coordinate transformations of ξ. They are interrelated by means of the following equations:

(CT1) $\quad g_{ij}(x) g_{jk}(x) = g_{ik}, \quad x \in U_i \cap U_j \cap U_k, \quad i, j, k \in J.$

2.4.2 Definition. Let G be a topological group, B a topological space, and $\mathcal{U} = \{U_j \mid j \in J\}$ an open cover of B. A *cocycle* (*of dimension one*) *for* $\mathcal{U}$ *with coefficients in* G [2] is a family $\{g_{ij} : U_i \cap U_j \longrightarrow G \mid i, j \in J\}$ of continuous maps that satisfy (CT1). From (CT1) one obtains the following two consequences:

1. $g_{ii}(x) = 1 \in G, \;\; x \in U_i, \quad i \in J.$
2. $g_{ji}(x) = g_{ij}(x)^{-1}, \; x \in U_i \cap U_j, \quad i, j \in J.$

To obtain them, it is enough to set $i = j = k$ in (CT1) for 1, and then $i = k$ in 1 to get 2.

The maps g_{ij} of a fiber bundle as given above, describe how the trivial portions of the determined fibration have to be "assembled"; they are, so to say, "assembly instructions". We have the following.

2.4.3 **Theorem**. *Let $\{g_{ij}\}$ be a cocycle for $\mathcal{U}$ with coefficients in G. Then, for every topological space F on which G acts effectively, there is a set bundle over B with fiber F and an atlas for the group G, whose coordinate transformations (as in 2.4.1) are the maps g_{ij} of the cocycle.*

Proof: For $x \in B$ we choose an index $k_x \in J$ such that $x \in U_{k_x}$. We define a set bundle $\mathcal{F}$ and a set $\mathcal{A}$ of local charts by

$$\begin{aligned} \mathcal{F} &= \{\mathcal{F}_x \mid x \in B\}\,, \quad \mathcal{F}_x = F\,, \\ \mathcal{A} &= \{\varphi_j \mid j \in J\}\,, \quad \varphi_j = \{\varphi_{j,x} \mid x \in U_j\}\,, \\ \varphi_{j,x} &= g_{k_x j}(x) : F \longrightarrow F = \mathcal{F}_x\,, \quad x \in U_j\,. \end{aligned}$$

By definition, $\varphi_{j,x}$ is an element of the group G, and so it is a bijective map $F \longrightarrow F$; it is therefore a local chart.

$\mathcal{A}$ is an atlas; namely,

$$\begin{aligned} \varphi_{i,x}^{-1} \circ \varphi_{j,x} &= g_{k_x i}(x)^{-1} g_{k_x j}(x) \\ &= g_{i k_x}(x)\; g_{k_x j}(x) \\ &= g_{ij}(x) \in G\,. \end{aligned}$$

Conditions 2.3.1, (B1)–(B3) for an atlas are satisfied thanks to Definition 2.4.2, and so one sees immediately that these are the desired coordinate transformations. □

[2] To be more precise, one should have to say, "a cocycle with coefficients in the sheaf of germs of continuous maps $B \longrightarrow G$", (cf. Hirzebruch [4, 2.6]). However, no confusion should arise by our short form of stating it.

2.4.4 DEFINITION. Two cocycles $g = \{g_{ij}\}$ and $\widetilde{g} = \{\widetilde{g}_{ij}\}$ for the cover $\mathcal{U} = \{U_j \mid j \in J\}$ are said to be *cohomologous* in $\mathcal{U}$ if there is a family of continuous maps $\{\lambda_j : U_j \longrightarrow G\}$ that satisfies the equations

$$\text{(CT2)} \quad \widetilde{g}_{ij}(x)\lambda_j(x) = \lambda_i(x)g_{ij}(x), \quad x \in U_i \cap U_j \quad i,j \in J.$$

2.4.5 **Theorem**. *Let ξ, $\widetilde{\xi}$ be fiber bundles over B, with fiber F and structure group G. Let $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ be the corresponding atlases with the same cover $\mathcal{U}$ and coordinate transformations $\{g_{ij}\}$, $\{\widetilde{g}_{ij}\}$. ξ and $\widetilde{\xi}$ are equivalent over B (see 2.3.12) if and only if the cocycles $g = \{g_{ij}\}$ and $\widetilde{g} = \{\widetilde{g}_{ij}\}$ are cohomologous in $\mathcal{U}$.*

In particular, a fiber bundle is characterized, up to equivalence over B, by its coordinate transformations.

Proof: Let $(f, \mathrm{id}_B) : \xi \longrightarrow \widetilde{\xi}$ be an equivalence. By means of the mapping $x \mapsto \widetilde{\lambda}_j(x) = \widetilde{\varphi}_{j,x}^{-1} \circ f_x \circ \varphi_{j,x}$ a continuous map $\lambda_j : U_j \longrightarrow G$ is determined (by 2.3.3). One has

$$\begin{aligned} \widetilde{g}_{ij}(x)\lambda_j(x) &= (\widetilde{\varphi}_{i,x}^{-1} \circ \widetilde{\varphi}_{j,x}) \circ (\widetilde{\varphi}_{j,x} \circ f_x \circ \varphi_{i,x}) \\ &= \widetilde{\varphi}_{i,x}^{-1} \circ f_x \circ (\varphi_{i,x} \circ \varphi_{i,x}^{-1}) \circ \varphi_{j,x} \\ &= (\widetilde{\varphi}_{i,x}^{-1} \circ f_x \circ \varphi_{i,x}) \circ (\varphi_{i,x}^{-1} \circ \varphi_{j,x}) \\ &= \lambda_i(x) g_{ij}(x) \,. \end{aligned}$$

Thus the cocycles $g = \{g_{ij}\}$ and $\widetilde{g} = \{\widetilde{g}_{ij}\}$ are cohomologous in $\mathcal{U}$.

Conversely, let $g = \{g_{ij}\}$ and $\widetilde{g} = \{\widetilde{g}_{ij}\}$ be cohomologous in $\mathcal{U}$. The map

$$f_x = \widetilde{\varphi}_{j,x} \circ \lambda_j(x) \circ \varphi_{j,x}^{-1} : \mathcal{F}_x \longrightarrow \widetilde{\mathcal{F}_x}$$

is independent of j; namely, the right hand side is equal to

$$\begin{aligned} \widetilde{\varphi}_{j,x} \circ \lambda_j(x) \circ g_{ji}(x) \circ \varphi_{i,x}^{-1} &= \widetilde{\varphi}_{j,x} \circ \widetilde{g}_{ji}(x) \circ \lambda_i(x) \circ \varphi_{i,x}^{-1} \\ &= \widetilde{\varphi}_{i,x} \circ \lambda_i(x) \circ \varphi_{i,x}^{-1} \,, \quad x \in U_i \cap U_j \,. \end{aligned}$$

The pair $(\{f_x\}, \mathrm{id}_B)$ is a bundle map, since conditions 2.3.3 (C1) and (C2) are obtained from

$$\begin{aligned} \widetilde{\varphi}_{i,x} \circ f_x \circ \varphi_{j,x} &= \widetilde{g}_{ij}(x) \circ \widetilde{\varphi}_{j,x}^{-1} \circ \widetilde{\varphi}_{j,x} \circ \lambda_j(x) \circ \varphi_{j,x}^{-1} \circ \varphi_{j,x} \\ &= \widetilde{g}_{ij}(x)\lambda_j(x) \in G \,. \end{aligned}$$

□

Not every fiber bundle has an atlas for a given cover. For this reason, we wish to compare different covers.

Let $\mathcal{U} = \{U_j \mid j \in J\}$ and $mathcalV = \{V_k \mid k \in K\}$ be open covers of B. Let $mathcalV$ be a *refinement* of $\mathcal{U}$, i.e., there exists a function $\alpha : K \longrightarrow J$ with $V_k \subset U_{\alpha(k)}$ for every $k \in K$. Let $g = \{g_{ij} \mid i, j \in J\}$ be a cocycle for $\mathcal{U}$ with coefficients in G. By

$$h_{kl} = g_{\alpha(k)\alpha(l)}|_{V_k \cap V_l}\,, \quad k, l \in K\,,$$

we define a new cocycle $\alpha^{\#}(g) = \{h_{kl} \mid k, l \in K\}$ for $mathcalV$ with coefficients in G. This is the cocycle *induced* by the refinement.

2.4.6 Definition. Let g and $\widetilde{g}$ be cocycles for the covers $\mathcal{U} = \{U_j \mid j \in J\}$ and $\widetilde{\mathcal{U}} = \{\widetilde{U}_i \mid i \in \widetilde{J}\}$ with coefficients in G. We say that g and $\widetilde{g}$ are *cohomologous* in B if there exists a common refinement $mathcalV = \{V_k \mid k \in K\}$ of $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ and "refining functions" $\alpha : K \longrightarrow J$ and $\widetilde{\alpha} : K \longrightarrow \widetilde{J}$ such that $\alpha^{\#}(g)$ and $\widetilde{\alpha}^{\#}(\widetilde{g})$ are cohomologous in $mathcalV$.

"Cohomology in B" is an equivalence relation. Reflexivity and symmetry are clear. Transitivity will be proved inside the proof of Theorem 2.4.7, although it is an easy *exercise* to prove it directly.

We denote by $[g]$ the corresponding equivalence class and call it *cohomology class* of g.

Let $H^1(B; G)$ be the set of cohomology classes of cocycles for covers of B with coefficients in G. Let $k_G(F, B)$ be the set of equivalence classes (over B) of fiber bundles over B with fiber F and structure group G.

2.4.7 **Theorem**. *If to each fiber bundle, the cohomology class of the cocycle consisting of its coordinate transformations is assigned, there is a bijection*

$$\gamma : k_G(F, B) \longrightarrow H^1(B; G)\,,$$

induced by mapping each fiber bundle ξ to the cohomology class of the cocycle determined by its coordinate transformations.

For the proof, we need some previous considerations.

Let $\mathcal{F}$ be a set bundle over B with atlas $\mathcal{A} = \{\varphi_j \mid j \in J\}$ for the cover $\mathcal{U} = \{U_j \mid j \in J\}$. Let $mathcalV$ $=\{V_k \mid k \in A\}$ be an open refinement of $\mathcal{U}$ and $\alpha : K \longrightarrow J$ the refining function, (i.e. $V_k \subset U_{\alpha(k)}$).

Define

$$\begin{aligned}\psi_k &= \{\varphi_{\alpha(k),x} \mid x \in V_k\} \\ \alpha^{\#}\mathcal{A} &= \{\psi_k \mid k \in K\}\,.\end{aligned}$$

2.4.8 **Lemma**. *The following statements hold:*

(a) *$\alpha^{\#}\mathcal{A}$ is an atlas equivalent to $\mathcal{A}$.*

(b) *If the cocycle g consists of the coordinate transformations of $\mathcal{A}$, then $\alpha^{\#}g$ consists of those of $\alpha^{\#}\mathcal{A}$.*

Since the *proof* is simple, we leave it to the reader. □

Proof of 2.4.7:

γ is well defined:

Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be set bundles over B with fiber F. Let $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ be atlases for $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ with respect to the group G and with covers $\mathcal{U} = \{U_j \mid j \in J\}$ and $\widetilde{\mathcal{U}} = \{\widetilde{U}_i \mid i \in \widetilde{J}\}$. Then

$$mathcalV = \{U_j \cap \widetilde{U}_i \mid (j,i) \in J \times \widetilde{J}\}$$

is an open refinement of $\mathcal{U}$ and $\widetilde{\mathcal{U}}$. As refining functions we have the projections

$$\begin{array}{lll} & \nearrow^{\widetilde{\alpha}} \ \widetilde{J}\,, & \widetilde{\alpha}(j,i) = i \\ J \times \widetilde{J} & & \\ & \searrow_{\alpha} \ J\,, & \alpha(j,i) := j\,. \end{array}$$

Let now g and $\widetilde{g}$ be the cocycles consisting of the coordinate transformations of $\mathcal{A}$ and $\widetilde{\mathcal{A}}$, respectively.

One has that, since $(\mathcal{F},\mathcal{A})$ and $(\widetilde{\mathcal{F}},\widetilde{\mathcal{A}})$ are equivalent over B, by 2.4.8 (a), $(\mathcal{F},\alpha^{\#}\mathcal{A})$ and $(\widetilde{\mathcal{F}},\widetilde{\alpha}^{\#}\widetilde{\mathcal{A}})$ are also equivalent over B. By 2.4.8 (b) and 2.4.5, $\alpha^{\#}g$ is cohomologous to $\widetilde{\alpha}^{\#}\widetilde{g}$ in $mathcalV$; and by Definition 2.4.6, g is cohomologous to $\widetilde{g}$ en B.

γ is injective:

Let $\mathcal{F}$, $\mathcal{A}$, $\mathcal{U}$, g and $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{U}}$, $\widetilde{g}$ be as in the first part of the proof. Let g and $\widetilde{g}$ be cohomologous in B. By definition, there exists an common open refinement $mathcalV = \{V_k \mid k \in K\}$ of $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ with refining maps $\alpha : K \longrightarrow J$ and $\widetilde{\alpha} : K \longrightarrow \widetilde{J}$ such that $\alpha^{\#}g$ and $\widetilde{\alpha}^{\#}\widetilde{g}$ are cohomologous in $mathcalV$. Consequently, $(\mathcal{F},\alpha^{\#}\mathcal{A})$ and $(\widetilde{\mathcal{F}},\widetilde{\alpha}^{\#}\widetilde{\mathcal{A}})$ are equivalent over B (see

2.4.8 (b) and 2.4.5) and so, by 2.4.8 (a), $(\mathcal{F}, \mathcal{A})$ and $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{A}})$ are equivalent over B too.

γ is surjective:

This is exactly the statement of Theorem 2.4.3. □

Since the "cohomology set" $H^1(B; G)$ is independent of the fiber F, we may use 2.4.7 to establish a relationship among bundles with different fibers, but the same structure group.

2.4.9 Definition. Two fiber bundles

$$\xi = (F, G, B; \mathcal{F}, \mathcal{A}), \qquad \widetilde{\xi} = (\widetilde{F}, G, B; \widetilde{\mathcal{F}}, \widetilde{\mathcal{A}})$$

are called *associated* if the cocycles consisting of their coordinate transformations are cohomologous in B; that is, if the images of their equivalence classes under

$$\gamma : k_G(F, B) \longrightarrow H^1(B; G) \quad \text{and} \quad \widetilde{\gamma} : k_G(\widetilde{F}, B) \longrightarrow H^1(B; G)$$

coincide.

2.4.10 Definition. Let $\theta : G \longrightarrow H$ be a continuous homomorphism of topological groups. If $\{g_{ij}\}$ is a G-cocycle, then $\{\theta \circ g_{ij}\}$ is an H-cocycle, as one deduces from (CT1). The assignment $\{g_{ij}\} \mapsto \{\theta \circ g_{ij}\}$ is compatible with the cohomology relation and determines a function

$$\theta_* : H^1(B; G) \longrightarrow H^1(B; H).$$

A geometric interpretation of θ_* is the following. Let $\theta : G \hookrightarrow H$ be the inclusion of a subgroup. If H acts effectively on F and G acts by restricting the action of H, then one may clearly consider a bundle $\xi = (F, G, B; \mathcal{F}, \mathcal{A})$ as a bundle with structure group H. However, in this case, θ_* does not have to be injective. By passing to the larger group H two nonequivalent bundles may become equivalent, as we show below in the case of the twisted torus (cf. 2.4.11, 3).

2.4.11 Examples.

1. Using the method shown in 2.3.2 and the local trivializations of 1.2.9(a) one may assign to the Moebius strip a set bundle and two local charts.

These local charts constitute an atlas, if we consider $G = \mathbb{Z}_2$ as the structure group seen as the group whose elements are $1 = \mathrm{id}_I$ and the reflection $t \mapsto 1 - t$ in I, endowed with the discrete topology. G then acts continuously and effectively on I.

2. We may similarly consider the Klein bottle. The structure group $G = \mathbb{Z}_2$ consists here of $1 = \mathrm{id}_{\mathbb{S}^1}$ and the reflection on a diameter of $\mathbb{S}^1$.

   Both the Moebius strip and the Klein bottles are associated.

3. The twisted torus is a set bundle over the circle $\mathbb{S}^1$ with fiber $F = \mathbb{S}^1 = \mathcal{F}_x$, $x \in \mathbb{S}^1$, $\mathbb{S}^1 = I/\{0,1\}$. Let $U_0 = \mathbb{S}^1 - \{0\}$, $U_1 = \mathbb{S}^1 - \{b\}$, $0 < b < 1$, two open sets in the circle with the local charts φ_0 and φ_1 given by

$$\begin{aligned} \varphi_{0,x} : \mathbb{S}^1 &\longrightarrow \mathcal{F}_x \\ s &\longmapsto s \end{aligned}$$

$$\begin{aligned} \varphi_{1,x} : \mathbb{S}^1 &\longrightarrow \mathcal{F}_x \\ s &\longmapsto s\,, \quad \text{for } b < x \leq 1 \\ s &\longmapsto gs\,, \ \text{for } 0 < x < b \end{aligned}$$

where $g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a rotation by π.

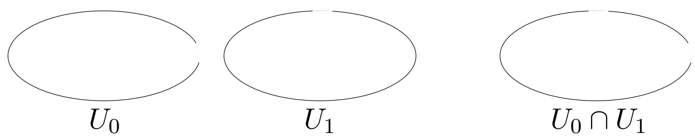


Figure 2.2

The only nontrivial coordinate transformation is

$$g_{0\,1}(x) = \varphi_{0,x}^{-1} \circ \varphi_{1,x} = \begin{cases} \mathrm{id}_{\mathbb{S}^1} & \text{if } b < x < 1, \\ g & \text{if } 0 < x < b. \end{cases}$$

The group $G = \{\mathrm{id}, g\} \cong \mathbb{Z}_2$ is again the structure group. The twisted torus is associated to the Klein bottle and therefore, it is nontrivial, since one of two associated bundles is trivial if and only if the other is also trivial.

If we take $\varphi_{1,x}$ to be also the identity, then instead of the twisted torus we obtain the trivial bundle. Defining

$$\begin{aligned} f_x : \mathcal{F}_x &\longrightarrow \mathcal{F}_x \\ s &\longmapsto d_x s\,, \end{aligned}$$

where d_x is the rotation in $\mathbb{S}^1$ by the angle πx, we obtain a map from the trivial bundle into the twisted torus that is compatible with the altases if we use as structure group not only $G = \{\mathrm{id}, g\}$, but the whole rotation group SO_2. By passing to the larger group SO_2, the twisted torus turns out to be equivalent to the trivial one. On the contrary, neither the Moebius strip, nor the Klein bottle can be trivialized by passing to a larger group, since their associated fibrations are nontrivial. (cf. 1.2.9).

2.4.1 Vector Bundles

A specially important role in algebraic topology, algebraic geometry, and differential geometry is played by the vector bundles, which constitute a special class with an interest of its own. See [1] for a more detailed exposition on them.

2.4.12 Definition. A *real* (resp. *complex*) *vector bundle* of dimension n is a fiber bundle $\xi = (\mathbb{R}^n, \mathrm{GL}_n(\mathbb{R}), B; \mathcal{F}, \mathcal{A})$, (resp. $\xi = (\mathbb{C}^n, \mathrm{GL}_n(\mathbb{C}), B; \mathcal{F}, \mathcal{A})$). By requiring that $\varphi_x : \mathbb{R}^n \longrightarrow \mathcal{F}_x$ be an isomorphism for every $\varphi \in \mathcal{A}$, we may furnish $\mathcal{F}_x$ with a vector space structure, independently of the local chart φ with $x \in U_\varphi$.

The usual operations of vector spaces can be extended to vector bundles.

Let V be a vector space and V^* be its dual. An isomorphism $f : V \longrightarrow W$ induces a dual isomorphism $f^t : W^* \longrightarrow V^*$ and this in turn induces

$$f^* = (f^t)^{-1} : V^* \longrightarrow W^* \,.$$

2.4.13 Definition. Given a vector bundle

$$\xi = (\mathbb{R}^n, \mathrm{GL}_n(\mathbb{R}), B; \mathcal{F}, \mathcal{A})\,,$$

its *dual vector bundle* is defined by

$$\xi^* = (\mathbb{R}^n = (\mathbb{R}^n)^*, \mathrm{GL}_n(\mathbb{R}), B; \mathcal{F}^*, \mathcal{A}^*)\,,$$

where

$$\mathcal{F}^* = \{(\mathcal{F}_x)^* \mid x \in B\}\,, \quad \varphi^* = \{(\varphi_x)^* : (\mathbb{R}^n)^* \longrightarrow (\mathcal{F}_x)^*\}$$

for $\varphi \in \mathcal{A}$. One has for $\varphi, \psi \in \mathcal{A}$

$$(\psi_x^*)^{-1} \circ \varphi_x^* = (\psi_x^{-1} \circ \varphi_x)^*$$

and $(\psi_x^{-1} \circ \varphi_x)^*$ is an automorphism of $(\mathbb{R}^n)^* = \mathbb{R}^n$ and hence it lies in $\mathrm{GL}_n(\mathbb{R})$. If we represent $\psi_x^{-1} \circ \varphi_x$ (with respect to the canonical basis of $\mathbb{R}^n$) by the matrix A_x, then $(\psi_x^*)^{-1} \circ \varphi_x^*$ is represented by $(A_x^t)^{-1}$, thus depending continuously on x.

2.4.14 Definition. Let

$$\xi_1 = (\mathbb{R}^n, \mathrm{GL}_n(\mathbb{R}), B; \mathcal{F}_1, \mathcal{A}_1) \quad \text{and} \quad \xi_2 = (\mathbb{R}^m, \mathrm{GL}_m(\mathbb{R}), B; \mathcal{F}_2, \mathcal{A}_2)$$

be vector bundles. Their *Whitney sum* is the vector bundle

$$\xi_1 \oplus \xi_2 = (\mathbb{R}^{n+m}, \mathrm{GL}_{n+m}(\mathbb{R}), B; \mathcal{F}, \mathcal{A})\,,$$

with

$$\mathcal{F}_x = \mathcal{F}_{1x} \oplus \mathcal{F}_{2x} \quad \text{and} \quad \varphi_{j,x} = \varphi_{1\,j,x} \oplus \varphi_{2\,j,x} : \mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m \longrightarrow \mathcal{F}_x\,.$$

$\varphi_{1,j}$ and $\varphi_{2,j}$ run independently along the atlases $\mathcal{A}_1$ and $\mathcal{A}_2$,

$$g_{ij}(x) = \varphi_{i,x}^{-1}\varphi_{j,x} = g_{1\,ij}(x) \oplus g_{2\,ij}(x) : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^{n+m}$$

is an element of $\mathrm{GL}_{n+m}(\mathbb{R})$.

2.4.15 Definition. Given two vector bundles

$$\xi_1 = (\mathbb{R}^n, \mathrm{GL}_n(\mathbb{R}), B; \mathcal{F}_1, \mathcal{A}_1) \quad \text{and} \quad \xi_2 = (\mathbb{R}^m, \mathrm{GL}_m(\mathbb{R}), B; \mathcal{F}_2, \mathcal{A}_2)\,,$$

we define their *tensor product* as the vector bundle

$$\xi_1 \otimes \xi_2 = (\mathbb{R}^{nm}, \mathrm{GL}_{nm}(\mathbb{R}), B; \mathcal{F}, \mathcal{A})$$

with

$$\mathcal{F}_x = \mathcal{F}_{1x} \otimes \mathcal{F}_{2x} \quad \text{and} \quad \varphi_{j,x} = \varphi_{1\,j,x} \otimes \varphi_{2\,j,x} : \mathbb{R}^{nm} = \mathbb{R}^n \otimes \mathbb{R}^m \longrightarrow \mathcal{F}_x\,.$$

Different isomorphisms $\mathbb{R}^{nm} \cong \mathbb{R}^n \otimes \mathbb{R}^m$ give origin to different equivalent vector bundles.

For other possible constructions see [1].

2.5 Principal Bundles

The previous considerations on coordinate transformations show that for the classification of fiber bundles over B with structure group G it is not necessary to know the fiber F (on which G acts effectively, (cf. 2.4.7, 2.4.8 (a), 2.4.9). Thus one may choose a convenient fiber, and the same structure group, namely G, is a good candidate. In this case (as we shall see) it is possible to endow the fibrations and their corresponding fiber bundles with an additional structure, (namely, an action of G on the total space).

If G is a topological group and $F = G$, we assume in this section that G acts by left translation on F.

2.5.1 Definition. Let G be a topological group. A *principal G-bundle* (or simply, a *principal bundle*) is a fiber bundle of the form

$$\xi = (G, G, B; \mathcal{G}, \mathcal{A})\,;$$

that is, a fiber bundle whose fiber coincides with its structure group with the effective action given by left translation.

For a principal bundle, using $\varphi_x : G \longrightarrow \mathcal{G}_x$ we may transform the right translations of G into a right action of G on $\mathcal{G}_x$ as follows.

Take $u \in G$ and $z \in \mathcal{G}_x$, and define the action by

$$zu = \varphi_x[(\varphi_x^{-1} z)u]\,.$$

It is easy to check that this action is independent of the choice of φ with $x \in U_\varphi$ and that the properties 2.2.12 (a) and (b) hold. These actions determine a right action

$$\rho_\xi : E \times G \longrightarrow E\,,$$

($E = \bigcup_{x \in B} \mathcal{G}_x$, if the $\mathcal{G}_x$ are disjoint to each other).

2.5.2 Definition. Let G be a topological group. A *principal G-fibration* (or simply, a *principal fibration*) is a pair (p, ρ) consisting of a fibration $p : E \longrightarrow B$ and a right action $\rho : E \times G \longrightarrow E$ such that the diagram

$$\begin{tikzcd} E \times G \arrow[r, "\rho"] \arrow[d, "\mathrm{proj}_1"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

is commutative; that is, for every $x \in E$ and $g \in G$, one has that $p(xg) = p(x)$.

A fiber map $(f, \overline{f}) : p \longrightarrow p'$ between principal G-fibrations is called a *principal map* if

$$f(zu) = f(z)u\,, \quad z \in E\,, \quad u \in G\,,$$

in other words, if the map $f : E \longrightarrow E'$ is *equivariant.*

Again, principal G-fibrations and principal maps build a category.

The trivial fibration $\mathrm{proj}_1 : B \times G \longrightarrow B$, together with the action $\rho : (B \times G) \times G \longrightarrow B \times G$ given by $\rho((x, v), u) = (x, vu)$ is a principal fibration called the *trivial principal G-fibration.* If $A \subset B$, then $\rho : E \times G \longrightarrow E$ induces a map $\rho_A : (p^{-1}A) \times G \longrightarrow p^{-1}A$ that equips $p_A : p^{-1}A \longrightarrow A$ with the structure of a principal fibration. We shall denote it again by p_A.

A principal fibration (p, ρ) is called *locally trivial*, if for every $z \in B$, there is a neighborhood U of z and a principal equivalence (that is, a principal map, that is an equivalence) between (p_U, ρ_U) and the trivial principal G-fibration over U.

2.5.3 **Theorem**. *The assignments*

$$\begin{aligned} \xi &\longmapsto (p_\xi,\ \rho_\xi) \\ (f, \overline{f}) &\longmapsto (\widehat{f}, \overline{f}) \end{aligned}$$

determine a functor from the category of principal G-bundles to the category of locally trivial principal G-fibrations. In particular, they assign to the trivial principal bundle, the trivial principal fibration (cf. 2.3.22).

Proof: First we show that ρ_ξ is continuous. For that we recall the definition of the topology of E, 2.3.14. Let $\mathcal{A}$ be an atlas for ξ. In the diagram

$$\begin{tikzcd} \bigcup_{\varphi\in\mathcal{A}}\{\varphi\} \times U_\varphi \times G \times G \arrow[r, "\rho'"] \arrow[d, "\Phi\times\mathrm{id}_G"'] & \bigcup_{\varphi\in\mathcal{A}}\{\varphi\} \times U_\varphi \times G \arrow[d, "\Phi"] \\ E \times G \arrow[r, "\rho"'] & E\,, \end{tikzcd}$$

let ρ' be given by $\rho'(\varphi, x, v, u) = (\varphi, x, vu)$. The diagram is commutative, Φ is an open map (see 2.3.18), and therefore, also $\Phi \times \mathrm{id}_G$ is open. Thus it is an identification. Since $\Phi\rho'$ is continuous, one proves the assertion.

The fibration $p = p_\xi$ is locally trivial, since for every $\varphi \in \mathcal{A}$, Φ induces a principal map

$$\begin{tikzcd} U_\varphi \times G \arrow[rr, "\Phi'"] \arrow[dr, "\mathrm{proj}_1"'] & & p^{-1}U_\varphi \arrow[dl, "p_{U_\varphi}"] \\ & U_\varphi \,. & \end{tikzcd}$$

If ξ is trivial, that is, if its atlas consists of just one chart φ, then the previous considerations imply that p_ξ is trivial, through the trivialization

$$\Phi' = \Phi : U_\varphi \times G = B \times G \longrightarrow p^{-1}U_\varphi = E \,.$$

By 2.3.22, we still have to prove that the determined fiber maps are principal maps, that is, that they are equivariant. Namely, $\widehat{f}(zu) = \widehat{f}(z)u$, $z \in E_\xi$, $u \in G$. Take $pz = p(zu) = x \in U_\varphi$ and $\overline{f}x = y \in U_\psi$. Then $f_x = \psi_y \circ g(x) \circ \varphi_x^{-1}$, and all three maps on the right-hand side are equivariant, i.e., compatible with the right action of G, (cf. 2.5.1). From 2.3.19 one has

$$\begin{aligned} \widehat{f}(zu) &= f_x(zu) \\ &= \psi_y \circ g(x) \circ \varphi_x^{-1}(zu) \\ &= (\psi_y \circ g(x) \circ \varphi_x^{-1}(z))u \\ &= f_x(z)u \\ &= \widehat{f}(z)u \,. \end{aligned}$$

$\square$

2.5.4 **Theorem**. *Let ξ and ξ' be principal G-bundles, and let $(h, \overline{h}) : p_\xi \longrightarrow p_{\xi'}$ be a principal map between their determined fibrations. Then there is a unique principal G-bundle map $(f, \overline{f}) : \xi \longrightarrow \xi'$ such that $\widehat{f} = h$ and $\overline{f} = \overline{h}$.*

Proof: Uniqueness is clear by the definition of $\widehat{f}$ (2.3.19). For the existence we have that f has to coincide with h on the fibers. Thus, take

$$f_x(z) = h(z) \quad \text{ for } \quad z \in \mathcal{G}_x \,.$$

If $y = \overline{f}(x)$, then one has $f_x(z) \in \mathcal{G}'_y$, since $(h, \overline{h})$ is a fiber map. We have to prove that f_x is bijective and that $f = \{f_x\}$ is compatible with the atlases. Take $\varphi \in \mathcal{A}$, $\psi \in \mathcal{A}'$, and $v \in G$. Then

$$\begin{aligned} (\psi_y^{-1} f_x \varphi_x)(v) &= (\psi_y^{-1} h \varphi_x)(ev) \\ &= (\psi_y^{-1} h \varphi_x(e))v \,; \end{aligned}$$

the last equality holds, since by assumption, ψ_y^{-1}, h, and φ_x are compatible with the right action of G, (see 2.5.1, 2.5.2). Now $g(x) = \psi_y^{-1} h \varphi_x(1) \in G$

defines a bijective map from G into itself (left translation by $g(x)$); since ψ_y and φ_x are bijective, so is also f_x. We still have to prove (cf. 2.3.3 (C2)) that the mapping $x \mapsto g(x)$ is continuous on $U_\varphi \cap \overline{f}^{-1} U_\psi$. This follows, since we may write g as the following composite of continuous maps:

$$\begin{array}{ccccc} U_\varphi \cap \overline{f}^{-1}U_\psi & \longrightarrow & (U_\varphi \cap \overline{f}^{-1}U_\psi) \times G & \xrightarrow{\Phi|} & p_\xi^{-1}(U_\varphi \cap \overline{f}^{-1}U_\psi) \\ x & \longmapsto & (x,1) & \longmapsto & \varphi_x(1) \end{array}$$

$$\begin{array}{ccccccc} \longrightarrow & p_\xi^{-1}(U_\psi) & \xrightarrow[\approx]{(\Phi|\ )^{-1}} & U_\psi \times G & \xrightarrow{\mathrm{proj}_2} & G \\ \longmapsto & f\varphi_x(1) & \longmapsto & (y, \psi_y^{-1} f\varphi_x(1)) & \longmapsto & \psi_y^{-1} f\varphi_x(1)\,. \end{array}$$

Finally, $\widehat{f} = h$ is clear. □

2.5.5 **Theorem**. *A principal bundle ξ is trivial if and only if the fibration p_ξ has a section.*

Proof: If ξ es trivial, so is also the determined fibration p_ξ (see 2.5.3). Therefore, p_ξ has a section.

Assume conversely that p_ξ has a section $s : B \longrightarrow E$. We define a map $f : B \times G \longrightarrow E$ by $f(b,v) = s(b)v$. Hence (f, id_B) is a principal map from the trivial principal fibration $\mathrm{proj}_1 : B \times G \longrightarrow B$ to p_ξ. proj_1 belongs to the trivial principal bundle over B, so (f, id_B) belongs to the bundle map (2.5.4), that, by 2.3.13 is an equivalence; but this means that ξ is trivial. □

2.5.6 REMARK. If ξ is not a principal bundle, then p_ξ may have a section, even though ξ is nontrivial. For example, for a vector bundle there is always a 0-section.

Other interesting (related) example is the following.

2.5.7 EXAMPLE. The Moebius strip has a section induced by the map $I \longrightarrow I \times I$, given by $s \mapsto (s, \frac{1}{2})$ (see 1.1.1 (b)). Associated bundles are simultaneosuly trivial. The associated principal bundle of the Moebius strip is the double covering map of the circle, which obviously does not have a section (see Figure 2.3).

Intuitively, we can say that the total space of a trivial fibration is composed of "layers", that are the images of sections. If a principal fibration has a section, we may "transport" it by means of the group action, so that each point of the total space lies in the image of a section (one says that the total space is "foliated"). Cf. the proof of 2.5.5.

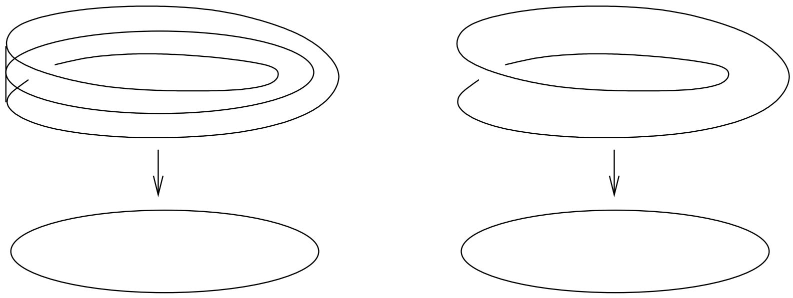


Figure 2.3

2.5.8 **Theorem**. *For each locally trivial principal fibration $p : E \longrightarrow B$, with group action $\rho : E \times G \longrightarrow E$, there is a unique principal bundle ξ such that $p = p_\xi$, $\rho = \rho_\xi$.*

Proof: Let $\{U_j \mid j \in J\}$ be an open cover of B and let p_{U_j} be trivial. Assume the principal maps

$$\Phi_j : U_j \times G \longrightarrow p^{-1}U_j$$

describe the local triviality. We have to define $\xi = (G, G, B; \mathcal{G}, \mathcal{A})$. Obviously, we have to set $\mathcal{G}_x = p^{-1}(x)$. $\mathcal{A} = \{\varphi_j\}$, $\varphi_j = \{\varphi_{j,x} \mid x \in U_j\}$, and $\varphi_{j,x} : G \longrightarrow \mathcal{G}_x$ will be given by $\varphi_{j,x}(u) = \Phi_j(x, u)$, $u \in G$. $\varphi_{j,x}$ is bijective, since Φ_j is a homeomorphism. Since Φ_j is compatible with the right action of G, $\varphi_{j,x}$ is also bijective, and one has

$$\varphi_{i,x}^{-1}\varphi_{j,x}(v) = \varphi_{i,x}^{-1}\varphi_{j,x}(e)v, \quad x \in U_i \cap U_j\,,$$

and we define $g_{ij}(x) = \varphi_{i,x}^{-1}\varphi_{j,x}(e)$. So, $g_{ij} : U_i \cap U_j \longrightarrow G$ is continuous, because

$$(x, g_{ij}(x)) = \Phi_i^{-1}\Phi_j(x, 1).$$

$\mathcal{A}$ is thus an atlas for $\mathcal{G}$ (where G acts on itself by left translation). Hence, ξ is a principal bundle.

If we now construct the fibration corresponding to ξ, we may ask if the space $E = \bigcup_{x \in B} \mathcal{G}_x$ recovers its original topology. This is, in fact, the case, since the map

$$\Phi : \bigcup U_j \times G \times \{j\} \longrightarrow E\,,$$

which according to 2.3.14 has to be constructed, must be an identification, which is even an open map if one takes in E the original topology. On the other hand, $\rho_\xi = \rho$, since $\varphi_{j,x}$ is compatible with the right action (see 2.5.1). All this shows the existence of a bundle ξ with the desired properties. Let

ξ' be another bundle with these properties. From $p_\xi = p_{\xi'}$ one obtains that both set bundles $\mathcal{G}$ and $\mathcal{G}'$ are equal. The equality of the corresponding fiber will be obtained after proving that the identity of the set bundles is a bundle map (2.3.5, 2.3.8). But the identity of p_ξ is a principal map. Therefore, 2.5.4 gives us the desired result. □

Let E be a topological group, G a subgroup of E, and $B = E/G$ the set of left cosets $\{zG \mid z \in E\}$. We endow B with the identification topology given by the natural projection $p : E \longrightarrow B$ (that is, B is a homogeneous space). The action $\rho : E \times G \longrightarrow E$ is given by $\rho(z, u) = zu$ and turns (p, ρ) into a principal fibration. We shall analyze under what conditions this is locally trivial.

2.5.9 **Theorem**. *If there exists an open neighborhood U of $p(e)$ (where $e = 1 \in E$ is the neutral element), and a map $s : U \longrightarrow p^{-1}U$ such that $p \circ s = \mathrm{id}_U$ (i.e., a section over U, or a local section), then (p, ρ) is a locally trivial principal fibration.*

Proof: Let $x \in B$ be any point, say $x = p(z)$. Then $U' = zU$ is an open neighborhood of x (E acts on B by left translation; 2.2.18).

The map $s' : U' \longrightarrow p^{-1}U'$ given by $y \mapsto zs(z^{-1}y)$ is continuous, and since

$$ps'(y) = p(zs(z^{-1}y)) = zps(z^{-1}y) = zz^{-1}y = y\,,$$

it is a section over U'.

Let U be open in B and s a section over U. Then

$$\begin{aligned} U \times G &\longrightarrow p^{-1}U \\ (x, v) &\longmapsto s(x)v \end{aligned}$$

is a principal map. It has as inverse the map

$$\begin{aligned} p^{-1}U &\longrightarrow U \times G \\ z &\longmapsto (p(z), sp(z)^{-1}\, z)\,, \end{aligned}$$

which is also a principal map. Thus, p is trivial over U. This, together with the first part of the proof, yields the desired statement. □

2.5.10 REMARK. The assumption of the previous theorem (the existence of a local section) holds, for example, if E is locally compact and finite dimensional (e.g. a finite CW-complex) and G is a closed subgroup (cf. [15, Appendix 1])

The following special case is easy to grasp. Take E to be a Lie group and G a closed subgroup, (cf. Chevalley [2, p. 110, Prop. 1]).

2.5.11 Definition. Let $\xi = (G, G, B; \mathcal{G}, \mathcal{A})$ be a principal bundle and H a closed subgroup of G. We define a bundle

$$\xi/H = (G/H, \widetilde{G}, B; \mathcal{F}, \widetilde{\mathcal{A}})$$

as follows. H acts on every $\mathcal{G}_x$ on the right. Consider the equivalence relation in $\mathcal{G}_x$ given by $z_1 \sim z_2$ if there exists $h \in H$ such that $z_1 = z_2 h$. Let $\mathcal{G}_x/H$ be the set of equivalence classes. Setting $\mathcal{F}_x = \mathcal{G}_x/H$ we may define

$$\widetilde{\mathcal{A}} = \{\widetilde{\varphi} \mid \varphi \in \mathcal{A}\}\,, \quad \widetilde{\varphi} = \{\widetilde{\varphi}_x \mid x \in U_\varphi\}$$

such that

$$\widetilde{\varphi}_x : G/H \longrightarrow \mathcal{F}_x$$

is the bijection canonically induced by φ_x. Thus $\widetilde{\psi}_x^{-1} \circ \widetilde{\varphi}_x : G/H \longrightarrow G/H$ is induced by $\psi_x^{-1} \circ \varphi_x = g(x) \in G$.

If the natural action (2.2.18) $G \times G/H \longrightarrow G/H$ is effective, then ξ/H with this action and structure group $\widetilde{G} = G$ is a fiber bundle.

If this action is not effective, then from $uvH = vH$ ($v \in G$, $u \in G$), one obtains $v^{-1}uv \in H$, so that $u \in vHv^{-1}$; thus, $u \in \bigcap_{v \in G} vHv^{-1}$. The group $H_0 = \bigcap_{v \in G} vHv^{-1}$ is the maximal normal subgroup of G contained in H. The natural action

$$G/H_0 \times G/H \longrightarrow G/H$$

is now effective and defining $\widetilde{G} = G/H_0$, ξ/H turns out to be a fiber bundle (considering $\widetilde{\psi}_x^{-1} \circ \widetilde{\varphi}_x$ as an element of the quotient group G/H_0).

2.5.12 Definition. Let $p : E \longrightarrow B$ be the fibration corresponding to ξ and $\widetilde{p} : \widetilde{E} \longrightarrow B$ the one corresponding to ξ/H. H acts, as a subgroup of G, on E. The set $\widetilde{E}$ is obtained from E by identifying with respect to the action of H (i.e., dividing out the H-action). Let $\pi : E \longrightarrow \widetilde{E}$ be the natural projection.

2.5.13 **Lemma**. *π is an identification.*

Proof: The topologies of both E and $\widetilde{E}$ are given through the open maps Φ and $\widetilde{\Phi}$ (2.3.14). Consider the diagram

$$\begin{tikzcd} \bigcup U_\varphi \times G \arrow[r, "\pi'"] \arrow[d, "\Phi"'] & \bigcup U_\varphi \times G/H \arrow[d, "\widetilde{\Phi}"] \\ E \arrow[r, "\pi"'] & \widetilde{E}\,. \end{tikzcd}$$

This diagram is commutative if one takes π' on each summand as the product of the identity with the natural projection $q : G \longrightarrow G/H$. q is an open map, therefore, also π'. Thus, $\pi \circ \Phi = \widetilde{\Phi} \circ \pi'$ is an identification and hence also Φ. Finally, π is also one. $\square$

2.5.14 DEFINITION. Let $\xi = (F, G, B; \mathcal{F}, \mathcal{A})$ be a fiber bundle. The principal bundle $\widetilde{\xi} = (G, G, B; \mathcal{G}, \widetilde{\mathcal{A}})$ associated to ξ is described as follows.

We say that a map $f : F \longrightarrow \mathcal{F}_x$ is *admissible* if $\varphi_x^{-1} \circ f \in G$ for $\varphi \in \mathcal{A}$ and $x \in U_\varphi$.

This definition is independent of the choice of φ such that $x \in U_\varphi$. Namely, take

$$\mathcal{G}_x = \{f \mid f : F \longrightarrow \mathcal{F}_x \text{ is admissible}\}\,,$$

$$\widetilde{\varphi}_x : G \longrightarrow \mathcal{G}_x \quad \text{given by} \quad v \mapsto (F \xrightarrow{v} F \xrightarrow{\varphi_x} \mathcal{F}_x)\,;$$

that is, $\widetilde{\varphi}_x(v) = \varphi_x \circ v$. Since $\varphi_x^{-1} \circ (\varphi_x \circ v) = v \in G$, $\varphi_x \circ v$ is admissible and thus it lies in $\mathcal{G}_x$. $\widetilde{\varphi}_x$ is bijective. From

$$\widetilde{\psi}_x^{-1} \widetilde{\varphi}_x(v) = \psi_x^{-1} \circ \varphi_x \circ v = g(x)v\,,$$

it follows that

$$\widetilde{\mathcal{A}} = \{\widetilde{\varphi} \mid \varphi \in \mathcal{A}\}$$

is an atlas and $\widetilde{\xi}$ is associated to ξ (2.4.9). $\square$

2.5.15 EXERCISE. Prove that $\widetilde{\xi}$ can be constructed using the coordinate transformations of ξ and G as fiber (where G acts on itself by right translation).

2.5.1 Stiefel Manifolds

We use the previous ideas to make some computations of the homotopy groups of the Stiefel manifolds, by defining adequate fibrations.

A *k-frame* $(x_1, \ldots, x_k)$ in $\mathbb{R}^n$ consists of k orthonormal vectors $x_i \in \mathbb{R}^n$, $1 \leq i \leq k$.

2.5.16 DEFINITION. The set $\mathcal{VS}_{n,k} = \{(x_1, \ldots, x_k) \mid (x_1, \ldots, x_k) \text{ is a } k\text{-frame in } \mathbb{R}^n\} \subset \mathbb{R}^{nk}$ with the relative topology induced by that of $\mathbb{R}^{nk}$ is called the *Stiefel manifold* (of k-frames in $\mathbb{R}^n$).

The orthogonal group O_n acts on $\mathcal{VS}_{n,k}$ (cf. Section 2.2) via

$$\begin{aligned} \mathrm{O}_n \times \mathcal{VS}_{n,k} &\longrightarrow \mathcal{VS}_{n,k} \\ (A,(x_1,\dots,x_k)) &\longmapsto (Ax_1,\dots,Ax_k)\,, \end{aligned}$$

since an orthogonal matrix A sends an (orthonormal) k-frame to an (orthonormal) k-frame. This action is transitive, but it is not effective. Let $(e_1,\dots,e_n)$ be the canonical basis in $\mathbb{R}^n$ and take $z_0 = (e_1,\dots,e_k) \in \mathcal{VS}_{n,k}$. The equation $Az_0 = z_0$ is equivalent to the fact that the matrix A has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix},$$

where 1 represents the identity matrix in O_k and $B \in \mathrm{O}_{n-k}$.

Via

$$B \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

we include O_{n-k} as a subgroup (!) of O_n. By 2.2.20, the mapping $A \mapsto Az_0$ induces a homeomorphism

$$\begin{aligned} \mathrm{O}_n/\mathrm{O}_{n-k} &\xrightarrow{\approx} \mathcal{VS}_{n,k}\,, \\ (v_1,\dots,v_n) &\longmapsto (v_1,\dots,v_k) \end{aligned}$$

since O_n is compact and $\mathcal{VS}_{n,k}$ is Hausdorff. We identify both spaces through this homeomorphism.

Take $k \le l$. Through the mapping

$$A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

we may consider O_{n-l} as a subgroup of O_{n-k}. Mapping $(x_1,\dots,x_l)$ to $(x_1,\dots,x_k)$ we obtain a map

$$\begin{tikzcd} \mathcal{VS}_{n,l} \arrow[rr] \arrow[d, equal] & & \mathcal{VS}_{n,k} \arrow[d, equal] \\ \mathrm{O}_n/\mathrm{O}_{n-l} & & \mathrm{O}_n/\mathrm{O}_{n-k}\,. \end{tikzcd} \tag{2.5.17}$$

By 2.5.9, one has, in particular, that

$$\mathrm{O}_n = \mathcal{VS}_{n,n} \longrightarrow \mathcal{VS}_{n,k} = \mathrm{O}_n/\mathrm{O}_{n-k}$$

is a principal fibration with structure group O_{n-k}. The only thing that one has to be convinced of, is that the map (2.5.17) corresponds to taking left cosets in O_n of the subgroup O_{n-k}.

By 2.5.11 and 2.5.12,

$$\mathrm{O}_n/\mathrm{O}_{n-l} \longrightarrow \mathrm{O}_n/\mathrm{O}_{n-k}$$

is a locally trivial fibration obtained from a fiber bundle with fiber $\mathrm{O}_{n-k}/\mathrm{O}_{n-l}$ and structure group O_{n-k}/H_0, where

$$H_0 = \bigcap_{B\in \mathrm{O}_{n-k}} B\mathrm{O}_{n-l}B^{-1}\,.$$

For $k = l$, $H_0 = \mathrm{O}_{n-l}$. For $k < l$, $H_0 = \{1\}$; namely, take $A \in H_0$, then we can consider A as a map from $\mathbb{R}^{n-k}$ into itself via $e_i \mapsto Ae_i$, $1 \le i \le n-k$. Since $k < l$, every $A_1 \in \mathrm{O}_{n-1}$ leaves the vector e_1 fixed, and since $BA_1B^{-1}Be_1 = Be_1$, every vector remains fixed under $A = BA_1B^{-1}$, that is, $A = 1$.

2.5.18 **Theorem**. *$\pi_i(\mathcal{VS}_{n,k}) = 0$ for $i < n-k$.*

Proof: By induction on k. For $k = 1$, $\mathcal{VS}_{n,1} = \mathbb{S}^{n-1}$. The map $\mathcal{VS}_{n,k+1} \longrightarrow \mathcal{VS}_{n,k}$ is a locally trivial fibration with fiber $\mathcal{VS}_{n-k,1} = \mathbb{S}^{n-k-1}$. From its exact homotopy sequence we choose the exact portion

$$\pi_i(\mathbb{S}^{n-k-1}) \longrightarrow \pi_i(\mathcal{VS}_{n,k+1}) \longrightarrow \pi_i(\mathcal{VS}_{n,k})\,.$$

The group on the left-hand side is zero if $i < n-k-1$, the one on the right-hand side, by induction hypothesis is zero if $i < n-k$. Thus, $\pi_i(\mathcal{VS}_{n,k+1}) = 0$ for $i < n-(k+1)$. $\square$

2.6 Twisted Products and Associated Bundles

In this section we show how the principal G-fibration determined by a principal G-bundle relates to the fibration determined by an associated G-bundle with an arbitrary fiber F.

2.6.1 Definition. Let G be a topological group, E a right G-space, and F a left G-space. There is a left action of G on $E \times F$ given by

$$\begin{aligned} E\times F\times G &\longrightarrow E\times F\,,\\ (x,y,g) &\longmapsto (xg^{-1},gy)\,. \end{aligned}$$

This action is called the *diagonal action* of G on $E \times F$. We define the *twisted product* of E and F to be the orbit space

$$E \times_G F = E \times F/G$$

given by identifying (x, y) with (xg^{-1}, gy) (see 2.2.21).[3] We denote the orbits, namely the elements of $E \times_G F$, by $[x, y]$. Observe that $[xg, y] = [x, gy]$.

2.6.2 Exercise. Prove that the twisted product is functorial. More precisely, show that there is a category G-$\mathcal{T}op$, whose objects are G-spaces and whose morphisms are *equivariant maps*, namely maps $f : X \longrightarrow Y$ such that $f(gx) = gf(x)$ (or $f(xg) = f(x)g$ in the case of a right action), where $x \in X$ and $g \in G$. Then prove that the twisted product is a two-variable functor from G-$\mathcal{T}op$ to $\mathcal{T}op$ such that if $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ are equivariant, then they induce a map $f \times_G f' : X \times_G X' \longrightarrow Y \times_G Y'$ given by $[x, x'] \mapsto [f(x), f'(x')]$.

2.6.3 **Lemma**. *Take $E = B \times G$ with the right action $\rho : E \times G \longrightarrow E$ given by $((b, g), h) \mapsto (b, gh)$, and let F be a (left) G-space. Then there is a canonical homeomorphism $\alpha : E \times_G F \longrightarrow B \times F$ such that* $\mathrm{proj}_1 \alpha[(b, g), y] = b$*; in other words, one has a commutative diagram*

$$\begin{tikzcd}
(B \times G) \times_G F \arrow[rr, "\alpha", "\approx"'] \arrow[dr] & & B \times F \arrow[dl, "\mathrm{proj}_1"] \\
& B\,, &
\end{tikzcd}$$

where the top arrow is a homeomorphism.

Proof: The properties of the (left) action of G on F (see 2.2.12 (a)) imply that the map $\alpha' : B \times G \times F \longrightarrow B \times F$ given by $\alpha'(b, g, y) \mapsto (b, gy)$ is compatible with the identification $B \times G \times F \longrightarrow (B \times G) \times_G F$, and thus it induces a map $\alpha : (B \times G) \times_G F \longrightarrow B \times F$. On the other hand, the map $\beta : B \times F \longrightarrow (B \times G) \times_G F$ given by $\beta(b, y) = [(b, e), y]$, where $e \in G$ is the neutral element, is the inverse of α. Hence α and β are inverse homeomorphisms with the desired property. $\square$

Assume that $p : E \longrightarrow B$ is a principal G-fibration. Then we have a right G-action on E such that for every $x \in E$, $p(xg) = p(x)$ (see 2.5.2). Consider the map $q' = p \circ \mathrm{proj}_1 : E \times F \longrightarrow B$. One has that $q'(xg, g^{-1}y) = p(xg) = p(x) = q'(x, y)$; therefore, q' is compatible with the identification

$$E \times F \longrightarrow E \times F/G = E \times_G F\,,$$

and thus it induces a map $q : E \times_G F \longrightarrow B$. We have the following.

[3] This construction is sometimes known as the *Borel construction.*

2.6.4 **Proposition**. *Let $p : E \longrightarrow B$ be a locally trivial principal G-fibration and F a (left) G-space. Then $q : E \times_G F \longrightarrow B$ is a locally trivial fibration with fiber F.*

Proof: It is enough to find an open cover $\{U_\varphi\}$ of B and homeomorphisms $\widetilde{\varphi} : U_\varphi \times F \longrightarrow q^{-1}(U_\varphi)$ such that $q\widetilde{\varphi}(b, y) = b$.

Since $p : E \longrightarrow B$ is locally trivial, there is an open cover $\{U_\varphi\}$ of B and homeomorphisms $\varphi : U_\varphi \times G \longrightarrow p^{-1}(U_\varphi)$ such that $p\varphi(b, g) = b$ and, for every $h \in G$, $\varphi(b, gh) = \varphi(b, g)h$.

Observe that $q^{-1}U_\varphi = p^{-1}U_\varphi \times_G F$. By Lemma 2.6.3, there is a homeomorphism

$$\alpha : (U_\varphi \times G) \times_G F \longrightarrow U_\varphi \times F\,.$$

Since φ is an equivariant homeomorphism, by 2.6.2 we can define

$$\widetilde{\varphi} = (\varphi \times_G \mathrm{id}_F) \circ \alpha^{-1} : U_\varphi \times F \longrightarrow q^{-1}U_\varphi = p^{-1}U_\varphi \times_G F\,.$$

Since both α^{-1} and $\varphi \times_G \mathrm{id}_F$ are homeomorphisms, $\widetilde{\varphi}$ is one too. Using 2.6.3, one easily verifies that $q \circ \widetilde{\varphi} = \mathrm{proj}_1 : U_\varphi \times F \longrightarrow U_\varphi$; in other words, the diagram

$$\begin{tikzcd} U_\varphi \times F \arrow[rr, "\widetilde{\varphi}"] \arrow[dr, "\mathrm{proj}_1"'] & & p^{-1}U_\varphi \times_G F \arrow[dl] \\ & U_\varphi & \end{tikzcd}$$

commutes. □

2.6.5 Definition. Given a locally trivial, principal G-fibration $p : E \longrightarrow B$, we call the locally trivial fibration $q : E' = E \times_G F \longrightarrow B$ its *associated fibration* with fiber F.

We have the following, that is the main result of this section.

2.6.6 **Theorem**. *Take a principal G-bundle $\xi_G = (G, G, B; \mathcal{G}, \mathcal{A})$, and an associated G-bundle $\xi = (F, G, B; \mathcal{F}, \mathcal{A})$ with fiber F. Let $p = p_{\xi_G} : E \longrightarrow B$ be the locally trivial fibration determined by ξ_G and $q' = p_\xi : E' \longrightarrow B$ the locally trivial fibration determined by ξ. Then $q' : E' \longrightarrow B$ is the fibration with fiber F associated to the principal fibration $p : E \longrightarrow B$.*

Proof: By 2.3.14 and 2.3.18, we have open identifications

$$\Phi : \bigcup_{\varphi \in \mathcal{A}} U_\varphi \times G \times \{\varphi\} \longrightarrow E\,,$$

$$\Phi' : \bigcup_{\varphi \in \mathcal{A}} U_\varphi \times F \times \{\varphi\} \longrightarrow E' .$$

Since Φ is an open identification that is also equivariant, it is an easy *exercise* to prove that

$$\Phi \times_G \mathrm{id}_F : \left(\bigcup_{\varphi \in \mathcal{A}} U_\varphi \times G \times \{\varphi\} \right) \times_G F \longrightarrow E \times_G F ,$$

is also an open surjective map, thus an identification.

Since by 2.6.3, $(U_\varphi \times G \times \{\varphi\}) \times_G F$ is canonically homeomorphic to $U_\varphi \times F \times \{\varphi\}$, then one has a canonical homeomorphism

$$\Psi : \left(\bigcup_{\varphi \in \mathcal{A}} U_\varphi \times G \times \{\varphi\} \right) \times_G F \longrightarrow \bigcup_{\varphi \in \mathcal{A}} U_\varphi \times F \times \{\varphi\}$$

such that $\Psi[(b, g, \varphi), y] = (b, gy, \varphi)$, for $\varphi \in \mathcal{A}$, $b \in U_\varphi$, $g \in G$ and $y \in F$. Since $\Phi' \circ \Psi$ is obviously compatible with the identification $\Phi \times_G \mathrm{id}_F$, it induces a homeomorphism $\psi : E \times_G F \longrightarrow E'$ such that the triangle

$$\begin{tikzcd} E \times_G F \arrow[rr, "\psi", "\approx"'] \arrow[dr, "q"'] & & E' \arrow[dl, "q'"] \\ & B & \end{tikzcd}$$

commutes, where $q : E \times_G F \longrightarrow B$ is as in Proposition 2.6.4. □

2.7 Induced Bundles

Given a fiber bundle over a space B and a map $A \longrightarrow B$, we study here how this map induces a fiber bundle over A.

2.7.1 Definition. Let $\xi = (F, G, B; \mathcal{F}, \mathcal{A})$ be a fiber bundle and $\alpha : A \longrightarrow B$ a continuous map. We define a new fiber bundle

$$\alpha^*(\xi) = (F, G, A; \widetilde{\mathcal{F}}, \widetilde{\mathcal{A}})$$

by the following:

$$\begin{aligned} \widetilde{\mathcal{F}}_a &= \mathcal{F}_{\alpha(a)} \\ \widetilde{\varphi} &= \{\widetilde{\varphi}_a : F \longrightarrow \widetilde{\mathcal{F}}_a \mid a \in \alpha^{-1}(U_\varphi)\} \\ \widetilde{\varphi}_a &= \varphi_{\alpha(a)} \\ \widetilde{\mathcal{A}} &= \{\widetilde{\varphi} \mid \varphi \in \mathcal{A}\} \end{aligned}$$

It is easy to check that 2.3.1 (B1)–(B3) hold. The fiber bundle $\alpha^*(\xi)$ is called the fiber bundle *induced* by ξ through the map α.

In case that $\alpha : A \hookrightarrow B$ is an inclusion, the induced bundle $\alpha^*(\xi)$ is called the *restriction* of ξ to A and is usually denoted by $\xi|A$.

2.7.2 Exercise. Prove that through a constant map, a trivial bundle is induced.

In what follows, we analyze the relationship between theprevious definition and that of an induced fibration (see 1.4.22).

Let $\widetilde{E} \longrightarrow A$ be the fibration induced by $p = p_\xi$ through $\alpha : A \longrightarrow B$. Then, as sets, $\widetilde{E} = \bigcup_{a\in A}\{a\} \times \mathcal{F}_a$ is equal to $E_{\alpha^*(\xi)}$. (Observe that for the construction of the fibration corresponding to ξ, one has to provide that the fibers are mutually disjoint; see 2.3.14). However, it is also true that both $\widetilde{E}$ and $E^*_{\widetilde{\alpha}}(\xi)$ have the same topology. For seeing this, one has to prove that the map $\widetilde{\Phi} : U_{\widetilde{\varphi}} \times F = \alpha^{-1}U_\varphi \times F \longrightarrow \widetilde{E}$ is a homeomorphism over some open set; thus the map $\bigcup_{\widetilde{\varphi}} U_{\widetilde{\varphi}} \times F \longrightarrow \widetilde{E}$ is an identification (cf. 2.3.14).

The image set $\widetilde{\Phi}(\alpha^{-1}U_\varphi \times F) = \widetilde{p}^{-1}\alpha^{-1}U_\varphi$ is open in $\widetilde{E}$, and one has $\widetilde{\Phi}(a,v) = (a,z)$ with $z = \widetilde{\varphi}_a(v) = \varphi_{\alpha(a)}(v) = \Phi(\alpha(a),v)$. Therefore, $\widetilde{\Phi}$ is continuous. The inverse map $\widetilde{\Phi}^{-1}$ is also continuous, as one deduces from $v = \mathrm{proj}_2\Phi^{-1}(z)$.

We have proved the following result.

2.7.3 **Theorem**. $\alpha^*(p_\xi) = p_{\alpha^*(\xi)}$. □

We can define a bundle map

$$(\alpha^\star, \alpha) : \alpha^*(\xi) \longrightarrow \xi \quad \text{by} \quad \alpha^\star_a = \mathrm{id} : \widetilde{\mathcal{F}_a} \longrightarrow \mathcal{F}_{\alpha(a)}\,.$$

In fact, this is a bundle map, since

$$\psi^{-1}_{\alpha(a)} \circ \alpha^\star_a \circ \widetilde{\varphi}_a = \psi^{-1}_{\alpha(a)} \circ \varphi_{\alpha(a)} = g_{\psi\varphi}(\alpha(a))$$

lies in G and depends continuously on $a \in A$. The next result follows immediately.

2.7.4 **Theorem**. *Let $(\widetilde{\alpha}^\star, \alpha) : \alpha^*(p_\xi) = p_{\alpha^*(\xi)} \longrightarrow p_\xi$ be the fiber map corresponding to the bundle map $(\alpha^\star, \alpha) : \alpha^*(\xi) \longrightarrow \xi$ (cf. 2.3.19). Then, $\widehat{\alpha}^\star = \beta$ (β as in 1.4.22).* □

2.7.5 NOTE. Let ξ be a fiber bundle. If $\{U_i \mid i \in I\}$ is the associated cover and $\{g_{ij}\}$ are the corresponding coordinate transformations, then the associated cover of $\alpha^*(\xi)$ is $\{\alpha^{-1}U_i \mid i \in I\}$, and the corresponding coordinate transformations are $\{g_{ij}\alpha\}$.

2.7.6 NOTE. If ξ is a principal bundle, then also $\alpha^*(\xi)$ is a principal bundle. Analogously, $\alpha^*(p)$ is a principal fibration if p is a principal fibration. For this last, one has to define the right action of G on $\widetilde{E}$ by means of $((a,z),v) \mapsto (a,zv)$, $(a,z) \in \widetilde{E}$, $v \in G$. Then (β,α) becomes a principal map. In $(p_{\alpha^*(\xi)}, \rho_{\alpha^*(\xi)})$ one obtains also the same structure as a principal bundle (cf. 2.5.3).

2.7.7 **Theorem**. *Let $(f,\alpha) : \widehat{\xi} \longrightarrow \xi$ be a bundle map between*

$$\widehat{\xi} = (F,G,A;\widehat{\mathcal{F}},\widehat{\mathcal{A}}) \quad \text{and} \quad \xi = (F,G,B;\mathcal{F},\mathcal{A})\,.$$

Then there exists a unique bundle map $(h,\mathrm{id}_A) : \widehat{\xi} \longrightarrow \alpha^*(\xi)$ such that the diagram of bundles

$$\begin{tikzcd} \widehat{\xi} \arrow[d, "(h{,}\mathrm{id}_A)"'] \arrow[dr, "(f{,}\alpha)"] & \\ \alpha^*(\xi) \arrow[r, "(\alpha^\star{,}\alpha)"'] & \xi \end{tikzcd}$$

is commutative.

Proof: The commutativity of the diagram requires to define $h_a = f_a$, from where the uniqueness of h follows. The so-defined map h determines a bundle map, since

$$\widetilde{\psi}_a^{-1} \circ h_a \circ \widehat{\varphi}_a = \psi_{\alpha(a)}^{-1} \circ f_a \circ \widehat{\varphi}_a$$

lies in G and depends continuously on $a \in A$, because (f,α), by assumption, is a bundle map. $\square$

There are some consequences of the previous result.

2.7.8 **Corollary**. *$\widehat{\xi}$ is equivalent to $\alpha^*(\xi)$ over* id_A *(2.3.13).* $\square$

2.7.9 **Corollary**. *If ξ is equivalent to ξ' over* id_B*, then $\alpha^*(\xi)$ is equivalent to $\alpha^*(\xi')$ over* id_A *for any continuous map $\alpha : A \longrightarrow B$.*

Proof: Let $(f,\mathrm{id}_B) : \xi \longrightarrow \xi'$ be an equivalence. Then $(f \circ \alpha^\star, \alpha) : \alpha^*(\xi) \longrightarrow \xi'$ is a bundle equivalence. The assertion then follows from 2.7.8. $\square$

2.7.1 Functional Bundles

Let $\xi = (F, G, B; \mathcal{F}, \mathcal{A})$ and $\widehat{\xi} = (F, G, A; \widehat{\mathcal{F}}, \widehat{\mathcal{A}})$ be fiber bundles. We call a bundle map $h : \widehat{\mathcal{F}}_a \longrightarrow \mathcal{F}_b$, $a \in A$, $b \in B$ *admissible* if the homeomorphism $\varphi_b^{-1} \circ h \circ \widehat{\varphi}_a$ lies in G. This definition is independent of the choice of the charts φ and $\widehat{\varphi}$ such that $a \in U_{\widehat{\varphi}}$, $b \in U_\varphi$, as follows from

$$\psi_b^{-1} \circ h \circ \widehat{\varphi}_a = (\psi_b^{-1} \circ \varphi_b) \circ \varphi_b^{-1} \circ h \circ \widehat{\varphi}_a \circ (\widehat{\varphi}_a^{-1} \circ \widehat{\psi}_a)\,,$$

since each of the compositions in parentheses lies in G, by definition of an atlas.

2.7.10 Definition. The *functional bundle* $\mathrm{Apl}(\widehat{\xi}, \xi)$ is a bundle $\widetilde{\xi} = (G, \widetilde{G}, A \times B; \widetilde{\mathcal{F}}, \widetilde{\mathcal{A}})$, where

$$\widetilde{\mathcal{F}}_{(a,b)} = \{h : \widehat{\mathcal{F}}_a \longrightarrow \mathcal{F}_b \mid h \text{ is admissible}\}$$

with atlas

$$\widetilde{\mathcal{A}} = \{(\widehat{\varphi}, \varphi) \in \widehat{\mathcal{A}} \times \mathcal{A}\}$$

such that

$$(\widehat{\varphi}, \varphi)_{(a,b)} : G \longrightarrow \widetilde{\mathcal{F}}_{(a,b)}\,, \quad (a, b) \in U_{\widehat{\varphi}} \times U_\varphi$$

is given by

$$v \longmapsto \varphi_b \circ v \circ \widehat{\varphi}_a^{-1}\,.$$

In order to check that the bundle $\mathrm{Apl}(\widehat{\xi}, \xi)$ is well defined, we have to prove that $\varphi_b \circ v \circ \widehat{\varphi}_a^{-1} \in \widetilde{\mathcal{F}}_{(a,b)}$; that is, that it is admissible, and that $(\widehat{\varphi}, \varphi)_{(a,b)}$ is bijective.

The former follows from

$$\varphi_b^{-1} \circ (\varphi_b \circ v \circ \widehat{\varphi}_a^{-1}) \circ \varphi_a = v \in G\,,$$

and the latter from the fact that one has an inverse of $(\widehat{\varphi}, \varphi)_{(a,b)}$ given by

$$v \longmapsto \varphi_b^{-1} \circ v \circ \widehat{\varphi}_a\,.$$

Now the question is if $\mathcal{A}$ is an atlas for some adequate group $\widetilde{G}$. We have

$$\begin{aligned}
(\widehat{\psi}, \psi)_{(a,b)}^{-1} (\widehat{\varphi}, \varphi)_{(a,b)}(v) &= \psi_b^{-1} \circ \varphi_b \circ v \circ \widehat{\varphi}_a^{-1} \circ \widehat{\psi}_a \\
&= g_{\psi\varphi}(b)\ v\ g_{\widehat{\psi}\circ\widehat{\varphi}}(a)^{-1} \\
&= \lambda\left(g_{\psi\circ\varphi}(b), g_{\widehat{\psi}\circ\widehat{\varphi}}(a), v\right)
\end{aligned}$$

if we define

$$\lambda : (G \times G) \times G \longrightarrow G$$

as the left action of the product group $G \times G$ on G given by

$$((\widehat{u}, u), v) \longmapsto uv\widehat{u}^{-1}.$$

As a matter of fact, this action is not always effective; namely, if $uv\widehat{u}^{-1} = v$ for every $v \in G$, one has, in particular, $u\widehat{u}^{-1} = e$, that is, $u = \widehat{u}$ and thus, $u \in Z(G)$, where $Z(G)$ denotes the center of the group G.

Take $H = \{(z, z) \in G \times G \mid z \in Z(G)\}$ and

$$\widetilde{G} = G \times G/H\,.$$

Since $H \subset G \times G$ is a normal subgroup, $\widetilde{G}$ is a group. The action of $\widetilde{G}$ on G induced by λ (and again denoted by λ) is effective and $\widetilde{\mathcal{A}}$ becomes an atlas for $\widetilde{G}$, since the mapping

$$U_{\widehat{\varphi}} \times U_\varphi \cap U_{\widehat{\psi}} \times U_\psi \ni (a, b) \longmapsto \Big(g_{\widehat{\psi}\circ\widehat{\varphi}}(a), g_{\psi\circ\varphi}(b)\Big) \in \widetilde{G}$$

is obviously continuous.

Let $(f, \alpha) : \widehat{\xi} \longrightarrow \xi$ be a bundle map. If $\widetilde{\xi} = \mathrm{Apl}(\widehat{\xi}, \xi)$, the bundle map determines a map $s : A \longrightarrow E_{\widetilde{\xi}}$ given by

$$A \ni a \longmapsto \Big(f_a : \widehat{\mathcal{F}}_a \longrightarrow \mathcal{F}_{\alpha(a)}\Big) \in \widetilde{\mathcal{F}}_{(a,\alpha(a))}$$

that, by definition of a bundle map, is admissible. One has that $\widetilde{p}s(a) = (a, \alpha(a))$.

2.7.11 **Lemma**. *Let $\alpha : A \longrightarrow B$ be a continuous map. The assignment $(f, \alpha) \mapsto s$ given above yields a one-to-one relation between bundle maps $(f, \alpha) : \widehat{\xi} \longrightarrow \xi$ and continuous maps $s : A \longrightarrow E_{\widetilde{\xi}}$ such that $\widetilde{p}s(a) = (a, \alpha(a))$.*

Proof: We prove that f is compatible with the atlases $\widehat{\mathcal{A}}$ and $\mathcal{A}$ if and only if s is continuous. That the map is bijective is obvious. The diagram

$$\begin{tikzcd} U_{\widehat{\varphi}} \cap \alpha^{-1}U_\varphi \arrow[rr, "g"] \arrow[dr, "s"'] & & U_{\widehat{\varphi}} \times U_\varphi \times G \arrow[dl, "\widetilde{\Phi}"] \\ & \widetilde{p}^{-1}(U_{\widehat{\varphi}} \times U_\varphi) & \end{tikzcd}$$

commutes if

$$g(a) = (a, \alpha(a), \varphi^{-1}_{\alpha(a)} \circ f_a \circ \widehat{\varphi}_a) \text{ and } \widetilde{\Phi}(a, b, v) = (\widehat{\varphi}, \varphi)_{(a,b)}(v) = \varphi_b \circ v \circ \widehat{\varphi}_a^{-1}\,.$$

The map $\widetilde{\Phi}$ defines the topology on $E_{\widetilde{\xi}}$ (see 2.3.14) and is a homeomorphism. The fact that s is continuous at a is equivalent to the fact that $\varphi_{\alpha(a)}^{-1} \circ f_a \circ \widehat{\varphi}_a$ is continuous at a. This last means that f is compatible with the atlases $\widehat{\mathcal{A}}$ and $\mathcal{A}$. Since the sets $U_{\widehat{\varphi}} \cap \alpha^{-1} U_\varphi$ constitute a cover of A, the assertion follows. $\square$

2.7.12 **Theorem**. *Let $\alpha_0, \alpha_1 : A \longrightarrow B$ be homotopic maps. If A is a* CW-*complex and ξ is a bundle over B, then the induced bundles $\alpha_0^*(\xi)$ and $\alpha_1^*(\xi)$ are equivalent over* id_A.

Proof: Let $\widehat{\xi} = \alpha_0^*(\xi)$. We shall prove that there is a bundle map $(f, \alpha_1) : \widehat{\xi} \longrightarrow \xi$. By 2.7.7, the bundle $\alpha_1^*(\xi)$ is equivalent to $\widehat{\xi}$ over id_A. Let α_t be a homotopy between α_0 and α_1. Consider

$$\begin{tikzcd}
& E_{\widetilde{\xi}} \arrow[d, "\widetilde{p}"] \\
A \arrow[ur, dashed, "s_0"] \arrow[r, "h_t"'] & A \times B \,,
\end{tikzcd}$$

with the homotopy $h_t(a) = (a, \alpha_t(a))$. For $t = 0$ the diagram commutes if s_0 corresponds to the bundle map $(\alpha_0^\star, \alpha_0) : \widehat{\xi} \longrightarrow \xi$ as in 2.7.11 (see 2.7.4). If A is a CW-complex and $\widetilde{p}$ is locally trivial, then by 1.4.8 and 1.4.9, we can lift h_t, with the initial condition s_0, say to a map s_t. s_1 gives us by 2.7.11 a bundle map $(f, \alpha_1) : \widehat{\xi} \longrightarrow E_{\widetilde{\xi}}$. $\square$

We have the following consequences of the previous result.

2.7.13 **Corollary**. *If $\alpha : A \longrightarrow B$ is nullhomotopic, then $\alpha^*(\xi)$ is a trivial bundle (A is a* CW-*complex).* $\square$

2.7.14 **Corollary**. *Every fiber bundle over a contractible* CW-*complex is trivial.*

Proof: Since $\xi = (\mathrm{id}_A)^*(\xi)$, and since $\mathrm{id}_A \simeq 0$, by 2.7.13, ξ es trivial. $\square$

2.7.15 NOTE. The proof of Theorem 2.7.12 required the lifting of a certain homotopy. The theorems of Dold [3], recalled in 1.4.14, allow us to weaken the assumptions. One may either assume the space A to be paracompact,[4] or the fiber bundle ξ to be *numerable*, i.e., such that it has an atlas whose corresponding cover is numerable. In this latter case, the induced bundle $\alpha_0^*(\xi)$ and the functional bundle are numerable.

[4]Every CW-complex is paracompact, as shown in [8].

2.8 Universal Bundles

In 2.7 we saw that homotopic maps induce equivalent fiber bundles. We now ask the opposite question. Namely, if there is a bundle ξ over an adequate space B such that *every* bundle over A is induced through a map $\alpha : A \longrightarrow B$. Moreover, we ask if it is possible to choose ξ and B in such a way, that the equivalence of the induced bundles implies that the maps through which they are induced are homotopic.

2.8.1 Existence and Extension of Sections

Let $\xi = (F, G, B; \mathcal{F}, \mathcal{A})$ be a fiber bundle and $p : E \longrightarrow B$ the determined fibration. Let B be a CW-complex, $A \subset B$ a subcomplex and $s : A \longrightarrow E$ a *section of p over A*, that is, a map such that the composite $p \circ s$ is the inclusion $i_A : A \hookrightarrow B$.

2.8.1 Question. When can s be extended to a section $\widetilde{s} : B \longrightarrow E$ of p over B (i.e., a section $\widetilde{s}$ such that $\widetilde{s}|_A = s$)?

The following result answers this question giving a sufficient condition.

2.8.2 **Theorem**. *If F is $(n-1)$-connected, i.e., if $\pi_i(F) = 0$ for all $i < n$ $(\leq \infty)$, and $\dim(B) \leq n$, then every section s over a subcomplex A of B can be extended to all of B.*

For $n = 0$ the theorem is trivial, since in this case B is discrete. For $n \geq 1$ one has $\pi_0(F) = 0$; namely, F is path connected. Thus $\pi_i(F)$ is, up to isomorphism, independent of the base point. Before passing to the proo, we need some preparation.

2.8.3 Remark. Let B^k be the k-skeleton of B, $k \geq 0$, and $B^{-1} = \emptyset$. Take a section s_{k-1} of $p : E \longrightarrow B$ over $A \cup B^{k-1}$, $k \geq 0$, and e^k a k-cell of $B - A$ with characteristic map $\alpha : \mathbb{B}^k \longrightarrow B$, where $\mathbb{B}^k$ is the unit k-disk. Before passing to the proof, we need some preparation.

We have the following situation:

$$\begin{tikzcd}
\mathbb{S}^{k-1} \arrow[r, "s_{k-1}\circ(\alpha|_{\mathbb{S}^{k-1}})"] \arrow[d, hook] & E \arrow[d, "p"] \\
\mathbb{B}^k \arrow[r, "\alpha"'] \arrow[ur, dashed, "r"] & B\,.
\end{tikzcd}$$

The composition $s_{k-1} \circ (\alpha|_{\mathbb{S}^{k-1}})$ is well defined because the section s_{k-1} is defined on $\alpha(\mathbb{S}^{k-1})$, since $\alpha(\mathbb{S}^{k-1}) \subset B^{k-1}$. In case that we can find the lifting r, we can extend s_{k-1} to a section s' over $A \cup B^{k-1} \cup e^k$ by giving it by

$$s'(x) = \begin{cases} s_{k-1}(x) & \text{if } x \in A \cup B^{k-1}, \\ r\alpha^{-1}(x) & \text{if } x \in \overline{e}^k. \end{cases}$$

One can easily check that s' is well defined and continuous, since α is an identification, and that it is a section. We need conditions in order for r to exist.

2.8.4 **Lemma**. *In the diagram*

$$\begin{tikzcd} \widetilde{E} \arrow[r, "\beta"] \arrow[d, "\widetilde{p}"'] & E \arrow[d, "p"] \\ A \arrow[r, "\alpha"'] & B\,, \end{tikzcd}$$

let $\widetilde{p} = \alpha^*(p)$ be the fibration induced by p through α and (β, α) the corresponding fiber map (1.4.22). The assignment $\widetilde{s} \mapsto \beta \circ \widetilde{s}$ defines a bijective function

$$\{\textit{sections of } \widetilde{p}\} \longrightarrow \{\textit{liftings of } \alpha\}.$$

Proof: If $r : A \longrightarrow E$ is a lifting of α, namely, if $p \circ r = \alpha$, let $\widetilde{s}_r : A \longrightarrow \widetilde{E}$ be the section $a \mapsto (a, r(a))$, $((a, r(a)) \in \widetilde{E}$, since $\alpha(a) = pr(a))$. The mapping $r \mapsto \widetilde{s}_r$ is the inverse of $\widetilde{s} \mapsto \beta \circ \widetilde{s}$. $\square$

Now we come back to 2.8.3. Consider the diagram

$$\begin{tikzcd} & \widetilde{E} \arrow[d, "\widetilde{p}"'] & E \arrow[d, "p"] \\ \mathbb{S}^{k-1} \arrow[ur, dashed, "t"] \arrow[urr, "s_{k-1}(\alpha|_{\mathbb{S}^k})"' near end] \arrow[r, hook, "i"'] & \mathbb{B}^k \arrow[r, "\alpha"'] & B\,. \end{tikzcd}$$

where $\widetilde{p} = \alpha^*(p)$

According to 2.8.4 the lifting $s_{k-1} \circ (\alpha|_{\mathbb{S}^k})$ of $\alpha \circ i$ corresponds to a section $t : \mathbb{S}^{k-1} \longrightarrow \widetilde{E}$ of $\widetilde{p}$ over $\mathbb{S}^{k-1}$ (more precisely, to a section in the fibration induced by $\alpha \circ i$, which can be interpreted as restriction of $\widetilde{p}$).

$\alpha^*(\xi)$ is a bundle over a contractible CW-complex; therefore, it is trivial (2.7.14). Thus, also $\widetilde{p}$ is trivial. Consequently, there is a homeomorphism f

that makes the following diagram commutative:

$$\begin{tikzcd}
\widetilde{E} \arrow[r, "f", "\approx"'] \arrow[dr, "\widetilde{p}"'] & \mathbb{B}^k \times F \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\mathrm{proj}_1"] & F \\
& \mathbb{B}^k \arrow[ur, dashed, "g"'] & \\
& \mathbb{S}^{k-1} \arrow[u, hook] \arrow[uul, bend left, "t"] &
\end{tikzcd}$$

2.8.5 **Lemma**. *The section t can be extended to a section over $\mathbb{B}^k$ if and only if* $\mathrm{proj}_2 \circ f \circ t : \mathbb{S}^{k-1} \longrightarrow F$ *can be extended to* $\mathbb{B}^k$.

Proof: Let $t' : \mathbb{B}^k \longrightarrow \widetilde{E}$ be an extension of t; then $\mathrm{proj}_2 \circ f \circ t'$ is an extension of $\mathrm{proj}_2 \circ f \circ t$. Conversely, let $g : \mathbb{B}^k \longrightarrow F$ be an extension of $\mathrm{proj}_2 \circ f \circ t$. Let $t' : \mathbb{B}^k \longrightarrow \widetilde{E}$ be given by $t'(z) = f^{-1}(z, g(z))$. Clearly, t' is a section that extends t. □

Proof of 2.8.2: We proceed by induction over (the dimension of) the skeletons.

There exists always an extension s_0 of s to $A \cup B^0$, since the 0-cells of $B^0 - A$ constitute a discrete subspace. Let s_{k-1} be a section over $A \cup B^{k-1}$, $(k \geq 1)$, that extends s.

There exists a section s_k over $A \cup B^k$ that extends s:

If $k > \dim(B)$, we simply set $s_k = s_{k-1}$.

If $k \leq \dim(B) \leq n$, then, by assumption, $\pi_{k-1}(F) = 0$. Thus, every map $\mathbb{S}^{k-1} \longrightarrow F$ is nullhomotopic and can thus be extended to $\mathbb{B}^k$. By 2.8.3–2.8.5, s_{k-1} can be extended to every k-cell of $B - A$. Since B is a CW-complex, the so extended map $s_k : A \cup B^k \longrightarrow E$ is well defined and continuous. This proves the theorem. □

The following theorem generalizes 2.8.2.

2.8.6 **Theorem**. *Assume that ξ is a numerable bundle (for instance, if B is paracompact), F is contractible, and $A \subset B$. Assume, moreover, that there is a continuous map $\tau : B \longrightarrow [0,1]$ with $A \subset \tau^{-1}(1)$ such that a section s over A can be extended to $\tau^{-1}(0,1]$. Then s can be extended to B.*

For the *proof* see Dold [3, 2.7, 2.8]. □

2.8.7 Notation. If ξ is any bundle, in what follows we shall write ξ as an (upper or sub-) index to indicate the parts that define it. So we have $\xi = (F^\xi, B_\xi, G^\xi; \mathcal{F}^\xi, \mathcal{A}^\xi)$, and $p_\xi : E_\xi \longrightarrow B_\xi$ will denote the determined fibration.

2.8.8 Construction. Let ξ and η be fiber bundles with $B_\xi = A$, $B_\eta = B$, fiber F and structure group G. We want to construct a bundle α with the property that all bundle maps $\xi \longrightarrow \eta$ are in one-to-one correspondence with the sections of $E_\alpha \longrightarrow B_\alpha$ (in analogy to 2.7.11).

Let $\gamma = \mathrm{Apl}(\xi, \eta)$ be the functional bundle (2.7.10) and ω the principal bundle determined by η (2.5.14). Let

$$E_\gamma \xrightarrow{p_\gamma} A \times B \xrightarrow{\mathrm{proj}_1} A$$

be the fibration determined by α.

The bundle α is defined as follows:

$$\begin{aligned}
F^\alpha &= \bigcup_{b\in B} \mathcal{F}_b^\omega = E_\omega \text{ (as topological spaces)},\\
B_\alpha &= A\\
\mathcal{F}_a^\alpha &= \bigcup_{b\in B} \mathcal{F}_{(a,b)}^\gamma = \{h : \mathcal{F}_a^\xi \longrightarrow \mathcal{F}_b^\eta \mid h \text{ is admissible and } b \text{ is arbitrary}\},\\
G^\alpha &= G,\\
\varphi_a^\alpha &: F^\alpha = E_\omega \longrightarrow \mathcal{F}_a^\alpha \text{ given by}\\
&\quad (v : F \longrightarrow \mathcal{F}_b^\eta) \mapsto (v\varphi_a^{-1} : \mathcal{F}_a^\xi \longrightarrow \mathcal{F}_b^\eta),\\
\varphi^\alpha &= \{\varphi_a^\alpha \mid a \in U_\varphi^\xi\}.
\end{aligned}$$

We have to check again that all these elements give us a fiber bundle. On the way, we shall describe the action of G on $E_\omega = F^\alpha$.

First assume that v is admissible; then also $v \circ \varphi_a^{-1}$, ($\varphi_a \in \varphi^\xi$, $a \in U_\varphi^\xi$), is admissible (cf. 2.7.1). Now,

$$\begin{aligned}
\psi_a^{-1}\varphi_a^\alpha(v) &= v \circ \varphi_a^{-1} \circ \psi_a\\
&= v \circ (g_{\psi\varphi}^\xi(a))^{-1}\\
&= \lambda^\alpha(g_{\psi\varphi}^\xi(a), v)
\end{aligned}$$

if we define the action $\lambda^\alpha : G \times E_\omega \longrightarrow E_\omega$ by $\lambda^\alpha(u, v) = v \circ u^{-1}$ (u^{-1} considered as a map $F \longrightarrow F$). Moreover, λ^α is continuous; namely, if $\rho : E_\omega \times G \longrightarrow E_\omega$ is the right action of G corresponding to the principal bundle ω, then

$$\lambda^\alpha(u, v) = \rho(v, u^{-1}).$$

2.8.9 Definition. The fiber bundle α is called the *partial functional bundle* of (ξ, η). We denote it by $\mathrm{Apl}_1(\xi, \eta)$.

The equation

$$E_\alpha = \bigcup_{a\in A} \mathcal{F}^\alpha_a = \bigcup_{a\in A}\bigcup_{b\in B} \mathcal{F}^\gamma_{(a,b)} = E_\gamma \tag{2.8.10}$$

is a set equality, and the diagram

$$\begin{tikzcd} E_\gamma \arrow[r, equal] \arrow[d, "p_\gamma"'] & E_\alpha \arrow[d, "p_\alpha"] \\ A \times B \arrow[r, "\mathrm{proj}_1"'] & A \end{tikzcd}$$

commutes.

2.8.11 **Lemma**. *The spaces E_γ and E_α have the same topology.*

Proof: Let φ^ξ and φ^η be local charts corresponding to the open sets $U \subset A$ and $V \subset B$. Consider the diagram

$$\begin{tikzcd} U \times V \times G \arrow[r, "\mathrm{id}\times\Phi_\omega"] \arrow[d, "\Phi_\gamma"', "\approx"] & U \times p_\omega^{-1}(v) \arrow[r, hook] & U \times E_\omega \arrow[d, "\Phi_\alpha"] \\ p_\gamma^{-1}(U \times V) \arrow[r, hook] & p_\gamma^{-1}(U \times B) \arrow[r, equal] & p_\alpha^{-1}(U)\,. \end{tikzcd} \tag{2.8.12}$$

The maps Φ determine the topologies on the total spaces E.

We already had the equalities

$$\begin{aligned} G &= F^\gamma = F^\omega\,, \\ \Phi_\gamma(a,b,w) &= \varphi^\gamma_{(a,b)}(w) = \varphi^\eta_b \circ w \circ (\varphi^\xi_a)^{-1}\,, \\ \Phi_\omega(b,w) &= \varphi^\omega_b(w) = \varphi^\eta_b \circ w\,, \\ \Phi_\alpha(a,v) &= \varphi^\alpha_a \circ v = v \circ (\varphi^\xi_a)^{-1}\,, \end{aligned}$$

that show that the diagram is commutative. If we endow E_γ with the topology determined by γ, we have to show that $p_\alpha^{-1}(U)$ is open in E_γ and that Φ_α is a homeomorphism over $p_\alpha^{-1}(U)$ (this way, the α-topology on E_α will be determined). The map $p_\alpha = \mathrm{proj}_1 \circ p_\gamma$ is continuous. Thus, $p_\alpha^{-1}(U)$ is open. Moreover, the map $\Phi_\alpha|_{U\times p_\omega^{-1}(V)}$ is a homeomorphism over $p_\gamma^{-1}(U \times V)$, see Diagram (2.8.12). If φ^η varies along the atlas $\mathcal{A}^\eta$, we obtain that the sets $U \times p_\omega^{-1}(V)$ build an open cover of $U \times E_\omega$, and that the sets $p_\gamma^{-1}(U \times V)$ build an open cover of $p_\alpha^{-1}(U)$. Hence, we obtain the assertion of the lemma. □

2.8.13 Note. In 2.7.11 we assigned to every bundle map $(f, \overline{f}) : \xi \longrightarrow \eta$ a map $s : A \longrightarrow E_\gamma = E_\alpha$ which is obviously a section of p_α. It is easy to see that this assignment yields a bijection between bundle maps $\xi \longrightarrow \eta$ and sections of p_α.

2.8.2 n-Universal Bundles

We characterize here bundles that are universal for fiber bundles over spaces B of bounded dimension.

2.8.14 Definition. Let η be a fiber bundle and ω the associated principal bundle. η will be called *n-universal* $(n \leq \infty)$ if $\pi_i(E_\omega) = 0$ for $i < n$. The determined fibration p_η is also called *n-universal.*

2.8.15 **Theorem**. *Let η be an n-universal fiber bundle, A a* CW*-complex of dimension $\leq n$, $A_0 \subset A$ a subcomplex and ξ a fiber bundle over A. Then any bundle map $\xi|A_0 \longrightarrow \eta$ can be extended to a bundle map $\xi \longrightarrow \eta$.*

Proof: Note that $\mathrm{Apl}_1(\xi|A_0, \eta) = \mathrm{Apl}_1(\xi, \eta)|A_0$. With this remark and 2.8.13, using the bundle map $\xi|A_0 \longrightarrow \eta$, we obtain a section $s_0 : A_0 \longrightarrow E_\alpha$ $(\alpha = \mathrm{Apl}_1(\xi, \eta))$ of p_α over A_0. By assumption, $\pi_i(F^\alpha) = 0$ for $i < n$, since $F^\alpha = E_\omega$. Theorem 2.8.1 guarantees that we can extend s_0 to a section $s : A \longrightarrow E_\alpha$. From s, we obtain a bundle map (2.8.13) that extends the given bundle map $\xi|A_0 \longrightarrow \eta$. □

2.8.16 Definition. Let η be a fiber bundle. We may assign to each homotopy class $[\alpha] \in [A, B_\eta]$ an equivalence class of fiber bundles over A; namely

$$[\alpha]^*(\eta) = [\alpha^*(\eta)]\,,$$

where $[\alpha]$ denotes the homotopy class of the map α and $[\xi]$ denotes the equivalence class of the bundle ξ.

By 2.7.12, this assignment is well defined (if A is a CW-complex). Denote by $k_G(A)$ the set of equivalence classes of fiber bundles over A with fiber F and structure group G, and let $\Omega(\eta) : [A, B_\eta] \longrightarrow k_G(A)$ be the function just defined.

2.8.17 **Theorem**. *Let η be an $(n+1)$-universal fiber bundle and A a* CW*-complex such that* $\dim(A) \leq n$. *Then the function $\Omega(\eta) : [A, B_\eta] \longrightarrow k_G(A)$ is bijective.*

Proof: $\Omega(\eta)$ is surjective: Let ξ be any fiber bundle over A. In Theorem 2.8.15, choose $A_0 = \emptyset$; thus, it gives us a bundle map $\xi \longrightarrow \eta$. From 2.7.8 and the definition of $\Omega(\eta)$ one obtains the assertion.

$\Omega(\eta)$ is injective: Let $\alpha_0, \alpha_1 : A \longrightarrow B_\eta$ be maps such that $\alpha_0^*(\eta)$ and $\alpha_1^*(\eta)$ are equivalent over id_A. Take $\xi = \alpha_0^*(\eta)$; the assumed equivalence of this bundle with $\alpha_1^*(\eta)$ gives us a bundle map $(f_1, \alpha_1) : \xi \longrightarrow \eta$. Take $f_0 = \alpha_0^\star$, and let $\mathrm{proj}_1 : A \times I \longrightarrow A$, $\zeta = (\mathrm{proj}_1)^*(\xi)$, and $i_\nu : A \longrightarrow A \times I$ be such that $i_\nu(a) = (a, \nu)$, $\nu = 0, 1$. We have bundle maps

$$\begin{tikzcd} \zeta|A \times \{\nu\} & \xi \arrow[l, "(i_\nu^\star, i_\nu)"'] \arrow[r, "(f_\nu, \alpha_\nu)"] & \eta \arrow[ll, bend left, leftarrow] \end{tikzcd},$$

since $(i_\nu^\star, i_\nu)$ is an equivalence, (cf. 2.3.13), and from there, a bundle map

$$\zeta|(A \times \{0\}) \cup (A \times \{1\}) \longrightarrow \eta\,.$$

By Theorem 2.8.15, there is an extension $(f, \alpha) : \zeta \longrightarrow \eta$ of this last bundle map, since $\dim(A \times I) = \dim(A) + 1 \leq n+1$, $A \times (\{0\} \cup \{1\})$ is a subcomplex of $A \times I$, and η is $(n+1)$-universal. In particular, $\alpha : A \times I \longrightarrow B_\eta$ is a map such that $\alpha \circ i_\nu = \alpha_\nu$, that is, α_0 and α_1 are homotopic. $\square$

2.8.18 DEFINITION. If η is *universal*, namely ∞-universal, or n-universal for all n, then the space B_η is called *classifying space* for the group G. It is frequently denoted by BG.

2.8.19 REMARK. Let $g : A' \longrightarrow A$ be continuous and ξ a bundle over A. The bundle map $\xi \mapsto g^*(\xi)$ induces a function

$$g^* = k(g) : k_G(A) \longrightarrow k_G(A')$$

converting k_G into a functor. If η is a fiber bundle, then the diagram

$$\begin{tikzcd} {[A, B_\eta]} \arrow[r, "\Omega(\eta)"] \arrow[d, "g^\#"'] & k_G(A) \arrow[d, "g^*"] \\ {[A', B_\eta]} \arrow[r, "\Omega(\eta)"'] & k_G(A') \end{tikzcd}$$

is commutative, that is, $\Omega(\eta)$ is a *natural transformation* of functors. If η is universal, then the functors $[\ \ , B_\eta]$ and k_G are naturally equivalent. Given two universal G-bundles η and η', we obtain a natural equivalence of functors

$$[\ \ , B_\eta] \longrightarrow [\ \ , B_{\eta'}]\,.$$

If both B_η and $B_{\eta'}$ are CW-complexes, from our previous theorems, we have maps $\alpha : B_{\eta'} \longrightarrow B_\eta$ and $\beta : B_\eta \longrightarrow B_{\eta'}$ such that $\eta' = \alpha^*(\eta)$ and $\eta = \beta^*(\eta')$.

Then one has $\eta = \beta^*\alpha^*(\eta) = (\alpha\beta)^*(\eta)$ and $\eta' = (\beta\alpha)^*(\eta')$, and since η and η' are universal, $\alpha \circ \beta \simeq \mathrm{id}_{B_\eta}$ and $\beta \circ \alpha \simeq \mathrm{id}_{B_{\eta'}}$. Thus, B_η and $B_{\eta'}$ have the same homotopy type. This shows, in particular, that the classifying space BG is well defined, up to homotopy type. For further generalizations of this see Dold [3, §7.].

2.9 Construction of Universal Bundles

In this section, we shall construct universal bundles in several instances.

2.9.1 Grassmann Manifolds

We construct here universal bundles for the groups $G = \mathrm{O}_k$.

As we did in section 2.5, via the mappings

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{resp.} \quad B \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix},$$

we shall consider the groups O_k, resp. O_{n-k}, as subgroups of O_n. Thus, also $\mathrm{O}_k \times \mathrm{O}_{n-k}$ is a subgroup of O_n. From 2.5.9–2.5.12, taking $E = \mathrm{O}_n$, $G = \mathrm{O}_k \times \mathrm{O}_{n-k}$, $H = \mathrm{O}_{n-k}$, we obtain the following result.

2.9.1 **Proposition**. *The canonical projection*

$$q : \mathrm{O}_n/\mathrm{O}_{n-k} \longrightarrow \mathrm{O}_n/\mathrm{O}_k \times \mathrm{O}_{n-k}$$

is a locally trivial fibration. Corresponding to it there is a fiber bundle $\eta = \eta_{n,k}$ with fiber

$$G/H \cong \mathrm{O}_k \tag{2.9.2}$$

and structure group

$$G/H_0 \cong \mathrm{O}_k\,. \tag{2.9.3}$$

This is true, since O_{n-k} is the maximal normal subgroup of $\mathrm{O}_k \times \mathrm{O}_{n-k}$ contained in O_{n-k}. □

The action $G/H_0 \times G/H \longrightarrow G/H$ corresponds, via the canonical identifications (2.9.2)–(2.9.3), to the group multiplication. Thus, η is a principal

O_k-bundle and $q = p_\eta$ is a principal fibration. One can easily check that the right action of O_k on $\mathrm{O}_n/\mathrm{O}_{n-k}$ associated to ρ_n (cf. 2.5.1) is given by $([A], B) \mapsto [AB]$, where $[A] \in \mathrm{O}_n/\mathrm{O}_{n-k}$ represents the (left) coset of the matrix A.

2.9.4 **Theorem**. *The map* $q : \mathrm{O}_n/\mathrm{O}_{n-k} \longrightarrow \mathrm{O}_n/\mathrm{O}_k \times \mathrm{O}_{n-k}$ *is an* $(n-k)$*-universal principal fibration; that is, the fiber bundle* $\eta_{n,k}$ *is an* $(n-k)$*-universal* O_k*-bundle.*

Proof: Recall that $\mathrm{O}_n/\mathrm{O}_{n-k} = \mathcal{VS}_{n,k}$. By 2.5.18, $\pi_i(\mathrm{O}_n/\mathrm{O}_{n-k}) = 0$ for $i < n-k$, thus the result. □

2.9.5 Remark. We describe the fibration q in a different way. For that, consider the diagram

$$\begin{tikzcd} \mathrm{O}_n/\mathrm{O}_{n-k} \arrow[r, "f"] \arrow[d, "q"'] & \mathcal{VS}_{n,k} \arrow[d, "p_{n,k}"] \\ \mathrm{O}_n/\mathrm{O}_k \times \mathrm{O}_{n-k} \arrow[r, "\overline{f}"'] & \mathrm{Gr}_{n,k}\,, \end{tikzcd} \tag{2.9.6}$$

where $\mathrm{Gr}_{n,k}$, as a set, consists of the k-dimensional subspaces of $\mathbb{R}^n$. If $(x_1, \ldots, x_k) \in \mathcal{VS}_{n,k}$ we denote by $[x_1, \ldots, x_k]$ the generated subspace, and we define

$$p_{n,k}(x_1, \ldots, x_k) = [x_1, \ldots, x_k]\,.$$

We furnish $\mathrm{Gr}_{n,k}$ with the identification topology. The space $\mathrm{Gr}_{n,k}$ is called the *Grassmann manifold* of k-planes in $\mathbb{R}^n$.

The map f is induced by the mapping

$$A \longmapsto (Ae_1, \ldots, Ae_k)\,,$$

(cf. 2.5.16) and the map $\overline{f}$ by the mapping

$$A \longmapsto [Ae_1, \ldots, Ae_k]\,,$$

$A \in \mathrm{O}_n$. One can easily be convinced that $A[e_1, \ldots, e_k] = [e_1, \ldots, e_k]$ if and only if $A \in \mathrm{O}_n \times \mathrm{O}_{n-k}$. With these definitions, Diagram (2.9.6) is commutative. The map f is a homeomorphism; the map $\overline{f}$ is bijective and, consequently, it is also a homeomorphism, since both q and $p_{n,k}$ are identifications.

The pair $(f, \overline{f})$ is a principal map if one defines the action

$$\mathcal{VS}_{n,k} \times \mathrm{O}_k \overset{m_{n,k}}{\longrightarrow} \mathcal{VS}_{n,k} \tag{2.9.7}$$

by

$$((x_1, \ldots, x_k), A) \longmapsto (x_1, \ldots, x_k)A\,,$$

where the k-tuple $(x_1, \ldots, x_k)$ should be seen as a matrix with k columns x_i; the columns of the product form an orthonormal k-frame that generates the same plane as $(x_1, \ldots, x_k)$, and this is the k-frame we refer to by $(x_1, \ldots, x_k)A$.

Since the pair $(f, \overline{f})$ is a principal map, by definition of the actions one has

$$(ABe_1, \ldots, ABe_k) = (Ae_1, \ldots, Ae_k)B\,,$$

and, obviously, both sides are the first k columns of the product matrix AB.

The (∞-)universal O_k-bundles are obtained by passing to the colimit. Let us consider the diagram

$$\begin{tikzcd}
\mathcal{VS}_{n,k} \arrow[r, hook] \arrow[d, "p_{n,k}"'] & \mathcal{VS}_{n+1,k} \arrow[r, hook] \arrow[d, "p_{n+1,k}"'] & \cdots \\
\mathrm{Gr}_{n,k} \arrow[r, hook] & \mathrm{Gr}_{n+1,k} \arrow[r, hook] & \cdots\,,
\end{tikzcd}$$

where the inclusions are induced by the canonical inclusion $\mathbb{R}^n \cong \mathbb{R}^n \times \{0\} \hookrightarrow \mathbb{R}^{n+1}$. One maps the k-frame $(x_1, \ldots, x_k)$ to its image under said inclusion. For the Grassmann manifolds $\mathrm{Gr}_{n,k}$ and $\mathrm{Gr}_{n+1,k}$, the inclusion is similarly induced. Thus, each square in the diagram commutes.

2.9.8 Definition. Define

$$\begin{aligned}
\mathcal{VS}_{\infty,k} &= \operatorname*{colim}_{n\to\infty} \mathcal{VS}_{n,k}\,,\\
\mathrm{Gr}_{\infty,k} &= \operatorname*{colim}_{n\to\infty} \mathrm{Gr}_{n,k}\,,\\
p_{\infty,k} &= \operatorname*{colim}_{n\to\infty} p_{n,k} : \mathcal{VS}_{\infty,k} \longrightarrow \mathrm{Gr}_{\infty,k}\,.
\end{aligned}$$

The space $\mathcal{VS}_{\infty,k}$ is the (∞-dimensional) *Stiefel manifold* of k-frames in $\mathbb{R}^\infty$, and the space $\mathrm{Gr}_{\infty,k}$ is the (∞-dimensional) *Grassmann manifold* of k-planes in $\mathbb{R}^\infty$.

2.9.9 **Theorem**. *The map $p_{\infty,k} : \mathcal{VS}_{\infty,k} \longrightarrow \mathrm{Gr}_{\infty,k}$ is a universal principal O_k-fibration.*

Proof: We divide the proof in several parts.

(a) First we prove that $p_{\infty,k}$ is continuous. The union topology in $\mathcal{VS}_{\infty,k} = \bigcup_n \mathcal{VS}_{n,k}$ is such that the canonical map from the topological sum $\coprod_n \mathcal{VS}_{n,k}$

into $\mathcal{VS}_{\infty,k}$ is an identification. Similarly for $\mathrm{Gr}_{\infty,k}$ (and for any colimit). From the commutativity of the diagram

$$\begin{tikzcd}
\coprod \mathcal{VS}_{n,k} \arrow[r] \arrow[d, "\coprod p_{n,k}"'] & \mathcal{VS}_{\infty,k} \arrow[d, "p_{\infty,k}"] \\
\coprod \mathrm{Gr}_{n,k} \arrow[r] & \mathrm{Gr}_{\infty,k}
\end{tikzcd}$$

one obtains the continuity of $p_{\infty,k}$.

(b) The actions $m_{n,k}$ given in (2.9.7) are compatible with the inclusions $\mathcal{VS}_{n,k} \subset \mathcal{VS}_{n+1,k}$. Namely, the diagram

$$\begin{tikzcd}
\mathcal{VS}_{n,k} \times \mathrm{O}_k \arrow[r, hook] \arrow[d, "m_{n,k}"'] & \mathcal{VS}_{n+1,k} \times \mathrm{O}_k \arrow[d, "m_{n+1,k}"] \\
\mathcal{VS}_{n,k} \arrow[r, hook] & \mathcal{VS}_{n+1,k}
\end{tikzcd}$$

is commutative, as one may easily verify. Define

$$m_{\infty,k} = \operatorname*{colim}_{n\to\infty} m_{n,k} : \operatorname*{colim}_{n\to\infty}(\mathcal{VS}_{n,k} \times \mathrm{O}_k) \longrightarrow \mathcal{VS}_{\infty,k}\,.$$

As sets, there is an equality

$$\operatorname*{colim}_{n\to\infty}(\mathcal{VS}_{n,k} \times \mathrm{O}_k) = \mathcal{VS}_{\infty,k} \times \mathrm{O}_k\,.$$

Besides, both spaces have the same topology, as one sees in the commutative diagram

$$\begin{tikzcd}
(\coprod \mathcal{VS}_{n,k}) \times \mathrm{O}_k \arrow[r, "a"] \arrow[d, equal] & \mathcal{VS}_{\infty,k} \times \mathrm{O}_k \arrow[d, leftrightarrow] \\
\coprod(\mathcal{VS}_{n,k} \times \mathrm{O}_k) \arrow[r] & \operatorname{colim}(\mathcal{VS}_{n,k} \times \mathrm{O}_k)\,,
\end{tikzcd}$$

because a is an identification, since O_k is compact. Thus O_k acts on $\mathcal{VS}_{\infty,k}$ on the right.

(c) We now prove that $p_{\infty,k}$ is locally trivial. Let us consider inside $\mathrm{Gr}_{n,k}$ the set $U_{n,k}$ of the planes E that are mapped onto $\mathbb{R}^n$ under the projection $\mathbb{R}^k \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^k$ (see Figure 2.4).

One has that $U_{n,k} = U_{n+1,k} \cap \mathrm{Gr}_{n,k}$. $U_{n,k}$ is open in $\mathrm{Gr}_{n,k}$ (cf. Milnor [11, 2.25 (a)]). Thus, $U_{\infty,k} = \bigcup_n U_{n,k}$ is open in $\mathrm{Gr}_{\infty,k}$. (Analogous considerations hold for k-planes that are mapped surjectively onto any other product of k factors $\mathbb{R}$ inside $\mathbb{R}^n$, and not necessarily the first k of them.) The fibration $p_{n,k}$ is trivial over $U_{n,k}$. Namely, we shall construct a particular trivialization

$$\varphi_{n,k} : U_{n,k} \times \mathrm{O}_k \longrightarrow p_{n,k}^{-1} U_{n,k}$$

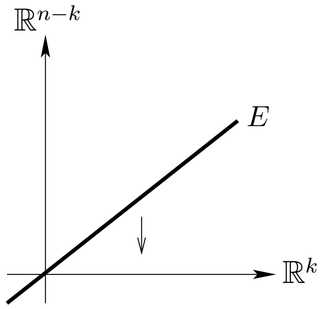


Figure 2.4

as follows (cf. 2.5.5). First we need a section

$$s : U_{n,k} \longrightarrow p_{n,k}^{-1} U_{n,k}\,,$$

and we define

$$\varphi_{n,k}([x_1, \dots, x_k], B) = (s[x_1, \dots, x_k])B\,;$$

the section is obtained as follows. Each plane $E \in U_{n,k}$ is generated by exactly a k-tuple $x_1(E), \dots, x_k(E)$ of vectors of the form

$$\begin{aligned} x_1(E) &= (1, 0, \dots, 0, x_{k+1,1}, \dots, x_{n,1}) \\ x_2(E) &= (0, 1, \dots, 0, x_{k+1,2}, \dots, x_{n,2}) \\ &\vdots \qquad\qquad\quad \ddots \end{aligned}$$

Observe that the n-tuples $(0, \dots, 1, \dots, 0, x_{k+1,i}, \dots, x_{n,i})$ are the solutions of a system of $n-k$ linear equations with n unknowns. They are clearly linearly independent.

The assignment $E \mapsto (x_1(E), \dots, x_k(E)) \in \mathbb{R}^{nk}$ is continuous (cf. Milnor, op. cit). Moreover, the orthonormalization $(x_1(E), \dots, x_k(E)) \mapsto (\widetilde{x}_1(E), \dots, \widetilde{x}_k(E))$ given by the Gram-Schmidt process is also continuous (namely, one can give explicit formulas for the orthonormalized basis; cf. for instance, the formulas given by Langwitz [**?**, p.74]). We thus may define

$$s(E) = (\widetilde{x}_1(E), \dots, \widetilde{x}_k(E)) \in \mathcal{VS}_{n,k}.$$

It is an easy matter to convince oneself that all maps $\varphi_{n,k}$ for different values of n are compatible and glue together to yield a map

$$\varphi_{\infty,k} : U_{\infty,k} \times \mathrm{O}_k \longrightarrow p_{\infty,k}^{-1} U_{\infty,k}$$

and the map $\varphi_{\infty,k}$, as a colimit of homeomorphisms, is also a homeomorphism. Of course, it is also a principal map over the identity map of $U_{\infty,k}$. Thus, $p_{\infty,k}$ is trivial over $U_{\infty,k}$.

(d) We prove that $\pi_i(\mathcal{VS}_{\infty,k}) = 0$ for every i. This follows from the next lemma.

2.9.10 **Lemma**. *Any compact set $K \subset \mathcal{VS}_{\infty,k}$ lies inside some $\mathcal{VS}_{n,k}$.*

Before proving this lemma, we come back to statement (d) in the proof of 2.9.9.

Let $f : \mathbb{S}^i \longrightarrow \mathcal{VS}_{\infty,k}$ represent any element of $\pi_i(\mathcal{VS}_{\infty,k})$. Its image $f(\mathbb{S}^i)$ is a compact set, and thus, by 2.9.10, it lies inside $\mathcal{VS}_{n,k}$ for some n. Let n be large enough that $n - k > i$. Hence, f is nullhomotopic as a map into $\mathcal{VS}_{n,k}$, and thus also as a map into $\mathcal{VS}_{\infty,k}$. This proves the statement.

Proof of 2.9.10: If the statement of the lemma were false, then there would be a sequence $p_1, p_2, \ldots$ of points of K such that $p_n \notin \mathcal{VS}_{n,k}$ for all n. But since K is compact, the sequence $\{p_i \mid i \in \mathbb{N}\}$ has an accumulation point p_0. Take any subset $S \subset Q = \bigcup_{i=0}^{\infty}\{p_i\}$. For every n, $S \cap \mathcal{VS}_{n,k}$ consists of only finitely many points, and therefore, it is closed in $\mathcal{VS}_{\infty,k}$ and also in Q. Thus, Q is discrete, which is a contradiction of the fact that p_0 is an accumulation point of the sequence. □

We now pass to the last part of the proof of 2.9.9.

(e) By 2.5.8, the Definition 2.8.14, and the previous parts (a)–(d) of the proof, we have that $p_{\infty,k}$ is a universal principal O_k-fibration. □

2.9.2 The Milnor Construction

Let G be an arbitrary topological group. We want to construct a universal principal G-fibration $p_G : EG \longrightarrow BG$. First we shall give a formal description of p_G and then we shall explain the geometrical meaning of the construction.

2.9.11 Construction. First we describe EG as a set.

Consider sequences

$$(t_1, v_1, t_2, v_2, \ldots, t_i, v_i, \ldots)$$

such that

$$t_i \in I = [0,1]\,, \quad v_i \in G\,, \quad i = 1, 2, 3, \ldots,$$

and $t_i \neq 0$ only for finitely many values of i and $\sum_{i=1}^{\infty} t_i = 1$. Two such sequences $(t_1, v_1, t_2, v_2, \ldots)$ and $(t'_1, v'_1, t'_2, v'_2, \ldots)$ are *equivalent* if

(a) $t_i = t'_i$ for every i, and

(b) for every i, $v_i = v'_i$ or $t_i = t'_i = 0$.

We denote by $t_1v_1 + t_2v_2 + \cdots = z$ the equivalence class of

$$(t_1, v_1, t_2, v_2, \ldots)\,.$$

(We just have to be careful to respect the ordering of the terms, and of course, write the terms with coefficient $t_i = 0$.) Let EG be the set of all these equivalence classes z.

We shall now furnish EG with a topology.

There are (coordinate) maps

$$\begin{aligned} t_i : EG &\longrightarrow I\,, \\ t_1v_1 + t_2v_2 + \cdots &\longmapsto t_i\,, \\ v_i : t_i^{-1}(0,1] &\longrightarrow G\,, \\ t_1v_1 + t_2v_2 + \cdots &\longmapsto v_i\,. \end{aligned}$$

Observe that the maps t_i and v_i are well defined. An element of EG is determined by its *coordinates*, namely, by its images under the maps t_i and v_i. We endow EG with the coarsest (smallest) topology that makes all these maps continuous. The meaning of this method of generating a topology is explained in the next lemma, that characterizes the topology and is easy to prove.

2.9.12 **Lemma**. *A map $f : X \longrightarrow EG$ is continuous if and only if the composed maps $t_i \circ f$ and $v_i \circ (f|_{(t_i\circ f)^{-1}(0,1]})$ are continuous.* □

We now define a right action $\rho : EG \times G \longrightarrow EG$.

This action is given by

$$\rho(t_1v_1 + t_2v_2 + \cdots, u) = t_1(v_1u) + t_2(v_2u) + \cdots\,.$$

This action ρ is continuous, as one easily proves using Lemma 2.9.12; namely, the diagram

$$\begin{tikzcd} EG \times G \arrow[r, "\rho"] \arrow[d, "\mathrm{proj}_1"'] & EG \arrow[d, "t_i"] \\ EG \arrow[r, "t_i"'] & I \end{tikzcd}$$

commutes. Thus $t_i \circ \rho = t_i \circ \mathrm{proj}_1$ is continuous, and

$$\begin{array}{ccccc}
\rho^{-1}t_i^{-1}(0,1] & = & t_i^{-1}(0,1] \times G & \xrightarrow{\rho|\dots} & t_i^{-1}(0,1] \\
\downarrow{\scriptstyle v_i \times \mathrm{id}} & & & & \downarrow{\scriptstyle v_i} \\
G \times G & & \xrightarrow[\mu]{} & & G\,,
\end{array}$$

commutes, where μ is the group multiplication. Thus $v_i \circ (\rho|_{\rho^{-1}t_i^{-1}(0,1]})$ is continuous.

BG is obtained from EG by passing to the orbit space under the group action ρ, that is, taking the quotient space under the equivalence relation

$$a, b \in EG; \quad a \sim b \Leftrightarrow \exists\ u \in G \ \text{ with } \ \rho(a,u) = b\,,$$

(see 2.2.21).

Let $p_G : EG \longrightarrow BG$ be the quotient map. Then p_G is a principal fibration.

2.9.13 **Theorem**. *p_G is a locally trivial principal fibration.* We denote by η_G the corresponding fiber bundle.

Proof: Let us consider the sets $W_i = t_i^{-1}(0,1]$ and $V_i = p_G W_i$. W_i is open in EG by the definition of the topology in EG. V_i is open in BG, since $p_G^{-1}(V_i) = W_i$. $\{V_i \mid i = 1, 2, \ldots\}$ is an open cover of BG. We shall prove that p_G is trivial over V_i.

We define maps $\Phi_i : V_i \times G \longrightarrow W_i$ by

$$\Phi_i\left(p_G(t_1v_1 + t_2v_2 + \cdots), u\right) = t_1(v_1v_i^{-1}u) + t_2(v_2v_i^{-1}u) + \cdots\,.$$

We show that they are homeomorphisms.

Φ_i is well defined.

Namely, if $p_G(t_1v_1 + \cdots) = p_G(t_1'v_1' + \cdots) \in V_i$, then one has $t_j = t_j'$, and there exists $w \in G$ such that $v_j' = v_jw$ for every j such that $t_j \neq 0$. Thus $v_i' = v_iw$ and $v_j'v_i'^{-1}u = v_jww^{-1}v_i^{-1}u = v_jv_i^{-1}u$ for such values of j.

Φ_i is bijective.

Namely,

$$(p_G, v_i) : W_i = t_i^{-1}(0,1] \longrightarrow V_i \times G$$

is an inverse of Φ_i and is continuous.

Φ_i is compatible with the action of G.

Namely,

$$\begin{aligned}\Phi_i(p_G(t_1v_1+\cdots),u_1u_2) &= t_1(v_1v_i^{-1}u_1u_2)+\cdots\\ &= t_1(v_1v_i^{-1}u_1+\cdots)u_2\\ &= \Phi_i(p_G(t_1v_1+\cdots),u_1)u_2\,.\end{aligned}$$

Φ_i is continuous.

Namely, $\Phi|_{V_i\times\{e\}}$ is continuous, since

$$\Phi_i(p_G(a),e)=\rho(a,v_i(a)^{-1}),\quad a\in W_i\,,$$

and since

$$\Phi_i(x,u)=\rho(\Phi_i(x,e),u),$$

Φ_i is also continuous. □

We still have to prove that $\pi_i(EG)=0$ for every i. Before doing it, we explain the construction of EG.

Consider inside EG the subset

$$E^kG=\{t_1v_1+\cdots+t_kv_k+\cdots+t_jv_j+\cdots\mid t_j=0\ \text{ if }\ j>k\}\,.$$

For example, a point $t_1v_1+t_2v_2\in E^2G$ can be described by a triple (t,v_1,v_2), $t=t_1$ $(1-t=t_2)$, since $t_1+t_2=1$, where the triples

$$(0,v_1,v_2)\ \text{ and }\ (0,v_1',v_2)$$

are identified, as well as are the triples

$$(1,v_1,v_2)\ \text{ and }\ (0,v_1,v_2')\,.$$

In other words, E^2G, as a set, is the *join* $G*G$ (see page 10), up to the fact that the topology of $G*G$ might be finer (larger). Analogously, one may see that, up to topology, E^3G can be considered as $(G*G)*G$, and so on.

2.9.14 **Theorem**. $\pi_i(EG)=0$ *for every* $i\geq 0$.

Proof: Defining

$$s_k(t_1v_1+t_2v_2+\cdots)=\sum_{j=1}^{k}t_j\,,$$

we have a continuous map

$$s_k:EG\longrightarrow I\,.$$

Take $U_k = s_k^{-1}(0,1]$. Then, $U_k \subset U_{k+1}$ for $k \geq 1$, and $\bigcup_{k=1}^{\infty} U_k = EG$, since for every $t_1v_1 + t_2v_2 + \cdots \in EG$, $\sum_{j=1}^{\infty} t_j = 1$. Because U_k is open in EG and $\{U_k\}$ is a cover of EG, each compact set in EG lies in some adequate U_n.

The image of the sphere $\mathbb{S}^i$ under a continuous map is compact and thus lies in U_n for some n. Thus, the theorem will be proved, if we prove that every U_k is contractible in EG.

We define a homotopy $h : U_k \times I \longrightarrow EG$ by

$$t_j \circ h : (a,t) \longmapsto \begin{cases} \frac{t+(1-t)s_k(a)}{s_k(a)} t_j(a) & \text{if } j \leq k, \\ (1-t)t_j(a) & \text{if } j > k, \end{cases}$$

$$v_j \circ h(a,t) \longmapsto v_j(a) \qquad \text{if } \; t_j h(a,t) > 0\,.$$

One has that

$$\begin{aligned} \sum_{j=1}^{\infty} t_j\left(h(a,t)\right) &= \frac{t+(1-t)s_k(a)}{s_k(a)} s_k(a) + (1-t)(1-s_k(a)) \\ &= 1\,, \end{aligned}$$

so that, indeed, $h(a,t) \in EG$.

By 2.9.12, h is continuous. One has that $h(a,0) = a$ and $h(a,1) \in E^kG$ $(= s_k^{-1}(1))$. We now attach to h another homotopy $d : E^kG \times I \longrightarrow E^{k+1}G$ given by

$$t_j \circ d : (a,t) \longmapsto \begin{cases} (1-t)t_j(a) & \text{if } j \leq k, \\ t & \text{if } j = k+1, \\ 0 & \text{if } j \geq k+2; \end{cases}$$

$$v_j \circ d : (a,t) \longmapsto \begin{cases} v_j(a) & \text{if } j \leq k, \\ e & \text{if } j \geq k+1. \end{cases}$$

One has that $d(a,0) = a$ and that

$$\begin{aligned} d(a,1) &= 0v_1(a) + 0 + \cdots + 0v_k(a) + 1e + 0e + 0 + \cdots \\ &= 0e + 0e + \cdots + 0e + 1e + 0e + \cdots\,; \end{aligned}$$

in other words, d contracts E^kG inside $E^{k+1}G$ in one point, and since $h(a,1) \in E^kG$ we have the desired result. □

2.9.15 REMARK. In fact, EG is contractible. Cf. Dold [3, §8].

If $\theta : G \longrightarrow G'$ is a continuous homomorphism between topological groups, then we have a fiber map

$$(\widetilde{\theta}, \overline{\theta}) : p_G \longrightarrow p_{G'}$$

given by

$$\begin{aligned} \widetilde{\theta}(t_1 v_1 + \cdots) &= t_1\theta(v_1) + t_2\theta(v_2) + \cdots\,, \\ \overline{\theta} p_G(t_1 v_1 + \cdots) &= p_{G'}(t_1\theta(v_1) + t_2\theta(v_2) + \cdots)\,. \end{aligned}$$

$\overline{\theta}$ is well defined.

In 2.4.10 we assigned to θ a natural transformation

$$\theta_* : k_G \longrightarrow k_{G'}\,,$$

(see 2.8.16 for the notation).

The next diagram

$$\begin{tikzcd} {[A, BG]} \arrow[r, "\Omega(\eta_G)"] \arrow[d, "\overline{\theta}_\#"'] & k_G(A) \arrow[d, "\theta_*"] \\ {[A, BG']} \arrow[r, "\Omega(\eta_{G'})"'] & k_{G'}(A)\,. \end{tikzcd}$$

Let $V_i \subset BG$ be the open set defined in the proof of 2.9.13, and correspondingly, $V_i' \subset BG'$. One has that $\overline{\theta}^{-1}(V_i^{-1}) = V_i$. Let $[f] = [A, BG]$. $\theta_*\Omega(\eta_G)[f]$ is represented by a bundle with open cover $\{f^{-1}(V_i)\}$ and coordinate transformations

$$f^{-1}(V_i) \cap f^{-1}(V_j) \xrightarrow{f} V_i \cap V_j \xrightarrow{g_{ij}} G \xrightarrow{\theta} G'\,,$$

and $\Omega(\eta_{G'})\overline{\theta}_\#$ is represented by a bundle with cover $\{f^{-1}\overline{\theta}^{-1}(V_i')\}$ and coordinate transformations

$$f^{-1}\overline{\theta}^{-1}(V_i') \cap f^{-1}\overline{\theta}^{-1}(V_j') \xrightarrow{f} \overline{\theta}^{-1}(V_1') \cap \overline{\theta}^{-1}(V_2') \xrightarrow{\overline{\theta}} V_1' \cap V_2' \xrightarrow{g_{ij}'} G'\,,$$

where g_{ij} and g_{ij}' are the coordinate transformations of η_G and $\eta_{G'}$, respectively.

Since $\overline{\theta}^{-1}(V_i') = V_i$, both covers coincide. It is, therefore, enough to check that $\theta \circ g_{ij} = g_{ij}' \circ \overline{\theta}$. But

$$\theta g_{ij} p_G(t_1 v_1 + t_2 v_2 + \cdots) = \theta(v_i v_j^{-1})$$

$$\begin{aligned} g_{ij}'\overline{\theta} p_G(t_1 v_1 + \cdots) &= g_{ij}' p_{G'}(t_1\theta(v_1) + t_2 v_2 + \cdots) \\ &= \theta(v_i)\theta(v_j)^{-1}\,. \end{aligned}$$

Chapter 3

Singular Homology of Fibrations

3.1 Introduction

3.2 Spectral Sequences

3.2.1 Additive Relations

We shall introduce here the concept of the subtitle, since it is a very convenient formalism for studying spectral sequences.

3.2.1 Definition. Let A and B be abelian groups (which we write additively). A *relation* $f : A \longrightarrow B$ is a triple $f = (A, B, F)$ such that $F \subset A \times B$ (cf. 2.3.23). We say that the relation f is *additive* if f is a subgroup of $A \times B$. If f is on the one hand a function of sets, and on the other, an additive relation, then f is a group homomorphism.

In what follows, we shall only consider additive relations. Let $f = (A, B, F)$ and $g = (B, C, H)$ be relations. We define the *composition* $g \circ f$ as the triple (A, C, H), where

$$H = \{(a, c) \in A \times C \mid \exists\, b \in B \ \text{ with } \ (a, b) \in F,\ (b, c) \in G\}\,.$$

The relation $g \circ f$ is additive again. Abelian groups, together with additive relations, constitute a category. Given a relation $f = (A, B, F)$, we define its *inverse relation* by

$$f^{-1} = (B, A, F^{-1})\,, \ \text{ where } \ F^{-1} = \{(b, a) \in B \times A \mid (a, b) \in F\}\,.$$

This relation f^{-1} is also additive. One has the following formulas:

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}\,, \quad (f^{-1})^{-1} = f\,.$$

3.2.2 EXERCISE. Prove that given additive relations $f : A \longrightarrow B$ and $g : B \longrightarrow C$, then the relations $g \circ f : A \longrightarrow C$ and $f^{-1} : B \longrightarrow A$ are, indeed, additive.

3.2.3 EXERCISE. Prove that, indeed, the abelian groups and the additive relations constitute a category. (Hint: Observe that for an abelian group A, the identity relation is $\mathrm{id}_A = (A, A, \Delta_A)$, where $\Delta_A = \{(a,b) \in A \times A \mid a = b\}$ is the diagonal subgroup.)

3.2.4 NOTE. If $f = (A, B, F)$ is additive, in general $f^{-1} \circ f \neq \mathrm{id}_A$.

3.2.5 EXERCISE. Give an example of an additive relation $f : A \longrightarrow B$ such that $f^{-1} \circ f \neq \mathrm{id}_A$. Analyze under what conditions one has $f^{-1} \circ f = \mathrm{id}_A$; in other words, characterize the isomorphisms in the category of abelian groups and additive relations.

Let A be an abelian group and $U \subset A$ a subset, and let $f = (A, B, F)$ be an additive relation. We define

$$\begin{aligned} f(U) &= \{b \in B \mid \exists\ a \in A \ \text{ with } \ (a,b) \in F\} \\ &= \mathrm{proj}_2((U \times B) \cap F)\,. \end{aligned}$$

If U ia a subgroup of A, then $f(U)$ is a subgroup of B. One has the following facts:

(a) If $U_1 \subset U_2$, then $f(U_1) \subset f(U_2)$.

(b) $f(\bigcup_{j \in J} U_j) = \bigcup_{j \in J} f(U_j)$.

(c) If f is a function, then $f^{-1}(\bigcap_{j \in J} U_j) = \bigcap_{j \in J} f^{-1}(U_j)$.

3.2.6 DEFINITION. Let $f = (A, B, F)$ be an additive relation. We define the following concepts:

The *image of* f by $\mathrm{Im}(f) = f(A)$.

The *indeterminacy of* f by $\mathrm{Ind}(f) = f(0)$.

The *definition domain of* f by $\mathrm{Def}(f) = \mathrm{Im}(f^{-1})$.

The *kernel of* f by $\mathrm{Ker}(f) = \mathrm{Ind}(f^{-1})$.

3.2.7 **Proposition**. *There is a unique additive relation $\overline{f}$ that fits into the next commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d] \\ \mathrm{Def}(f) \arrow[u, hook] \arrow[r, dashed, "\overline{f}"'] & B/\mathrm{Ind}(f)\,, \end{tikzcd}$$

with the following properties:

(a) *$\overline{f}$ is a homomorphism.*

(b) *The mapping $f \mapsto \overline{f}$ establishes a one-to-one correspondence between additive relations $f : A \longrightarrow B$ and homomorphisms from a subgroup of A into a quotient of B.*

(c) *$\overline{f}$ induces canonically an isomorphism*

$$\overline{\overline{f}} : \mathrm{Def}(f)/\mathrm{Ker}(f) \longrightarrow \mathrm{Im}(f)/\mathrm{Ind}(f)\,.$$

The *proof* is routine and we leave it to reader. □

3.2.2 Exact Couples and their Spectral Sequences

There are several approaches to spectral sequences. We chose here the classical one through exact couples invented by Massey [10]. Before stating the definition, we need some previous concepts.

3.2.8 Definition. Let $A = \{A_{p,q} \mid (p,q) \in \mathbb{Z} \times \mathbb{Z}\}$ and $C = \{C_{p,q} \mid (p,q) \in \mathbb{Z} \times \mathbb{Z}\}$ be families of abelian groups. We define a *homomorphism*

$$h : A \longrightarrow C$$

of *bidegree* (r,s) as a family of homomorphisms

$$h_{p,q} : A_{p,q} \longrightarrow C_{p+r,q+s}\,;$$

we denote this fact by $\mathrm{bideg}(h) = (r,s)$. A and C are called *bigraded groups.* The elements of $A_{p,q}$ are said to have *bidegree* (p,q). Let $k : C \longrightarrow D$ be another homomorphism of bidegree (u,v). We say that a sequence

$$A \xrightarrow{h} C \xrightarrow{k} D$$

is *exact* at C if for every (p,q), the sequence

$$A_{p,q} \xrightarrow{h_{p,q}} C_{p+r,q+s} \xrightarrow{k_{p+r,q+s}} D_{p+r+u,q+s+v}$$

is exact at $C_{p+r,q+s}$.

3.2.9 DEFINITION. An *exact couple* is a triangle

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & C \arrow[ul, "k"] & \end{tikzcd} \tag{3.2.10}$$

of bigraded abelian groups and homomorphisms such that

$$\begin{aligned} \mathrm{bideg}(i) &= (1,-1)\,, \\ \mathrm{bideg}(j) &= (0,0)\,, \\ \mathrm{bideg}(k) &= (-1,0)\,, \end{aligned}$$

that is exact at each vertex.

A piece of the exact couple (3.2.10) looks like follows

$$\begin{tikzcd} & \vdots \arrow[d, "i"] & & \\ C_{p,q+1} \arrow[r, Rightarrow, "k"] & A_{p-1,q+1} \arrow[r, "j"] \arrow[d, Rightarrow, "i"] & C_{p-1,q+1} \arrow[r, "k"] & A_{p-2,q+1} \\ & A_{p,q} \arrow[r, Rightarrow, "j"] \arrow[d, "i"] & C_{p,q} \arrow[r, Rightarrow, "k"] & A_{p-1,q} \\ & A_{p+1,q-1} \arrow[d, dotted, "\bar{i}"] & & \\ & \overline{A}_{p+q} & & \end{tikzcd} \tag{3.2.11}$$

The double arrows show the intertwined exact sequences. Let $\overline{A}_{p,q}$ be the colimit of the sequence

$$\cdots \longrightarrow A_{p-1,q+1} \xrightarrow{i} A_{p+1,q-1} \xrightarrow{i} \cdots .$$

Thus we have a system of homomorphisms

$$\bar{i}_{p,q} = \bar{i} : A_{p,q} \longrightarrow \overline{A}_n\,, \quad p,q \in \mathbb{Z}\,, \quad n = p+q\,,$$

with the following properties:

(a) The diagram

$$\begin{tikzcd} A_{p,q} \arrow[rr, "i"] \arrow[dr, "\bar{i}"'] & & A_{p+1,q-1} \arrow[dl, "\bar{i}"] \\ & \overline{A}_n & \end{tikzcd}$$

is commutative.

(b) $\bigcup_{p+q=n}\mathrm{Im}(\bar{i} : A_{p,q} \longrightarrow \overline{A}_n) = \overline{A}_n$.

(c) $\mathrm{Ker}(\bar{i} : A_{p,q} \longrightarrow \overline{A}_n = \bigcup_{r=0}^{\infty}\mathrm{Ker}(i^r : A_{p,q} \longrightarrow A_{p+r,q-r})$, (where $i^\circ = \mathrm{id}$ and $i^r = i \circ i^{r-1}$, $r \geq 1$).

Recall that for each sequence of (abelian) groups, there exists a colimit, and it is unique up to isomorphism (see [1]).

3.2.12 DEFINITION. Let G be a group and $\{F_pG \mid p \in \mathbb{Z}\}$ a sequence of subgroups such that $F_pG \subset F_{p+1}G$; we refer to $\{F_pG\}$ as a *filtration* of the group G.

3.2.13 CONSTRUCTION. In what follows, for simplicity, we shall assume that

$$A_{p,q} = 0 \quad \text{if} \quad p < 0\,.$$

We have now from the relation

$$f = \bar{i}j^{-1} : C_{p,q} \xrightarrow{j^{-1}} A_{p,q} \xrightarrow{\bar{i}} \overline{A}_{p+q}\,,$$

the groups

$$\mathrm{Im}(f),\quad \mathrm{Ind}(f),\quad \mathrm{Ker}(f),\quad \text{and}\quad \mathrm{Def}(f)\,,$$

(as defined in 3.2.6).

$$\begin{aligned}\mathrm{Im}(f) &= \bar{i}j^{-1}(C_{p,q}) = \bar{i}(\mathrm{Def}(j)) = \bar{i}(A_{p,q})\\ \mathrm{Ind}(f) &= \bar{i}j^{-1}(0) = \bar{i}(\mathrm{Ker}(j)) = \bar{i}(i(A_{p-1,q+1}))\\ &= \bar{i}(A_{p-1,q+1})\,.\end{aligned}$$

By

$$\bar{i}(A_{p,q}) = F_p\overline{A}_{p+q}\,,$$

and since $\mathrm{Ind}(f) \subset \mathrm{Im}(f)$, we obtain a filtration of $\overline{A}_{p+q}$, because we have proved

$$\begin{aligned}\mathrm{Ind}(f) &= F_{p-1}\overline{A}_{p+q}\,,\\ \mathrm{Im}(f) &= F_p\overline{A}_{p+q}\,.\end{aligned} \tag{3.2.14}$$

We also have

$$\begin{aligned}\mathrm{Ker}(f) = \mathrm{Ind}\left(f^{-1}\right) = \left(j\bar{i}^{-1}\right)(0) &= j\left(\mathrm{Ker}\left(\bar{i}\right)\right)\\ &= j\left(\bigcup_{r=0}^{\infty}\left(i^r\right)^{-1}(0)\right) = \bigcup_{r=0}^{\infty} j\left(i^r\right)^{-1}(0)\\ &= \bigcup_{r=0}^{\infty} j\left(i^r\right)^{-1}k\,(0) = \bigcup_{r=0}^{\infty}\mathrm{Ind}\left(j\left(i^r\right)^{-1}k\right)\end{aligned} \tag{3.2.15}$$

by definition of colimit (see 3.2.9). Moreover, we have

$$\begin{aligned}
\mathrm{Def}(f) = \mathrm{Im}(f^{-1}) &= j\bar{i}^{-1}\left(\overline{A}_{p+q}\right) = j\left(A_{p,q}\right) \\
&= k^{-1}(0) = k^{-1}\bigcap_{r=0}^{\infty} i^r j^{-1}(0) \qquad (3.2.16)\\
&= \bigcap_{r=0}^{\infty} k^{-1} i^r j^{-1}(0) = \bigcap_{r=0}^{\infty} \mathrm{Ker}\left(j\left(i^r\right)^{-1} k\right)
\end{aligned}$$

and

$$\begin{aligned}
\bigcap_{r=0}^{\infty} i^r j^{-1}(0) &= \bigcap_{r=0}^{\infty} \mathrm{Ind}(i^r j^{-1} : C_{p-r-1,q+r} \longrightarrow A_{p-r-1,q+r} \longrightarrow A_{p-1,q}) \\
&= 0\,,
\end{aligned}$$

since by assumption $A_{p,q} = 0$ if $p < 0$. These computations can be figured out in a diagram similar to (3.2.11).

In the expressions for $\mathrm{Ker}(f)$ and $\mathrm{Def}(f)$ the relations $j(i^{r-1})^{-1}k$ play a role.

3.2.17 DEFINITION. Take $r \geq 1$, $r \in \mathbb{Z}$, $i^0 = \mathrm{id}_A$.

$$\begin{aligned}
d^r_{p,q} &= j(i^{r-1})^{-1}k : C_{p,q} \longrightarrow C_{p-r,q+r-1}\,, \\
Z^r_{p,q} &= \mathrm{Def}(d^r_{p,q})\,, \\
B^r_{p,q} &= \mathrm{Ind}(d^r_{p+r,q-r+1})\,, \\
Z^\infty_{p,q} &= \bigcap_{r=1}^{\infty} Z^r_{p,q}\,, \\
B^\infty_{p,q} &= \bigcup_{r=1}^{\infty} B^r_{p,q}\,, \\
E^r_{p,q} &= Z^r_{p,q}/B^r_{p,q}\,, \quad 1 \leq r \leq \infty\,.
\end{aligned}$$

As we already did with i, j, k, occasionally we shall omit the indexes p, q, in these objects, even though we shall not be dealing with the whole bigraded group, but just of one member of it.

3.2.18 **Proposition**. *The following equations hold:*

$$\begin{aligned}
\mathrm{Im}(d^r) &= \mathrm{Ind}(d^{r+1})\,, \\
\mathrm{Ker}(d^r) &= \mathrm{Def}(d^{r+1})\,, \\
\mathrm{Im}(f) &\subset \mathrm{Ker}(d^r)\,.
\end{aligned}$$

Proof: Consider the following immediate equalities:

$$\begin{aligned}
\mathrm{Im}(d^r{}_{p,q}) &= j(i^{r-1})^{-1}k(C_{p,q}) \\
&= j(i^{r-1})^{-1}j^{-1}(0) \qquad\qquad \text{(by the exactness)} \\
&= j(i^r)^{-1}(0) \\
&= j(i^r)^{-1}k(0) \\
&= \mathrm{Ind}(d^{r+1}_{p+1,q+1})\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ker}(d^r)_{p,q} &= (j(i^{r-1})^{-1}k)^{-1}(0) \\
&= k^{-1}(i^{r-1})j^{-1}(0) \\
&= k^{-1}(i^{r-1})i(A_{p-r-1,q+r}) \qquad \text{(by the exactness)} \\
&= k^{-1}(i^r)j^{-1}(C_{p-r-1,q+r}) \\
&= (d^{r+1}_{p,q})^{-1}(C_{p-r-1,q+r}) \\
&= \mathrm{Def}(d^{r+1}_{p,q})\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Im}(d^r)_{p,q} &= j(i^{r-1})^{-1}k(C_{p,q}) \\
&\subset j(A_{p-r,q+r-1}) \\
&= k^{-1}(0) \\
&\subset k^{-1}(i^{r-1})j^{-1}(0) \\
&= \mathrm{Ker}(d^r_{p-r,q+r-1})\,.
\end{aligned}$$

□

Applying again the complete diagram as in (3.2.11), one can rewrite what we just proved in the following chain of inclusions (since $\mathrm{Ind}(f) \subset \mathrm{Im}(f)$ and $\mathrm{Def}(f) \supset \mathrm{Ker}(f)$).

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, equal] & \mathrm{Im}(d^1) \arrow[r, hook] \arrow[d, equal] & \mathrm{Im}(d^2) \arrow[r, hook] \arrow[d, equal] & \cdots \arrow[r, hook] & \mathrm{Ker}(d^2) \arrow[r, hook] \arrow[d, equal] & \mathrm{Ker}(d^1) \arrow[r, hook] \arrow[d, equal] & C \arrow[d, equal] \\
\mathrm{Ind}(d^1) \arrow[r, hook] & \mathrm{Ind}(d^2) \arrow[r, hook] & \mathrm{Ind}(d^3) \arrow[r, hook] & \cdots \arrow[r, hook] & \mathrm{Def}(d^3) \arrow[r, hook] & \mathrm{Def}(d^2) \arrow[r, hook] & \mathrm{Def}(d^1)\,.
\end{tikzcd}$$

In other words,

$$0 = B^1 \subset B^2 \subset B^3 \subset \cdots \subset Z^3 \subset Z^2 \subset Z^1 = C\,.$$

In particular, one has

$$\begin{aligned}
B^{r+1} &= \mathrm{Im}(d^r) \subset \mathrm{Def}(d^r) = Z^r\,, \\
B^r &= \mathrm{Ind}(d^r) \subset \mathrm{Ker}(d^r) = Z^{r+1}\,,
\end{aligned}$$

and thus d^r (cf. 3.2.7) induces the following diagram:

$$
\begin{tikzcd}
\mathrm{Def}(d^r) \arrow[r, "\overline{d}^r"] \arrow[d, two heads] & C/\mathrm{Ind}(d^r) \\
E^r = \mathrm{Def}(d^r)/\mathrm{Ind}(d^r) \arrow[r, "\overline{d}^r"] \arrow[d, two heads] & \mathrm{Def}(d^r)/\mathrm{Ind}(d^r) = E^r \arrow[u, hook] \\
\mathrm{Def}(d^r)/\mathrm{Ker}(d^r) \arrow[r, "\cong"] & \mathrm{Im}(d^r)/\mathrm{Ind}(d^r) \arrow[u, hook]
\end{tikzcd}
$$

Hence $\overline{d}^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ is a homomorphism.

3.2.19 **Theorem**. *The pair* $(E^r, \overline{d}^r)$ *is a chain complex and its homology satisfies*

$$H_{p,q}(E^r, \overline{d}^r) \cong E^{r+1}_{p,q}\,.$$

Proof: That $\overline{d}^r \circ \overline{d}^r = 0$ follows simply from

$$
\begin{aligned}
\mathrm{Im}(\overline{d}^r) &= \mathrm{Im}(d^r)/\mathrm{Ind}(d^r) \\
&\subset \mathrm{Ker}(d^r)/\mathrm{Ind}(d^r) = \mathrm{Ker}(\overline{d}^r)\,.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
H_{p,q}(E^r, \overline{d}^r) &= \mathrm{Ker}(\overline{d}^r_{p,q})/\mathrm{Im}(\overline{d}^r_{p+r,q-r+1}) \\
&\cong \mathrm{Ker}(d^r_{p,q})/\mathrm{Im}(d^r_{p+r,q-r+1}) \\
&= \mathrm{Def}(d^{r+1}_{p,q})/\mathrm{Ind}(d^{r+1}_{p+r+1,q-r}) \\
&= Z^{r+1}_{p,q}/B^{r+1}_{p,q} = E^{r+1}_{p,q}\,.
\end{aligned}
$$

by Definition 3.2.17 and by 3.2.18. □

3.2.20 DEFINITION. A sequence $(E^r, \overline{d}^r)$ of chain complexes, together with isomorphisms

$$H(E^r, \overline{d}^r) \cong E^{r+1}$$

is called a *spectral sequence*.

3.2.21 REMARK. The isomorphism 3.2.7 (c) induced by an additive relation f

$$\mathrm{Def}(f)/\mathrm{Ker}(f) \longrightarrow \mathrm{Im}(f)/\mathrm{Ind}(f)\,,$$

by using our computations in 3.2.13 and Definition 3.2.17 yield an isomorphism

$$E^\infty_{p,q} \cong F_p\overline{A}_{p+q}/F_{p-1}\overline{A}_{p+q}\,, \tag{3.2.22}$$

since $F_p\overline{A}_{p+q}/F_{p-1}\overline{A}_{p+q} = \mathrm{Im}(f)/\mathrm{Ind}(f)$, as in (3.2.14), and

$$\mathrm{Def}(f) = \bigcap_{r=0}^{\infty} \mathrm{Ker}(d^{r+1}) = Z^{\infty}_{p,q}\,, \qquad \text{(see (3.2.16) and 3.2.18)},$$

$$\mathrm{Ker}(f) = \bigcup_{r=0}^{\infty} \mathrm{Ind}(d^{r+1}) = B^{\infty}_{p,q}\,, \qquad \text{(see (3.2.15) and 3.2.18)}.$$

3.3 The Homology Spectral Sequence of a Serre Fibration

3.3.1 Construction. Let $\pi : E \longrightarrow B$ be a Serre fibration over a CW-complex B. Denote by B^p the p-skeleton and by E^p its inverse image under π, $\pi^{-1}(B^p)$, $p \geq 0$. In particular, set $E^p = \emptyset$ if $p < 0$. We have an exact couple (see 3.2.9).

(3.3.2)
$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & C \arrow[ul, "k"] & \end{tikzcd}\,,$$

given by the definitions

$$\begin{aligned} A_{p,q} &= H_{p+q}(E^p)\,, \\ C_{p,q} &= H_{p+q}(E^p, E^q)\,, \\ i : H_{p+q}(E^p) &\longrightarrow H_{p+q}(E^{p+1})\,, \\ j : H_{p+q}(E^p) &\longrightarrow H_{p+q}(E^p, E^{p-1})\,, \end{aligned}$$

that are induced by the canonical inclusions, and

$$k : H_{p+q}(E^p, E^{p-1}) \longrightarrow H_{p+q-1}(E^{p-1})$$

given by the boundary homomorphism ∂.

The bidegrees of these homomorphisms clearly are:

$$\begin{aligned} \mathrm{bideg}(i) &= (1, -1)\,, \\ \mathrm{bideg}(j) &= (0, 0)\,, \\ \mathrm{bideg}(k) &= (-1, 0)\,, \end{aligned}$$

as in 3.2.9.

Take

$$\overline{A}_n = H_n(E)\,, \qquad \text{and let} \quad \bar{\imath} : H_n(E^p) \longrightarrow H_n(E)$$

be induced by the inclusion. Then $H_n(E)$, together with $\bar{\imath}$ (for every p) is a colimit of the sequence

$$\cdots \xrightarrow{i} H_n(E^p) \xrightarrow{i} H_n(E^{p+1}) \xrightarrow{i} H_n(E^{p+2}) \longrightarrow \cdots ,$$

so that we have to prove the following.

Each element $x \in H_n(E)$ lies inside the image of some $\bar{\imath}$, and if some element in $H_n(E^p)$ lies inside the kernel of $\bar{\imath}$, then it also lies inside the kernel of

$$i^r : H_n(E^p) \longrightarrow H_n(E^{p+r})$$

for r large enough.

Let $x = [z]$, where $z \in Z_n(E)$ is a cycle. The support $|z|$ of the cycle z, ($z = \sum_{i=1}^k \alpha_i \zeta_i$, $\zeta_i : \Delta^n \longrightarrow E$, $|z| = \bigcup_{i=1}^k \zeta_i(\Delta^n)$) is compact; hence, also $\pi(|z|)$ is compact, and since B has the weak topology, there exists p such that $\pi(|z|) \subset B^p$, and consecuently $|z| \subset E^p$ and so $x \in \operatorname{Im}(H_n(E^P) \longrightarrow H_n(E))$. Analogously, one can conclude that if a cycle has support in E^p and is a boundary (in E); thus it is a boundary in E^{p+r} for some r.

Finally, one has that (cf. 3.2.13) $A_{p,q} = 0$ for $p < 0$.

3.3.3 Note. We shall use any abelian group as group of coefficients in homology.

3.3.4 Remark. It is important to ponder what is happening with the formalism of Section 3.2 in the case of $E = B$, $\pi = \mathrm{id}_B$. We had $d^r = j(i^{r-1})^{-1}k$. Thus $d^1 = jk$,

$$d^1 : H_{p+q}(B^p, B^{p-1}) \xrightarrow{\partial} H_{p+q-1}(B^{p-1}) \longrightarrow H_{p+q-1}(B^{p-1}, B^{p-2}) .$$

Moreover,

$$\begin{aligned} Z^1_{p,q} &= \mathrm{Def}(d^1) = H_{p,q}(B^p, B^{p-1}) , \\ B^1_{p,q} &= \mathrm{Ind}(d^1) = 0 . \end{aligned}$$

Thus, we can identify the group $E^1_{p,q}$ with $H_{p,q}(B^p, B^{p-1})$, where $\overline{d}^1$ corresponds to d^1.

On the other hand, in this case one has (see [1, 7.3.1]) that

$$H_{p+q}(B^p, B^{p-1}) = \begin{cases} \bigoplus_{i \in J^p} \mathbb{Z} & \text{if } q = 0, \\ 0 & \text{if } q \neq 0, \end{cases}$$

where $\bigoplus_{i\in J^p}\mathbb{Z}$ represents the free group generated by the p-cells of B (or, instead of $\mathbb{Z}$ the coefficient group A); that is, it is the *group of cellular p-chains of B* (with coefficients in A), and d^1 is the usual boundary operator. Thus we obtain that

$$E^2_{p,q} \cong \begin{cases} H_p(B) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0, \end{cases}$$

Since $\overline{d}^r$ has bidegree $(-r, r-1)$, one has that $\overline{d}^r = 0$ for $r \geq 2$ and hence we have that $E^2_{p,q} \cong E^r_{p,q}$ for all $r \geq 2$. One has also, from $\overline{d}^r$ and 3.2.18, that $\mathrm{Def}(d^{r+1}) = \mathrm{Ker}(d^r) = \mathrm{Def}(d^r)$; hence $Z^2 = Z^\infty$, analogously, from $\mathrm{Ind}(d^r) = \mathrm{Im}(d^r) = \mathrm{Ind}(d^{r+1})$, we obtain $B^2 = B^\infty$. It is now easy to verify the formula (3.2.22) that ralates E^∞ with the filtration of $H_*(B)$.

We want to compute now E^1, E^2, and d^1 for the exact couple (4.2.2). For that, we need the following result.

3.3.5 **Lemma**. *Let A_0, A_1, B_0, and B_1 be subspaces of B such that*

$$\begin{array}{ccc} A_0 & \overset{\simeq}{\subset} & B_0 \\ \cap & & \cap \\ A_1 & \overset{\simeq}{\subset} & B_1 \end{array}$$

and assume that A_0, resp. A_1, is a strong deformation retract of B_0, resp. B_1. Then the inclusion induces isomorphisms

$$H_n(\pi^{-1}(A_1), \pi^{-1}(A_0)) \cong H_n(\pi^{-1}(B_1), \pi^{-1}(B_0))\,;$$

besides,

$$H_n(\pi^{-1}(B_\nu), \pi^{-1}(A_\nu)) = 0, \quad \nu = 0, 1\,.$$

To be able to prove this lemma, we need another one.

3.3.6 **Lemma**. *Let $\pi : X \longrightarrow Y$ be a Serre fibration. If $A \subset B$ is a strong deformation retract of B, then $S(\pi^{-1}(A))$ a chain deformation retract of $S(E)$, where S denotes the corresponding singular complex.*

Proof: Since A is a strong deformation retract of B, there is a map

$$\varphi : B \times I \longrightarrow B$$

with the following properties:

$$\begin{aligned} \varphi(b,0) &= b && \text{if } b \in B, \\ \varphi(b,1) &\in A && \text{if } b \in B, \\ \varphi(a,t) &= a && \text{if } a \in A. \end{aligned}$$

We now proceed in steps.

(a) Let $\gamma_n(E)$ be the set of singular n-simplexes. We want to assign to each $\sigma \in \gamma_n(E)$ a map

$$\widehat{\sigma} : \Delta_n \times I \longrightarrow E\,, \quad (\widehat{\sigma}(x,t) = \widehat{\sigma}_t(x))\,,$$

with the following properties:

(i) $\partial_i(\widehat{\sigma}_t) = (\widehat{\partial_i\sigma})_t$.

(ii) $\widehat{\sigma}_0 = \sigma$.

(iii) $\widehat{\sigma}_t(x) = \sigma(x)$ for all $t \in I$, if $\sigma \in \gamma_n(\pi^{-1}(A))$.

(iv) $\pi\widehat{\sigma}_t(x) = \varphi(\pi\sigma(x), t)$.

From (iv) one has also that $\widehat{\sigma_1}(x) \in \pi^{-1}(A)$, that is, $\widehat{\sigma_1} \in S(\pi^{-1}(A))$.

With respect to the notation, we have that $\widehat{\sigma_t}$ is a singular simplex of $\gamma_n(E)$. Let $\partial_i\sigma$ be the i-face of σ. If we denote by $\varepsilon_i : \Delta_{n-1} \longrightarrow \Delta_n$ the canonical inclusion into the i-face, then $\partial_i\sigma = \sigma \circ \varepsilon_i$.

(b) We now construct $\widehat{\sigma}$ by induction on n. Consider the problem

$$\begin{tikzcd} \Delta_n \times \{0\} \cup \varepsilon_i(\Delta_{n-1}) \times I \arrow[r, "h_0"] \arrow[d, hook] & E \arrow[d, "\pi"] \\ \Delta_n \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widehat{\sigma}"] & B \end{tikzcd} \tag{3.3.7}$$

where

$$\begin{aligned} h(x,t) &= \varphi(\pi\sigma(x), t) && \text{(guaranteed by (iv))},\\ h_0(x,0) &= \sigma(x) && \text{(guaranteed by (ii))},\\ h_0(\varepsilon_i(y), t) &= (\widehat{\partial_i\sigma})(y,t) && \text{(guaranteed by (i))}. \end{aligned}$$

$(\widehat{\partial_i\sigma})$ has already been constructed, by the induction hypothesis. h_0 is well defined:

$$\begin{aligned} h_0(\varepsilon_i(y), 0) &= (\widehat{\partial_i\sigma})(y,0)\,,\\ &= \partial_i\sigma(y) && \text{(by the induction hypothesis (ii))}\\ &= \sigma(\varepsilon_i(y)) && \text{(by the definition of } \partial_i\sigma)\\ &= h_0(\varepsilon_i(y), 0)\,. \end{aligned}$$

Diagram (3.3.7) is commutative:

$$\begin{aligned} \pi h_0(x,0) &= \pi\sigma(x) \\ &= \varphi(\pi\sigma(x),0) \\ &= h(x,0)\,; \end{aligned} \qquad \begin{aligned} \pi h_0(\varepsilon_i(y),t) &= \pi(\widehat{\partial_i\sigma})(y,t) \\ &= \varphi(\pi\partial_i\sigma(y),t) \\ &= \varphi(\pi\sigma(\varepsilon_i(y),t) \\ &= h(\varepsilon_i(y),t)\,. \end{aligned}$$

Since π is a Serre fibration, the problem has a solution and we obtain $\widehat{\sigma}$ that fulfills (a), (i) - (iv).

(c) For every topological space W there is a chain homotopy that is natural in W

$$F : S_n(W) \longrightarrow S_{n+1}(W \times I)$$

with the property

$$\partial F + F\partial = j^1_\# - j^0_\#\,,$$

where $j^\nu : W \longrightarrow W \times I$ is given by $j^\nu(w) = (w,\nu)$, $\nu = 0,1$ (cf. [9, II.8]). In particular, the diagram

$$\begin{array}{ccc} S_q(\Delta_{n-1}) & \xrightarrow{F} & S_{q+1}(\Delta_{n-1} \times I) \\ {\scriptstyle (\varepsilon_i)_\#}\downarrow & & \downarrow{\scriptstyle (\varepsilon_i\times\mathrm{id})_\#} \\ S_q(\Delta_n) & \xrightarrow[F]{} & S_{q+1}(\Delta_n \times I) \end{array}$$

is commutative.

(d) We define homomorphisms

$$\begin{aligned} r &: S_n(B) \longrightarrow S_n(\pi^{-1}(A))\,, \\ h &: S_n(B) \longrightarrow S_{n+1}(B)\,, \end{aligned}$$

by

$$r(\sigma) = \widehat{\sigma}_1\,, \quad h(\sigma) = \widehat{\sigma}_\# F(\iota_n) \quad (\iota_n = \mathrm{id}_{\Delta_n})\,,$$

if σ is an n-simplex, and then by extending linearly.

Statement: *r is a chain transformation, $r(\sigma) = \sigma$ if $\sigma \in S_n(\pi^{-1}(A))$, and h is a chain homotopy such that*

$$(\partial h + h\partial)\sigma = r\sigma - \sigma\,, \quad \sigma \in S_n(B)\,.$$

Proof: It is a chain transformation:

$$\begin{aligned}\partial r(\sigma) &= \partial(\widehat{\sigma}_1)\\ &= \sum_{i=0}^{n}(-1)^i\partial_i(\widehat{\sigma}_1)\\ &= \sum_{i=0}^{n}(-1)^i\widehat{(\partial_i\sigma)}_1\\ &= \sum_{i=0}^{n}(-1)^i r(\partial_i\sigma)\\ &= r\partial(\sigma)\,.\end{aligned}$$

By (a) (iii), $r(\sigma)=\widehat{\sigma}_1=\sigma$ if $\sigma\in S_n(\pi^{-1}(A))$. Thus

$$\begin{aligned}\partial h(\sigma) &= \partial\widehat{\sigma}_\# F(\iota_n)\\ &= \widehat{\sigma}_\#\partial F(i_n)\,,\\ h\partial(\sigma) &= h(\sum(-1)^i\partial_i\sigma)\\ &= \sum(-1)^i\widehat{(\partial_i\sigma)}_\# F(\iota_{n-1})\\ &= \sum(-1)^i\widehat{\sigma}_\#(\varepsilon_i\times id)_\# F(i_{n-1}) &&\text{(since by (a) (i) } \widehat{\partial_i\sigma}=\widehat{\sigma}(\varepsilon_i\times\mathrm{id}))\\ &= \sum(-1)^i\widehat{\sigma_\#}F\varepsilon_{i\#}(\iota_{n-1}) &&\text{(by (c))}\\ &= \widehat{\sigma_\#}F\partial(\iota_n) &&\text{(since } \partial_i(\iota_n)=\varepsilon_{i\#}\iota_{n-1}).\end{aligned}$$

Summarizing:

$$\begin{aligned}\partial h(\sigma)+h\partial(\sigma) &= \widehat{\sigma_\#}(\partial F+F\partial)\ (\iota_n)\\ &= \widehat{\sigma_\#}(j^1_\#-j^0_\#)(\iota_n)\\ &= \widehat{\sigma}_1-\widehat{\sigma}_0\\ &= r(\sigma)-\sigma\,.\end{aligned}$$

□

Proof of 3.3.5: By 3.3.6, we have

$$H_n(\pi^{-1}(A_\nu))\xrightarrow{\cong}H_n(\pi^{-1}(B_\nu))\,,\qquad \nu=0,1\,.$$

The long homology exact sequences of the pairs

$$(\pi^{-1}(A_1),\pi^{-1}(A_0))\quad\text{ and }(\pi^{-1}(B_1),\pi^{-1}(B_0))$$

fit together as follows.

$$\begin{array}{ccccccc}\cdots\to H_{n+1}(\pi^{-1}(A_0)) & \xrightarrow{\partial} & H_n(\pi^{-1}(A_1),\pi^{-1}(A_0)) & \to & H_n(\pi^{-1}(A_1)) & \to & \cdots\\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ \cdots\to H_{n+1}(\pi^{-1}(B_0)) & \xrightarrow{\partial} & H_n(\pi^{-1}(B_1),\pi^{-1}(A_0)) & \to & H_n(\pi^{-1}(B_1)) & \to & \cdots\,.\end{array}$$

Moreover, for the pair $(\pi^{-1}(B_\nu), \pi^{-1}(A_\nu))$ we have

$$H_{n+1}(\pi^{-1}(A_\nu)) \xrightarrow{\cong}$$

$$H_{n+1}(\pi^{-1}(B_\nu)) \xrightarrow{\partial=0} \underset{\substack{\| \\ 0\,,}}{H_n(\pi^{-1}(B_\nu), \pi^{-1}(A_\nu))} \xrightarrow{0} H_n(\pi^{-1}(A_\nu)) \to H_n(\pi^{-1}(B_\nu))$$

□

3.3.1 Computation of the E^1-term of the Spectral Sequence

Let $\pi : E \longrightarrow B$ be a Serre fibration over the *simplicial complex* B; let s_j^p, $j \in J_p$, be the closed p-simlpexes of B, ${e^p}_j$ the open p-simplex of s_j^p, m_j^p the baricenter and $\dot{s}_j^p$ the boundary of s_j^p. Consider the diagram
(3.3.8)

$$\begin{array}{ccc}
H_n(\pi^{-1}(B^p), \pi^{-1}(B^{p-1})) & \xleftarrow{(1)} & \oplus_{j\in J_p} H_n(\pi^{-1}(s_j^p), \pi^{-1}(\dot{s}_j^{p-1})) \\
{\scriptstyle (2)}\downarrow & & \downarrow{\scriptstyle (3)} \\
H_n(\pi^{-1}(B^p), \pi^{-1}(B^p - \cup\{m_j^p\})) & \xleftarrow{(4)} & \oplus H_n(\pi^{-1}(s_j^p), \pi^{-1}(s_j^p - m_j^p)) \\
{\scriptstyle (5)}\uparrow & & \uparrow{\scriptstyle (6)} \\
H_n(\cup\pi^{-1}(e_j^p), \cup\pi^{-1}(e_j^p - m_j^p)) & \xleftarrow[(7)]{} & H_n(\pi^{-1}(e_j^p), \pi^{-1}(e_j^p - m_j^p))\,.
\end{array}$$

All homomorphisms in this diagram are induced by inclusions. For instance, (7) is an isomorphism in singular homology. By excision, (5) and (6) are isomorphisms too, thus also (4) is an isomorphism. Finally, (2) and (3) are isomorphisms by 3.3.5; hence also (1) is an isomorphism.

3.3.9 REMARK. As in 3.3.4, we can identify $H_n(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$ with $E^1_{p,n-p}$. By isomorphism (1) in Diagram (3.3.1), we have already reduced the groups

$$H_n(\pi^{-1}(s_j^p), \pi^{-1}(\dot{s}_j^p))\,,$$

and now we are going to examine them.

For a p-simplex s^p of B, let $\partial_i s^p$ be the i-face, $\tau_i s^p = \bigcup_{j\neq i} \partial_j s^p$ the union of the remaining faces, and $\rho_i s^p = \partial_0 \cdots \widehat{\partial_i} \cdots \partial_p s^p$ [1] the ith vertex of s^p. (See Figure 3.1)

[1] $\widehat{\ }$ means that the corresponding symbol is omitted.

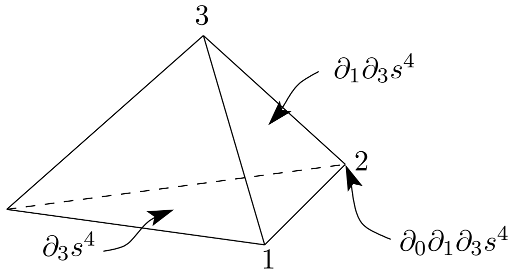


Figure 3.1

We adopt the abbreviation

$$h_n(A, A') = H_n(\pi^{-1}(A), \pi^{-1}(A'))$$

and consider the diagram

(3.3.9)
$$\begin{tikzcd} h_n(s^p, \dot{s}^p) \arrow[r, "\partial", "\cong"'] \arrow[d, dashed, "\alpha_i^p"'] & h_{n-1}(\dot{s}^p, \tau_i s^p) \\ h_{n-1}(\partial_i s^p, (\partial_i s^p)^{\cdot}) \arrow[r, "\cong", "(2)"'] & h_{n-1}(\dot{s}^p - \rho_i s^p, \tau_i s^p - \rho_i s^p) \arrow[u, "\cong"', "(1)"] \end{tikzcd}$$

The subspace $\tau_i s^p$ is a strong deformation retract of s^p, thus by 3.3.5,

$$h_n(s^p, \tau_i s^p) = 0\,,$$

and from the long homology exact sequence of the triple

$$(\pi^{-1}(s^p), \pi^{-1}(\dot{s^p}), \pi^{-1}(\tau_i s^p))$$

one obtains that ∂ in (3.3.9) is an isomorphism. (1) is an isomorphism by excision, and (2) is an isomorphism by 3.3.5. Define α_i^p by the commutativity of the diagram, and take

$$\widetilde{\beta}^p = \alpha_0^1 \cdots \alpha_0^{p-1}\alpha_0^p : h_n(s^p, \dot{s}^p) \xrightarrow{\cong} h_{n-p}(\rho_p s^p)\,.$$

Figures 3.2 and 3.3 show the geometry of these considerations.

3.3.2 Translation of the Homology of the Fiber

3.3.10 **Lemma**. *Let $\pi : E \longrightarrow B$ be a Serre fibration and $f : B' \longrightarrow B$ a continuous map. Then the induced fibration $f^*\pi$ is also a Serre fibration.*

The *proof* is an *exercise.* $\square$

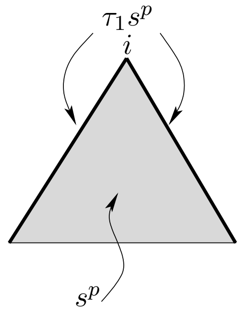


Figure 3.2

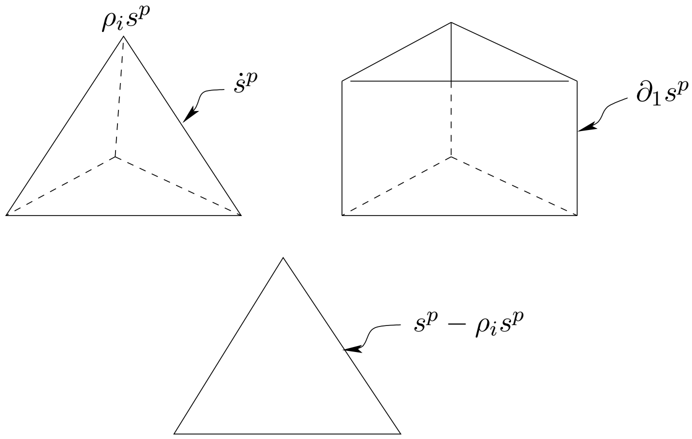


Figure 3.3

Let $\omega : I \longrightarrow B$ be a path. Consider the fibration $\widetilde{\pi}$ induced by π through ω.

$$\begin{tikzcd} E' \arrow[r, "\widetilde{\omega}"] \arrow[d, "\widetilde{\pi}"'] & E \arrow[d, "\pi"] \\ I \arrow[r, "\omega"'] & B\,. \end{tikzcd}$$

The map $\widetilde{\omega}$ induces a homeomorphism $\widetilde{\pi}^{-1}(t) \approx \pi^{-1}(\omega(t))$ with whose help we identify $H_n(\pi^{-1}(\omega(t)))$ and $H_n(\widetilde{\pi}^{-1}(t))$.

3.3.11 Definition. Let $\omega : I \longrightarrow B$ be a path. The *translation of the homology of the fiber* along ω is the homomorphism

$$\begin{aligned} \omega_\star : H_n(\pi^{-1}(\omega(0))) \;&\cong\; H_n(\pi^{-1}(0)) \xrightarrow{(1)} H_n(E') \longleftarrow \\ &\xleftarrow{(2)} H_n(\widetilde{\pi}^{-1}(1)) \;\cong\; H_n(\pi^{-1}(\omega(1)))\,. \end{aligned}$$

The homomorphisms (1) and (2) induced by the inclusion are isomorphisms by 3.3.10 and 3.3.5.

3.3.12 **Theorem**. *The translation of the homology of the fiber has the following properties:*

(a) $\omega_\star = \mathrm{id}$ *if* ω *is constant.*

(b) $(\omega_1\omega_2)_\star = \omega_{2\star} \circ \omega_{1\star}$.

(c) $\omega_0 \simeq \omega_1 \mathrm{rel}(\dot{I}) \Rightarrow \omega_{0\star} = \omega_{1\star}$.

In other words, the translation of the homology of the fiber is a functor from the fundamental grupoid of B into the category of abelian groups (and isomorphisms).

Proof: (a) In this case, (1) = (2).

(b) Let $\pi_{1\,2} : E_{1\,2} \longrightarrow I$ be the fibration induced through $\omega_1\omega_2$ and take the paths

$$\begin{aligned} \overline{h}_1 : I &\longrightarrow I \\ t &\longmapsto \left(\frac{1}{2}\right) t\,, \end{aligned} \qquad \begin{aligned} \overline{h}_2 : I &\longrightarrow I \\ t &\longmapsto \frac{1}{2} + \left(\frac{1}{2}\right) t\,, \end{aligned}$$

and the fibrations induced through them

$$\begin{tikzcd} E_i \arrow[r, "h_i"] \arrow[d, "\pi_i"'] & E_{1\,2} \arrow[d] \\ I \arrow[r, "\overline{h}_i"'] & I\,, \end{tikzcd}$$

then the fibration π_i is induced by π through $\omega_i = (\omega_1\omega_2)\overline{h}_i$. The following diagram commutes.

$$\begin{tikzcd} H_n(\pi^{-1}(\omega_1\omega_2(0))) \arrow[rrr] \arrow[rrrrrr, bend left, "(\omega_1\omega_2)_\star"] \arrow[d, equal] & & & H_n(E_{1\,2}) & & & H_n(\pi^{-1}(\omega_1\omega_2(1))) \arrow[lll] \arrow[d, equal] \\ H_n(\pi^{-1}(\omega_1(0))) \arrow[r] \arrow[drr, "\omega_{1\star}"'] & H_n(E_1) \arrow[urr, "h_{1\star}"] & & & & H_n(E_2) \arrow[ull, "h_{2\star}"'] & H_n(\pi^{-1}(\omega_2(1))) \arrow[l] \\ & & H_n(\pi^{-1}(\omega_1(1))) \arrow[ul] \arrow[rr, equal] & & H_n(\pi^{-1}(\omega_2(0))) \arrow[ur] \arrow[urr, "\omega_{2\star}"'] & & \end{tikzcd}$$

The homomorphisms $h_{1\star}$ y $h_{2\star}$ are bijective by 3.3.5, thus the diagram consists of nothing else but isomorphisms. Hence, we obtain the assertion.

(c) Let $h : I \times I \longrightarrow B$ be such that $h(s, 0) = \omega_0(s)$, $h(s, 1) = \omega_1(s)$, $h(0, t) = \omega_0(0) = \omega_1(0)$, $h(1, t) = \omega_0(1) = \omega_1(1)$. Let $\widetilde{\pi} : G \longrightarrow I \times I$

be induced by π through h, and let $k_0 : E_0 \longrightarrow G$ be given by $k_0(s,x) = ((s,0),x)$ (it lands in G!). From the commutative diagram

$$\begin{tikzcd} H_n(\pi^{-1}(\omega_0(0))) \arrow[r] \arrow[d, dashed] & H_n(E_0) \arrow[d, "k_{0*}", "\overset{3.3.6}{\cong}"'] & H_n(\pi^{-1}(\omega_0(1))) \arrow[l] \arrow[d, dashed] \\ H_n(\pi^{-1}(\{(0,0)\})) \arrow[r, dashed] & H_n(G) \arrow[r, dashed, leftrightarrow] & H_n(\pi^{-1}(\{(1,0)\})) \end{tikzcd}$$

that consists only of isomorphisms, one has that ω_{0*} can be defined through the dotted arrows. Analogously we may conclude for ω_1, and thus the homotopy between

$$\pi^{-1}(\omega_0(0)) \longrightarrow \pi^{-1}(0,0) \longrightarrow G \quad \text{and} \quad \pi^{-1}(\omega_1(0)) \longrightarrow \pi^{-1}(0,1) \longrightarrow G$$

grants us the assertion. $\square$

3.3.13 Definition. The fibration π is said to be *orientable* if $\omega_\star$ depends only on the end points of ω.

3.3.14 Exercise. Prove that if π fulfills the assumptions of 1.5.9 on the translation of the fiber, then $(\varphi_\omega)_* = \omega_\star$.

By the considerations made in 3.3.2, we may come back to the computation of the E^1-term of the spectral sequence.

Assume that B is path connected and take $b_0 \in B$ and $F = \pi^{-1}(b_0)$. For each point of the form $\underset{\sim}{\rho_p} s^p$ we choose a path ω from $\rho_p s^p$ to b_0 and together with the isomorphism $\widetilde{\beta}^p$ from 3.3.9, we obtain the isomorphism

$$\beta^p = \omega_\star \circ \alpha_0^1 \circ \ldots \circ \alpha_0^p : H_n(\pi^{-1}(s^p), \pi^{-1}(\dot{s}^p)) \longrightarrow H_{n-p}(F)\,. \tag{3.3.15}$$

Take $f_j^p : (\pi^{-1}(s_j^p), \pi^{-1}(\dot{s}_j^p)) \hookrightarrow (\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$. Together with 3.3.1, by defining

$$\gamma(\sum_j a_j s_j^p) = \sum_j (f_{j*}^p)(\beta_p)^{-1}(a_j)$$

we obtain an isomorphism

$$\gamma : C_p(B; H_{n-p}(F)) \longrightarrow H_n(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))\,. \tag{3.3.16}$$

We write an element of the group $C_p(B, H_{n-p}(F))$ of simplicial chains of B with coefficients in $H_{n-p}(F)$ as a linear combination $\sum_j a_j s_j^p$ (with a finite number of coefficients different from zero and $a_j \in H_{n-p}(F)$). In β^p we omit the index j. We have proved the following result.

3.3.17 **Theorem**. *For a Serre fibration $\pi : E \longrightarrow B$ on a path-connected simplicial complex B, one has*

$$E_{p,q}^1 = C_{p,q}^1 = H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1})) \cong C_p(B; H_q(F))\,,$$

where the isomorphism is given by γ. $\square$

3.3.3 Computation of the E^2-term of the Spectral Sequence

We define in $\{C_p(B; H_q(F)) \mid p \in \mathbb{Z}\}$ a boundary operator ∂ by $\gamma\partial = d^1\gamma$. We now want to determine ∂. First we decompose $h_n(B^p, B^{p-1})$ in a direct sum analogous to (3.3.1). In the diagram

$$\begin{tikzcd}
 & h_n(s_j^p, \dot{s}_j^p) \arrow[rr, "(1)"] \arrow[dl, "f_{j*}^p"'] \arrow[dr, "h_{j*}^p"] & & h_n(s_j^p, s_j^p - m_j^p) \arrow[dr, "(2)"] & \\
h_n(B^p, B^{p-1}) \arrow[rr, "g_{j*}^p"'] & & h_n(B^p, B^p - e_j^p) \arrow[rr, "(3)"'] & & h_n(B^p, B^p - m_j^p)
\end{tikzcd}$$

all homomorphisms are induced by inclusions. (1) and (3) are isomorphisms by 3.3.5, (2) is one by excision, and thus h_{j*}^p is also one, and we obtain

$$g_{k*}^p f_{j*}^p = \begin{cases} 0 & \text{if } k \neq j, \\ h_{j*}^p & \text{if } k = j, \end{cases}$$

(the second is an isomorphism). Thus the homomorphisms g_{k*}^p determine $h_n(B^p, B^{p-1})$ as a direct sum; that is, we can give (uniquely) an element in $h_n(B^p, B^{p-1})$ by its images under the homomorphisms g_{j*}^p.

Take $\partial(as_j^p) = \sum_{k \in J_{p-1}} a_k s_k^{p-1}$. We have to compute the coefficients a_k. From the definition of ∂ we have that

$$\sum_k f_{k*}^{p-1}(\beta^{p-1})^{-1}(a_k) = \gamma\partial(as_j^p)$$

and thus, applying g_{l*}^{p-1}, we have

$$\begin{aligned} h_{l*}^{p-1}(\beta^{p-1})\ (a_l) &= g_{l*}^{p-1}(\textstyle\sum f_{k*}^{p-1}(\beta^{p-1})^{-1}\ (a_k)) \\ &= g_{l*}^{p-1}\gamma\partial(as_j^p) \\ &= g_{l*}^{p-1} d^1 f_{j*}^p (\beta^p)^{-1}\ (a). \end{aligned} \tag{3.3.17}$$

The following is a commutative diagram.

$$\begin{tikzcd}
h_n(s_j^p, \dot{s}_j^p) \arrow[rrrr, "\alpha_i^p"] \arrow[dr, "\partial"] \arrow[ddd, "f_{j*}^p"'] & & & & h_{n-1}(s_l^{p-1}, \dot{s}_l^{p-1}) \arrow[dl, "(2)"] \arrow[ddd] \\
 & h_{n-1}(\dot{s}_j^p) \arrow[rr] \arrow[d] & & h_{n-1}(\dot{s}_j^p, \tau_i s_j^p) \arrow[ddr, "(1)"] & \\
 & h_{n-1}(B^{p-1}) \arrow[dr] & & & \\
h_n(B^p, B^{p-1}) \arrow[ur, "\partial"] \arrow[rr, "d^1"'] & & h_{n-1}(B^{p-1}, B^{p-2}) \arrow[rr, "g_{l*}^{p-1}"'] & & h_{n-1}(B^{p-1}, B^{p-1} - e_l^{p-1})\,,
\end{tikzcd}$$

where α_i^p is as in 3.3.9, and this, as well as (2), are considered only when $s_l^{p-1} = \partial_i s_j^p$. All other homomorphisms are induced by inclusions or are connecting homomorphisms of the corresponding homology sequence.

From this we have that, if s_l^{p-1} is not a face of s_j^p (see figure 3.4), then $\dot{s}_j^p$ lies in $B^{p-1} - e_l^{p-1}$; hence (1) is the trivial homomorphism and so $g_{l*}^{p-1} d^1 f_{j*}^p = 0$, and by the computation 3.3.17, we have $a_l = 0$, because h_{l*}^{p-1} as well as β^{p-1} are isomorphisms.

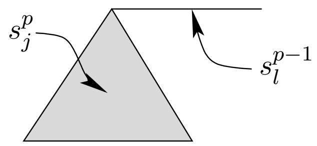


Figure 3.4

Let $s_l^{p-1} = \partial_i s_j^p$, be the i-face of s_j^p. Thus

$$\begin{aligned} h_{l*}^{p-1} \alpha_i^p (\beta^p)^{-1}\ (a) &= g_{l*}^{p-1} d^1 f_{j*}^p (\beta^p)^{-1}\ (a) \\ &= h_{l*}^{p-1} (\beta^{p-1})^{-1} (a_l); \end{aligned}$$

hence, $a_l = \beta_{p-1} \alpha_i^p (\beta_p)^{-1}\ (a) = \xi_i^p(a)$, and

$$\partial(as^p) = \sum_{i=0}^{p} \xi_i^p(a)(\partial_i s^p)\,,$$

where ξ_i^p is an automorphism of $H_q(F)$).

Now we determine the automorphism ξ_i^p, and for that we suppose again that the fibration $\pi : E \longrightarrow B$ is orientable (3.3.13).

First, by the definition of β_p (3.3.15), we have that $\beta^{p-1}\alpha_0^p = \beta^p$, that is,

$\xi_0^p = \mathrm{id}$. We shall analyze now $\xi_1^1 = \beta^0\alpha_1^1(\beta^1)^{-1}$ in the diagram

$$\begin{tikzcd}
 & h_q(\dot{s}^1, \tau_0 s^1) & h_q(\partial_0 s^1) \arrow[l] \arrow[r, "\omega_\star"] \arrow[dr, "f"] \arrow[dd, "s^1_*"'] & h_q(b_0) \\
h_{q+1}(s^1, \dot{s}^1) \arrow[ur, "\cong"] \arrow[dr] \arrow[urr, bend left, "\alpha_0^1"] \arrow[drr, bend right, "\alpha_1^1"'] \arrow[urrr, bend left=40, "\beta^1"] \arrow[drrr, bend right=40, "\beta^0"'] & & & h_q(s^1) \\
 & h_q(\dot{s}^1, \tau_1 s^1) & h_q(\partial_1 s^1) \arrow[l] \arrow[r, "\omega'_\star"'] \arrow[ur, "g"] & h_q(b_0)\,.
\end{tikzcd}$$

We have considered s^1 as a path from $\partial_0 s^1$ to $\partial_1 s^1$, and $s^1_\star$ is the translation along s^1 (see 3.3.2).

We shall prove that $f \circ \alpha_0^1 = -g \circ \alpha_1^1$. Namely, if z is a $(q+1)$-cycle of $\pi^{-1}s^1$ modulo $\pi^{-1}\dot{s}^1$, then ∂z decomposes as the sum $z_0 + z_1$ of two q-cycles such that the support of z_i lies in $\pi^{-1}\partial_i s^1$. $f\alpha_0^1(z)$ is thus represented by z_0 and $g\alpha_1^1(z)$ by z_1. Hence one has

$$(g\alpha_1^1 + f\alpha_0^1)[z] = [z_0 + z_1] = [\partial z] = 0\,.$$

And so,

$$\begin{aligned}
\xi_1^1 &= \beta^0\alpha_1^1(\beta^1)^{-1} \\
&= \beta^0\alpha_1^1(f\alpha_0^1)^{-1}f\omega_*^{-1} \\
&= -\beta^0\alpha_1^1(g\alpha_1^1)^{-1}f\omega_*^{-1} \\
&= -\beta^0 s_*^1\omega_*^1 \\
&= -\mathrm{id} \qquad\qquad \text{(by the orientability).}
\end{aligned}$$

Now, by induction on p, we prove

$$\xi_i^p = (-1)^i\mathrm{id}\,.$$

Before passing to the inductive proof we recall the following facts about boundary operators applied to s^p.

$$\begin{aligned} s^p &= [0, \ldots, p] \\ \partial_i s^p &= [0, \ldots, \widehat{i}, \ldots, p] \\ \partial_0 \partial_i s^p &= [1, \ldots, \widehat{i}, \ldots, p] \\ \partial_0 s^p &= [1, \ldots, p] \\ &= [0', \ldots, (p-1)'] \\ \partial_{i-1} \partial_0 s^p &= [1, \ldots, \widehat{i}, \ldots, p] \\ &= \partial_0 \partial_i s^p \end{aligned}$$

Now we pass to the proof. For $p = 1$ it has been already proved above. Take $p > 1$. ∂ is a boundary operator. Consider

$$0 = \partial\partial(as^p) = \sum_{i=0}^{p} \sum_{j=0}^{p-1} \xi_j^{p-1} \xi_i^p(a)(\partial_j \partial_i s^p)$$

The fact that $\xi_0^p = \mathrm{id}$ has been already proved. For $i \geq 1$ one has

$$\partial_0 \partial_1 s^p = \partial_{i-1} \partial_0 s^p \neq \partial_j \partial_k s^p \quad \text{if} \quad (j,k) \neq (0,i), (i-1,0)\,.$$

The double sum can be zero only if

$$\xi_i^p = \xi_0^{p-1} \xi_i^p = -\xi_{i-1}^{p-1} \xi_0^p = -\xi_{i-1}^{p-1}\,,$$

and by the induction hypothesis, we obtain from this the assertion. Thus we have proved that

$$\partial(as^p) = \sum_{i=0}^{p} (-1)^i a(\partial_i s^p)\,;$$

that is, that ∂ is the ordinary boundary homomorphism. In other words, this states that γ (3.3.16) is a chain isomorphism. Hence we have the following theorem, known as the Leray–Serre theorem.

3.3.18 **Theorem**. *Let $\pi : E \longrightarrow B$ be a homologically simple (orientable) Serre fibration over a* CW*-complex B. Then $(E^1_{*,q}, \overline{d}^1)$ is, through γ (3.3.16), isomorphic as a chain complex to $(C_*(B; H_q(F)), 0)$. Therefore, γ induces an isomorphism*

$$E^2_{pq} \cong H_p(B; H_q(F))\,.$$

□

The spectral sequence E^r_{pq} is known as the *Leray–Serre spectral sequence* of the Serre-fibration $\pi : E \longrightarrow B$.

3.3.4 Computation of the E^r-Terms for Large r

By the last theorem, we have that $E^1_{pq} \cong C_p(B; H_q(F)) = 0$ for $p < 0$ or $q < 0$. We say that the spectral sequence is *concentrated in the first quadrant*, and hence,

$$E^r_{p,q} = 0 \quad \text{if} \quad p < 0 \quad \text{or} \quad q < 0\,, \quad r \geq 1\,.$$

If we consider the differential

$$\overline{d}^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$$

the group on the right-hand side is zero for $r > p$ and thus the differential $\overline{d}^r$ is also zero. From the diagram that defines $\overline{d}^r$ in 3.2.17, we obtain that $\overline{d}^r = 0$ is equivalent to

$$\mathrm{Ker}(d^r)_{p,q} = \mathrm{Def}(d^r)_{p,q}\,.$$

Correspondingly, we have that for the differential

$$\overline{d}^r : E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q}$$

the group on the left-hand side is zero for $r > q+1$ and thus

$$\mathrm{Ind}(d^r)_{p,q} = \mathrm{Im}(d^r)_{p,q}\,.$$

From these two equalities, together the chains of inclusions in 3.2.17 and Definition 3.2.12, we have

$$\begin{aligned}
C_{p,q} &= Z^1_{p,q} \supset Z^2_{p,q} \supset \cdots \supset Z^{p+1}_{p,q} = Z^{p+2}_{p,q} = \cdots \\
0 &= B^1_{p,q} \subset B^2_{p,q} \subset \cdots \subset B^{q+2}_{p,q} = B^{q+3}_{p,q} = \cdots \\
Z^\infty_{p,q} &= \bigcap_r Z^r_{p,q} = Z^{p+1}_{p,q} = Z^r_{p,q} \quad \text{if} \quad r > p \\
B^\infty_{p,q} &= \bigcup_r B^r_{p,q} = B^{q+2}_{p,q} = B^r_{p,q} \quad \text{if} \quad r > q+1 \\
E^\infty_{p,q} &= Z^\infty_{p,q}/B^\infty_{p,q} = E^r_{p,q} \quad \text{if} \quad r > \max\{p, q+1\}\,.
\end{aligned}$$

We recall again that $H_n(E)$ is filtered by the groups

$$F_pH_n(E) = \mathrm{Im}(H_n(\pi^{-1}(B^p)) \longrightarrow H_n(E))$$

(see 3.3.1). We thus have

$$0 = F_{-1}H_n(E) \subset F_0H_n(E) \subset \cdots \subset F_nH_n(E) = H_n(E)\,,$$

where

$$F_pH_n(E)/F_{p-1}H_n(E) = E^\infty_{p,n-p} = E^r_{p,n-p}$$

for r large enough. In particular, we have $E^r_{p,n-p} = 0$ for $p > n$, so that, indeed, one has the mentioned filtration as indicated.

3.4 APPLICATIONS

3.4.1 GENERAL ASSUMPTIONS. In this section we shall always assume that $\pi : E \longrightarrow B$ is a homologically simple (orientable) Serre fibration over a path-connected CW-complex B.

We shall apply the results of Section 3.3.

3.4.1 Spherical Fibrations

We analyze here Serre fibrations with a sphere as fiber.

Assume that $\pi : E \longrightarrow B$ satisfies the general assumptions 3.4.1, and that

$$F = \pi^{-1}(b_0) \approx \mathbb{S}^{m-1}\,, \quad m \geq 2\,, \quad (b_0 \in B)\,.$$

Then

$$E^2_{p,q} \cong H_p(B; H_q(F)) = 0 \quad \text{if} \quad q \neq 0, m-1\,.$$

Hence, for these values of q, $E^r_{p,q} = 0$, $r \geq 2$, and moreover,

$$E^2_{p,0} = E^2_{p,m-1} = H_p(B; G)$$

if $G(\cong H_0(\mathbb{S}^{m-1}) = H_{m-1}(\mathbb{S}^{m-1}))$ is the coefficient group of the homology. We have that

$$\overline{d}^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$$

is nonzero, at most in case that

$$\overline{d}^m : E^m_{p,0} \longrightarrow E^m_{p-m,m-1}\,;$$

otherwise, the domain or codomain would be the trivial groups. Thus we have

$$E^2 = E^3 = \cdots = E^m\,,$$
$$E^{m+1} = E^{m+2} = \cdots = E^\infty\,.$$

From 3.2.18, we have the following exact sequence

$$\begin{array}{ccccccccc}
0 \longrightarrow H_{p,0}(E^m, \overline{d}^m) & \longrightarrow & E^m_{p,0} & \xrightarrow{\overline{d}^m} & E^m_{p-m,m-1} & \longrightarrow & H_{p-m,m-1}(E^m, \overline{d}^m) \longrightarrow 0 \\
\| & & \| & & \| & & \| \\
E^{m+1}_{p,0} & & E^2_{p,0} & & E^2_{p-m,m-1} & & E^{m+1}_{p-m,m-1} \\
\| & & \| & & \| & & \| \\
E^\infty_{p,0} & & H_p(B;G) & & H_{p-m}(B;G) & & E^\infty_{p-m,m-1}\,.
\end{array}$$

If we now consider the filtered homology of the total space E (see Subsection **3.3.4**)

$$0 \subset F_0H_n(E) \subset F_1H_n(E) \subset \cdots \subset F_nH_n(E) = H_n(E)\,,$$

and since

$$F_pH_n(E)/F_{p-1}H_n(E) = E^\infty_{p,n-p} = 0 \quad \text{if} \quad n-p \neq 0, m-1\,,$$

i.e., $p \neq n, n-m+1$, this filtration looks as follows:

$$0 = \cdots = F_{n-m}H_n(E) \subset F_{n-m+1}H_n(E) = \cdots = F_{n-1}H_n(E) \subset H_n(E)\,.$$

Thus we have the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & F_{n-1}(H_n(E)) \arrow[r] \arrow[d, equal] & H_n(E) \arrow[r] & H_n(E)/F_{n-1}H_n(E) \arrow[r] \arrow[d, equal] & 0 \\
& F_{n-m+1}H_n(E)/F_{n-m}H_n(E) \arrow[d, equal] & & E^\infty_{n,0} & \\
& E^\infty_{n-m+1,m-1}\,. & & &
\end{tikzcd}$$

Glueing together both exact sequences, we obtain

$$\begin{tikzcd}
\cdots \arrow[rr] & & H_p(E) \arrow[rr, dashed] \arrow[dr, rightsquigarrow] & & H_p(B) \arrow[r, dotted] & H_{p-m}(B) \arrow[rr, dashed] \arrow[dr, dotted] & & H_{p-1}(E) \arrow[r] & \cdots \\
& E^\infty_{p-m+1,m-1} \arrow[ur, rightsquigarrow] & & E^\infty_{p,0} \arrow[ur, dotted] \arrow[dr, rightsquigarrow] & & & E^\infty_{p-m,m-1} \arrow[ur, rightsquigarrow] \arrow[dr, dotted] & & \\
0 \arrow[ur, rightsquigarrow] & & 0 \arrow[ur, dotted] & & 0 & 0 \arrow[ur, rightsquigarrow] & & 0\,. &
\end{tikzcd}$$

The arrows $\dashrightarrow$ are so that the triangles commute; the arrows $\rightsquigarrow$ indicate the first of the exact sequences and the arrows $\cdots\!\!\rightarrow$ indicate the second. Then, it is an easy matter to check the exactness of the top horizontal sequence.

3.4.2 **Theorem**. *Under the general assumptions 3.4.1 on $\pi : E \longrightarrow B$, there is an exact sequence*

$$\cdots \longrightarrow H_p(E) \xrightarrow{\pi_*} H_p(B) \longrightarrow H_{p-m}(E) \longrightarrow H_{p-1}(E) \xrightarrow{\pi_*} \cdots\,,$$

that is known as the Gysin sequence.

Proof: After all done in 3.4.1, it is enough to check that the homomorphism

$$H_p(E) \longrightarrow E^\infty_{p,0} \longrightarrow H_p(B)$$

is indeed induced by π.

Let us consider the commutative square

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d, "\pi"'] & B \arrow[d, "\mathrm{id}"] \\ B \arrow[r, "\mathrm{id}"'] & B \end{tikzcd}$$

as a fiber map from π to id_B. If we denote with a tilde the spectral sequence associated to the fibration id_B (cf. 3.3.4), by the naturality of the spectral sequence, we have the following commutative diagram:

$$\begin{tikzcd} H_p(E) \arrow[r] \arrow[d, "\cong"'] & E^\infty_{p,0} \arrow[r, hook] \arrow[d] & E^2_{p,0} \arrow[r, "\cong"] \arrow[d] & H_p(B; H_0(F)) \arrow[d, "\pi_0^*"] \\ H_p(B) \arrow[r] & \widetilde{E}^\infty_{p,0} \arrow[r, equal] & \widetilde{E}^2_{p,0} \arrow[r, "\cong"] & H_p(B; G)\,, \end{tikzcd}$$

where $(\pi_0)_*$ is induced by the homomorphism $H_0(F) \longrightarrow H_0(b_0) = G$ an can be considered as the identity, since $F \approx \mathbb{S}^{m-1}$ is connected. Therefore, it remains to convince oneself that bottom line in the diagram is the identity of $H_p(B) = H_p(B; G)$, which follows immediately from the definitions. $\square$

We now consider the special case $E \approx \mathbb{S}^{l-1}$. We assume moreover that $H_p(B) = 0$ for $p > r > 0$ and $H_r(B) \neq 0$. From the exactness of

$$0 = H_{r+m}(B) \longrightarrow H_r(B) \xrightarrow{\cong} H_{r+m-1}(E) \longrightarrow H_{r+m-1}(B) = 0$$

it follows that the homomorphism in the middle is an isomorphism; but from $H_r(B) \neq 0$ and $E \approx \mathbb{S}^{l-1}$ one has that $r + m = l$. From the Gysin sequence for $1 < p < l - 1$ one obtains also that

$$H_p(B) \cong H_{p-m}(B)\,.$$

If $p-m > 1$ we may continue lowering the dimensions. There are two possible cases; namely, if m divides p we finish with $H_0(B) \cong G$.

On the other hand, we reach $H_q(B)$ with $0 < q < m$. From the exactness of

$$0 = H_q(E) \longrightarrow H_q(B) \longrightarrow H_{q-m}(B) = 0$$

one obtains that $H_q(B) = 0$. (Observe that $q < m = l - r \leq l - 1$ if $r \geq 1$; for $r = 0$ we have in any case that $H_q(B) = 0$ for $q > 0$). Thus we have the following.

3.4.3 **Theorem**. *Let $\pi : \mathbb{S}^{l-1} \longrightarrow B$ be a fibration that satisfies the general assumptions 3.4.1, with fiber $\mathbb{S}^{m-1}$ $(m \geq 2)$, and $H_p(B) = 0$ if p is large enough (for instance, if B is a finite-dimensional* CW*-complex). Then $l = (s+1)m$ for some $s \in \mathbb{Z}$ and*

$$H_p(B) = \begin{cases} G & \text{if } p = 0, m, 2m, \ldots, sm, \\ 0 & \text{otherwise.} \end{cases}$$

□

As a special case we we can compute the homology of the complex and quaternionic projective spaces. Namely, since we have fibrations

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \longrightarrow \mathbb{CP}^n\,,$$

$$\mathbb{S}^3 \hookrightarrow \mathbb{S}^{4n+3} \longrightarrow \mathbb{HP}^n\,,$$

we conclude the following.

3.4.4 **Corollary**.

$$H_p(\mathbb{CP}^n) = \begin{cases} G & \text{if } p = 0, 2, 4, \ldots, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

$$H_p(\mathbb{HP}^n) = \begin{cases} G & \text{if } p = 0, 4, 8, \ldots, 4n, \\ 0 & \text{otherwise.} \end{cases}$$

□

3.4.5 REMARK. For $\mathbb{RP}^n$ the problem is that the corresponding fibration

$$\mathbb{S}^0 \hookrightarrow \mathbb{S}^n \longrightarrow \mathbb{RP}^n$$

has disconnected fiber. See [1] for the corresponding computation.

3.4.2 Fibrations with Spherical Base Space

We shall now study fibrations of the form $\pi : E \longrightarrow \mathbb{S}^m$, $m \geq 2$.

One has

$$E^2_{p,q} \cong H_p(\mathbb{S}^m; H_q(F)) \cong \begin{cases} H_q(F) & \text{if } p = 0, m, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, again all terms $E^r_{p,q} = 0$ for $p \neq 0$, m and $r \geq 2$ and the differentials can only be nonzero in the case

$$\overline{d}^m : E^m_{m,q} \longrightarrow E^m_{0,q+m-1}\,,$$

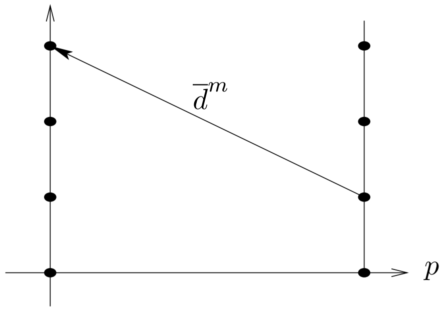


Figure 3.5

as shown in Figure 3.5.

So again we have

$$\begin{aligned} E^2 &= E^3 = \cdots = E^m \\ E^{m+1} &= E^{m+2} = \cdots = E^\infty \,. \end{aligned}$$

Moreover, the following sequence is exact:

$$\begin{array}{ccccccccc} 0 \longrightarrow & H_{p,q}(E^m, \overline{d}^m) & \longrightarrow & E^m_{m,q} & \xrightarrow{\overline{d}^m} & E^m_{0,q+m-1} & \longrightarrow & H_{0,q+m-1}(E^m, \overline{d}^m) & \longrightarrow 0 \\ & \| & & \| & & \| & & \| & \\ & E^{m+1}_{m,q} & & E^2_{m,q} & & E^2_{0,q+m-1} & & E^{m+1}_{0,q+m-1} & \\ & \| & & \| & & \| & & \| & \\ & E^\infty_{m,q} & & H_q(F) & & H_{q+m-1}(F) & & E^\infty_{0,q+m-1} \,. & \end{array}$$

On the other hand, from $F_pH_n(E)/F_{p-1}H_n(E) \cong E^\infty_{p,n-p} = 0$ for $p \neq 0, m$ one has that the filtration "collapses" as follows:

$$0 \subset F_0H_n(E) = F_{m-1}H_n(E) \subset F_mH_n(E) = \cdots = F_nH_n(E) = H_n(E)\,,$$

and from there, we obtain the exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & F_{m-1}(H_n(E)) & \longrightarrow & H_n(E) & \longrightarrow & H_n(E)/F_{m-1}H_n(E) & \longrightarrow 0 \\ & \| & & & & \| & \\ & E^\infty_{0,n} & & & & E^\infty_{m,n-m} & \end{array}$$

Analogously to 3.4.1 we glue both sequences together to obtain

$$\begin{tikzcd} \cdots \arrow[r] & H_{q+m}(E) \arrow[rr, dashed] \arrow[dr, squiggly] & & H_q(F) \arrow[r, dotted, "\overline{d}^m"] & H_{q+m-1}(F) \arrow[rr, dashed] \arrow[dr, dotted] & & H_{q+m-1}(E) \arrow[r] & \cdots \\ & & E^\infty_{m,q} \arrow[ur, dotted] \arrow[dr, squiggly] & & & E^\infty_{0,q+m-1} \arrow[ur, squiggly] \arrow[dr, dotted] & & \\ & 0 \arrow[ur, dotted] & & 0 & 0 \arrow[ur, squiggly] & & 0\,. & \end{tikzcd}$$

3.4.6 **Theorem**. *Under the general assumptions 3.4.1 on $\pi : E \longrightarrow \mathbb{S}^m$, there is an exact sequence*

$$\cdots \longrightarrow H_{q+m}(E) \longrightarrow H_q(F) \longrightarrow H_{q+m-1}(F) \xrightarrow{i^*} H_{q+m-1}(E) \longrightarrow \cdots,$$

where $i : F \hookrightarrow E$ is the inclusion of the fiber in the total space. This sequence is known as the Wang sequence*.*

Proof: After all done above, it is enough to check that the homomorphism

$$H_{q+m-1}(F) \longrightarrow E^\infty_{0,q+m-1} \longrightarrow H_{q+m-1}(E)$$

is indeed induced by i. Let $\{b_0\} \subset B$ be a 0-simplex and consider

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ \{b_0\} \arrow[r, hook] & B \end{tikzcd}$$

as a fiber map from π' to π. If we denote with a tilde the spectral sequence of π' we have, by the naturality, the commutative diagram

$$\begin{tikzcd} H_r(F) \arrow[r, "\cong"] \arrow[d, equal] & H_0(\{b_0\}; H_r(F)) \arrow[r, "\cong"] & \widetilde{E}^2_{0,r} \arrow[r, equal] \arrow[d] & \widetilde{E}^\infty_{0,r} \arrow[r] \arrow[d] & H_r(F) \arrow[d, "i_*"] \\ H_r(F) \arrow[r] \arrow[d, equal] & H_0(Y; H_r(F)) \arrow[r, "\cong"] & E^2_{0,r} \arrow[r] \arrow[d, equal] & E^\infty_{0,r} \arrow[r] \arrow[d, equal] & H_r(E) \arrow[d, equal] \\ H_r(F) \arrow[rr] & & E^m_{0,r} \arrow[r] & E^{m+1}_{0,r} \arrow[r] & H_r(E). \end{tikzcd}$$

From it, it is easy to convince oneself that the top row yields the identity. □

3.4.7 EXAMPLE. Let E be the path space in $\mathbb{S}^m$ that start in $b_0 \in \mathbb{S}^m$. $\pi : E \longrightarrow \mathbb{S}^m$ maps each path to its end point. This is the so-called *path fibration* and can be proved to be a Hurewicz fibration with fiber $\pi^{-1}(b_0) = \Omega\mathbb{S}^m$, the *loop space* of $\mathbb{S}^m$ (cf. 1.4.18 or [1, 3.3.17] and see Figure 3.6).

It is easy to prove that E is contractible. Thus $H_n(E) = 0$ for $n > 0$. The Wang sequence yields (for $m \geq 2$)

$$H_q(\Omega\mathbb{S}^m; G) = \begin{cases} G & \text{if } q \sim 0 \mod (m-1), \\ 0 & \text{otherwise.} \end{cases}$$

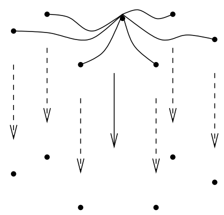


Figure 3.6

3.4.8 EXAMPLE. Let $\pi : \mathbb{S}^m \longrightarrow \mathbb{S}^k$ be a homologically trivial Serre fibration with fiber $\mathbb{S}^n$, $k \geq 1$, $n \geq 1$. The Gysin sequence with coefficients in $\mathbb{Z}$ and $p = k + n + 1$ looks as follows

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_{k+n+1}(\mathbb{S}^k) & \longrightarrow & H_k(\mathbb{S}^k) & \longrightarrow & H_{k+n}(\mathbb{S}^m) & \longrightarrow \cdots\,, \\ & \| & & & & & \\ & 0 & & & & & \end{array}$$

thus $H_{k+n}(\mathbb{S}^m) \neq 0$; therefore,

$$k + n = m\,. \tag{3.4.9}$$

On the other hand, the Wang sequence for $q = n - k + 1$ looks as follows

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_{n-k+1}(\mathbb{S}^n) & \longrightarrow & H_n(\mathbb{S}^n) & \longrightarrow & H_n(\mathbb{S}^m) & \longrightarrow \cdots\,. \\ & & & & & \| & \\ & & & & & 0 & \end{array}$$

Thus $H_{n-k+1}(\mathbb{S}^n) \neq 0$; therefore,

$$n - k + 1 = 0\,; \tag{3.4.10}$$

from (3.4.9) and (3.4.10) it follows that $n = k - 1$ and $m = 2k - 1$.

For $k = 2, 4, 8$ we have fibrations

$$\begin{aligned} \mathbb{S}^1 &\longrightarrow \mathbb{S}^3 \longrightarrow \mathbb{S}^2\,, \\ \mathbb{S}^3 &\longrightarrow \mathbb{S}^7 \longrightarrow \mathbb{S}^4\,, \\ \mathbb{S}^7 &\longrightarrow \mathbb{S}^{15} \longrightarrow \mathbb{S}^8\,, \end{aligned}$$

known as *Hopf fibrations* (see [5, 6]).

3.4.3 Fibrations in Small Dimensions

3.4.11 General Assumptions. Besides the general assumptions of 3.4.1 (π is a homologically simple Serre fibration over a CW-complex) we shall assume that $\pi : E \longrightarrow B$ satisfies

$$\begin{aligned} H_p(B;\mathbb{Z}) &= 0 \quad \text{if } 0 < p < r, \\ H_q(F;\mathbb{Z}) &= 0 \quad \text{if } 0 < q < s. \end{aligned}$$

By the universal coefficients formula (see [1, 7.4.8]) we have

$$\begin{aligned} E^2_{p,q} &\cong H_p(B; H_q(F;G)) \\ &\cong H_p(B;\mathbb{Z}) \otimes H_q(F;G) \oplus \mathrm{Tor}(H_{p-1}(B;\mathbb{Z}), H_q(F,G)) \\ &= 0 \quad \text{if} \quad 0 < p < r \quad \text{or} \quad 0 < q < s\,. \end{aligned}$$

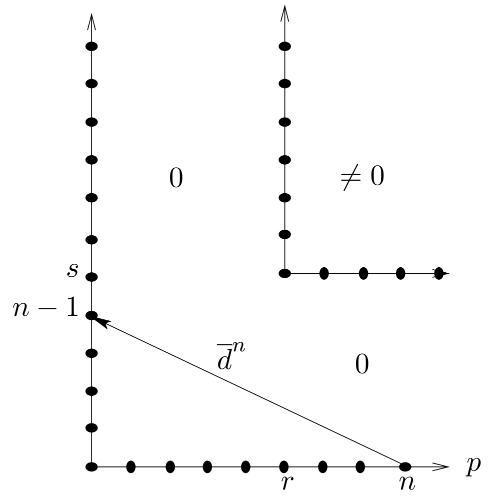


Figure 3.7

Thus the nonzero terms of the spectral sequence are distributed according to Figure 3.7. Again, we omit writing the coefficients. For the elements of the term $E^r_{p,q}$, we call $p+q$ their *total degree*. In what follows, we shall consider only elements of total degree $n < r+s$. Thus, a differential $\overline{d}^k$ for $k \geq 2$ will be nonzero at most in the case

$$\overline{d}^n : E^n_{n,0} \longrightarrow E^n_{0,n-1}\,.$$

We have

$$\begin{aligned} E^2_{n,0} &= E^3_{n,0} = \cdots = E^n_{n,0} \\ E^{n+1}_{n,0} &= E^{n+2}_{n,0} = \cdots = E^\infty_{n,0} \\ E^2_{0,n-1} &= \cdots = E^n_{0,n-1} \\ E^{n+1}_{0,n-1} &= \cdots = E^\infty_{0,n-1}\,, \end{aligned}$$

and analogously to the previous subsections, we obtain the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & E^{n+1}_{n,0} \arrow[r] \arrow[d, equal] & E^n_{n,0} \arrow[r, "\overline{d}^n"] \arrow[d, equal] & E^n_{0,n-1} \arrow[r] \arrow[d, equal] & E^{n+1}_{0,n-1} \arrow[r] \arrow[d, equal] & 0\,. \\ & E^\infty_{n,0} & E^2_{n,0} \arrow[d, equal] & E^2_{0,n-1} \arrow[d, equal] & E^\infty_{0,n-1} & \\ & & H_n(B) & H_{n-1}(F) & & \end{tikzcd}$$

On the other hand, $F_pH_n(E)/F_{p-1}H_n(E) \cong E^\infty_{p,n-p} = 0$ for $p \neq 0, n$. So one has

$$0 \subset F_0H_n(E) = \cdots = F_{n-1}H_n(E) \subset H_n(E)$$

and the exact sequence

$$0 \longrightarrow E^\infty_{0,n} \longrightarrow H_n(E) \longrightarrow E^\infty_{n,0} \longrightarrow 0\,.$$

Overlapping the exact sequences, as above, we obtain

3.4.12 **Theorem**. *For $n < r + s$ there is an exact sequence*

$$\cdots \longrightarrow H_n(F) \xrightarrow{i_*} H_n(E) \xrightarrow{\pi_*} H_n(B) \xrightarrow{\tau} H_{n-1}(F) \longrightarrow \cdots\,.$$

The fact that $H_n(F) \longrightarrow H_n(E)$ and $H_n(E) \longrightarrow H_n(B)$ are induced by i and π, respectively, can be proved in an analogous form to the previous subsections. □

The homomorphism τ is called the *transgression* and has a geometric interpretation (see, for instance, [**?**, 10.6]).

3.4.13 Remark. We saw in the first chapter that a Serre fibration yields an exact sequence of homotopy sets. This last theorem shows that, at least for some dimensions, one also has an exact sequence in homology. For the Hopf fibration $\mathbb{S}^1 \xrightarrow{i} \mathbb{S}^3 \xrightarrow{\pi} \mathbb{S}^2$, the sequence

$$H_3(\mathbb{S}^1) \longrightarrow H_3(\mathbb{S}^3) \longrightarrow H_3(\mathbb{S}^2)$$

is **not** exact. This shows that the inequality $n < r + s$ cannot be improved in general.

Chapter 4

Generalized Cohomology of Fibrations

In this chapter, we present different spectral sequences, according to the type of fibration we are dealing with (Leray-Serre-Whitehead; Rothenberg-Steenrod).

4.1 Introduction

In this section, we introduce the concept of a generalized cohomology theory and the properties that will be relevant for the spectral sequences that we construct. Then we introduce the concept of a system of local coefficients for ordinary cohomology.

4.1.1 Generalized Cohomology Theories

4.1.1 Definition. Let $\mathcal{T}op_2$ be some category of pairs (X,Y) of topological spaces and maps of pairs. Let, moreover, $\mathcal{A}b$ be the category of abelian groups and homomorphisms. A *cohomology theory* h^* on $\mathcal{T}op_2$ is a collection of contravariant functors and natural transformations indexed by $q \in \mathbb{Z}$,

$$h^q : \mathcal{T}op_2 \longrightarrow \mathcal{A}b \quad \text{and} \quad \delta^q : h^q \circ R \stackrel{\cdot}{\longrightarrow} h^{q+1},$$

these last called *connecting homomorphisms*, where $R : \mathcal{T}op_2 \longrightarrow \mathcal{T}op_2$ is the functor that sends a pair (X,Y) to the pair $(Y,\emptyset)$ and the map of pairs $f : (X',Y') \longrightarrow (X,Y)$ to $f|_{Y'}$, satisfying the following axioms:

Homotopy. If $f_0 \simeq f_1 : (X',Y') \longrightarrow (X,Y)$ is a homotopy of pairs, then

$$f_0^* = f_1^* : h^q(X,Y) \longrightarrow h^q(X',Y')$$

for all $q \in \mathbb{Z}$.

Excision. For every pair of spaces (X, Y) and a subset $U \subset Y$ satisfying $\overline{U} \subset \mathring{A}$, the inclusion $j : (X - U, Y - U) \longrightarrow (X, Y)$ induces an isomorphism

$$h^q(X, Y) \cong h^q(X - U, Y - U)$$

for all $q \in \mathbb{Z}$.

Exactness. For every pair of spaces (X, A) we have a long exact sequence

$$\cdots \xrightarrow{\delta^{q-1}} h^q(X, Y) \xrightarrow{i^*} h^q(X) \xrightarrow{i^*} h^q(Y) \xrightarrow{\delta^q} h^{q+1}(X, Y) \longrightarrow \cdots ,$$

where $i : (X, \emptyset) \hookrightarrow (X, Y)$ and $j : (Y, \emptyset) \hookrightarrow (X, \emptyset)$ are the inclusions, and we write $h^q(X)$ instead of $h^q(X, \emptyset)$.

4.1.2 Examples.

(a) The singular cohomology functors with coefficients in G, $(X, Y) \mapsto H^q(X, Y; G)$ constitute a cohomology theory for every abelian group G in the category $\mathcal{T}op_2$ of all pairs of spaces. (Here, $H^q(X, Y; G) = 0$ if $q < 0$.)

(b) The K-theory functors $(X, Y) \mapsto K^q(X, Y)$ form a cohomology theory in the category of pairs of paracompact spaces and closed subspaces. (See [1, 9.5.9, (9.5.8), and 9.5.10].)

4.1.3 Remark. There is also the dual concept of a *homology theory* h_* on $\mathcal{T}op_2$, which is a collection of covariant functors and natural transformations indexed by $q \in \mathbb{Z}$,

$$h_q : \mathcal{T}op_2 \longrightarrow \mathcal{A}b \quad \text{and} \quad \partial_q : h_q \stackrel{\cdot}{\longrightarrow} h_{q-1} \circ R ,$$

these last called *connecting homomorphisms*, where as before, $R : \mathcal{T}op_2 \longrightarrow \mathcal{T}op_2$ maps a pair of spaces to the second space of the pair, and they satisfy the same axioms as the cohomology with the obvious modifications.

Some examples we have of this are the ordinary homology groups with coefficients in an abelian group G as introduced in Section **??**, and given by $(X, A) \mapsto H_q(X, A; G)$.

We shall sometimes require two further axioms for a generalized cohomology theory h^*.

Weak homotopy equivalence. Given a weak homotopy equivalence $f : (X', Y') \longrightarrow (X, Y)$ (see [1, 5.1.17]), then $f^* : h^q(X, Y) \longrightarrow h^q(X', Y')$ is an isomorphism for all $q \in \mathbb{Z}$.

Additivity. For every collection $\{(X_\lambda, Y_\lambda)\}_{\lambda\in\Lambda}$ of pairs of topological spaces, the inclusions $i_\lambda : (X_\lambda, Y_\lambda) \hookrightarrow \coprod_{\mu\in\Lambda}(X_\mu, Y_\mu)$ induce an isomorphism

$$(i_\lambda^*) : h^q\left(\coprod_\lambda (X_\lambda, Y_\lambda)\right) \longrightarrow \prod_{\lambda\in\Lambda} h^q(X_\lambda, Y_\lambda)\,.$$

In what follows we analyze a very interesting example of how, given a generalized cohomology theory, one can produce a new cohomology theory associated to a given Hurewicz fibration.

4.1.4 DEFINITION. Let $\pi : E \longrightarrow B$ be a (fixed) Hurewicz fibration and $A \subset B$. For any map of pairs $(X, Y) \longrightarrow (B, A)$, let (E_X, E_Y) be the pair such that $E_X \longrightarrow X$ and $E_Y \longrightarrow Y$ are the fibrations induced over X and Y, respectively, through the given map (no confusion should arise if a different map of pairs is taken, since as a "pair" over B it is different and thus should be denoted differently). Set

$$\bar{h}^*(X, Y) = h^*(E_X, E_Y)\,.$$

4.1.5 **Theorem**. *$\bar{h}^*$ is a cohomology theory on the category $\mathcal{T}op_{B2}$ of pairs of spaces over B and maps over B.*

Proof: We check first that $\bar{h}^*$ is a functor, and hence we have to see how it applies to maps. Let $f : (X', Y') \longrightarrow (X, Y)$ be a map over B, namely, such that the triangle

$$\begin{tikzcd} X' \arrow[rr, "f"] \arrow[dr] & & X \arrow[dl] \\ & B & \end{tikzcd}$$

commutes. Then f induces a map of pairs $\widehat{f} : (E_{X'}, E_{Y'}) \longrightarrow (E_X, E_Y)$ such that the square

$$\begin{tikzcd} (E_{X'}, E_{Y'}) \arrow[r, "\widehat{f}"] \arrow[d] & (E_X, E_Y) \arrow[d] \\ (X', Y') \arrow[r, "f"'] & (X, Y) \end{tikzcd}$$

commutes, and is given by $\widehat{f}(x',e) = (f(x'),e)$, where $(x',e) \in E_{X'} \subset X' \times E$.

Define the homomomorphism $\bar{f}^* : \bar{h}^*(X,Y) \longrightarrow \bar{h}^*(X',Y')$ induced by f by

$$\widehat{f}^* : h^*(E_X, E_Y) \longrightarrow h^*(E_{X'}, E_{Y'})\,.$$

One easily verifies that this is a functorial construction.

We prove now that the functors $\bar{h}^*$ fulfill the axioms of a cohomology theory.

Homotopy. If $H : f_0 \simeq f_1 : (X',Y') \longrightarrow (X,Y)$ is a homotopy over B, then $\bar{f}_0^* = \bar{f}_1^*$.

Namely, consider the diagram, where for simplicity we omit writing the second member of each pair of spaces

$$\begin{tikzcd}
 & & E_X \arrow[dr] \arrow[dd] & \\
E_{X'} \times I = E_{X' \times I} \arrow[urr, dashed, "\bar{H}"] \arrow[rrr] \arrow[d] & & & E \arrow[dd] \\
 & & X \arrow[dr] & \\
X' \times I \arrow[urr, "H"] \arrow[rrr] & & & B\,,
\end{tikzcd}$$

where $\bar{H}(e,x',t) = (e, H(x',t))$. Then

$$\begin{aligned}
\bar{H}(e,x',0) &= (e, f_0(x')) = \bar{f}_0(x') \\
\bar{H}(e,x',1) &= (e, f_1(x')) = \bar{f}_1(x')
\end{aligned}$$

This proves the homotopy. (Under the assumption that $\pi : E \longrightarrow B$ is a Hurewicz fibration, one may assume that H is any homotopy and not only a homotopy over B.)

Exactness. Given a pair of spaces (X,Y), there is a long exact sequence

$$\cdots \longrightarrow \bar{h}^q(X,Y) \longrightarrow \bar{h}^q(X) \longrightarrow \bar{h}^q(Y) \xrightarrow{\bar{\delta}} \bar{h}^{q+1}(X,Y) \longrightarrow \cdots\,.$$

Namely, the given sequence is in fact the following:

$$\cdots \longrightarrow h^q(E_X,E_Y) \longrightarrow h^q(E_X) \longrightarrow h^q(E_Y) \xrightarrow{\delta} h^{q+1}(E_X,E_Y) \longrightarrow \cdots\,,$$

which is obviously exact. Note that, in particular, this provides the definition of $\bar{\delta}$.

Excision. If $\overline{U} \subset \overset{\circ}{Y}$, then the inclusion induces an isomorphism

$$\bar{h}^q(X,Y) \cong h^q(X-U, Y-U)$$

for all q.

Namely, $E_U \subset E_Y$ and clearly

$$\overline{E_U} \subset E_{\overline{U}} \subset E_{\overset{\circ}{Y}} \subset \overset{\circ}{E_Y} \;.$$

Thus the assertion follows from the excision axiom for h^*. □

4.1.6 REMARK. If h^* satisfies the additivity axiom, then also $\bar{h}^*$.

Namely, If $(X,Y) = \coprod_\lambda (X_\lambda, Y_\lambda) \longrightarrow B$, then

$$(E_X, E_Y) = \coprod_\lambda (X_\lambda, Y_\lambda) \,,$$

thus the axiom for $\bar{h}^*$ follows from the corresponding one for h^*.

The next is a useful result.

4.1.7 **Lemma**. *Let $\pi : E \longrightarrow B$ be a fibration and $f : X \longrightarrow B$, $g : Y \longrightarrow B$ be spaces over B. Let moreover $X' \subset X$, $Y' \subset Y$ and $\varphi : (X, X') \longrightarrow (Y, Y')$ be a map over B that is also a relative homeomorphism, that is, it is a map of pairs such that $\varphi|_{X-X'} : X - X' \longrightarrow Y - Y'$ is a homeomorphism. Then the induced map $\widetilde{\varphi} : (E_X, E_{X'}) \longrightarrow (E_Y, E_{Y'})$ is also a relative homeomorphism.*

Proof: Recall that

$$E_X = \{(x, e) \mid f(x) = \pi(e)\} \quad \text{and} \quad E_Y = \{(y, e) \mid g(x) = \pi(e)\} \,.$$

Then $\widetilde{\varphi}(x, e) = (\varphi(x), e)$. If $\psi : Y - Y' \longrightarrow X - X'$ is the inverse homeomorphism of $\varphi|_{X-X'}$, then the map

$$\widetilde{\psi} : E_Y - E_{Y'} \longrightarrow E_X - E_{X'}$$

given by $\widetilde{\psi}(y, e) = (\psi(y), e)$ is well defined, since $y \in Y - Y'$, and is the inverse of $\widetilde{\varphi}|_{E_X - E_{X'}}$. □

Using 1.4.20 and 4.1.7 we have the following.

4.1.8 **Theorem**. *Let $\pi : E \longrightarrow B$ be a Hurewicz fibration and $f : X \longrightarrow B$, $g : Y \longrightarrow B$ be spaces over B. Let moreover $X' \subset X$, $Y' \subset Y$ be cofibrations, and $\varphi : (X, X') \longrightarrow (Y, Y')$ a map over B that is also a relative homeomorphism. Then*

$$\bar{\varphi} : \bar{h}^*(Y, Y') \longrightarrow \bar{h}^*(X, X')$$

is an isomorphism.

Proof: Since by 1.4.20, $E_{X'} \subset E_X$ and $E_{Y'} \subset E_Y$ are cofibrations, it follows that

$$\bar{h}^*(X, X') = h^*(E_X, E_{X'}) \cong \widetilde{h}^*(E_X/E_{X'})$$

and

$$\bar{h}^*(Y, Y') = h^*(E_Y, E_{Y'}) \cong \widetilde{h}^*(E_Y/E_{Y'})\,,$$

where the isomorphisms are induced by the corresponding quotient maps. Moreover, by 4.1.7, we have that $\widetilde{\varphi} : (E_X, E_{X'}) \longrightarrow (E_Y, E_{Y'})$ is a relative homeomorphism. We have a commutative diagram

$$\begin{tikzcd}
h^*(E_Y/E_{Y'}) \arrow[r, "\widehat{\varphi}^*", "\cong"'] \arrow[d, "\cong"'] & h^*(E_X/E_{X'}) \arrow[d, "\cong"] \\
h^*(E_Y, E_{Y'}) \arrow[r, "\widetilde{\varphi}^*"'] & h^*(E_X, E_{X'})\,,
\end{tikzcd}$$

where the map $\widehat{\varphi} : E_X/E_{X'} \longrightarrow E_Y/E_{Y'}$ is the homeomorphism induced by the relative homeomorphism $\widetilde{\varphi}$. Thus one has that $\widetilde{\varphi}^*$ on the bottom is also an isomorphism. □

4.1.2 Systems of Local Coefficients

4.1.9 DEFINITION. Let B be a topological space. A *system of local coefficients* on B is a contravariant functor

$$\mathcal{G} : \Pi_1(B) \longrightarrow \mathcal{A}b\,,$$

where $\Pi_1(B)$ denotes the fundamental groupoid of B (1.5.5) and $\mathcal{A}b$ is the category of abelian groups (and isomorphisms). In other words, a system of local coefficients maps every point $b \in B$ to an abelian group $\mathcal{G}(b)$, and every path $\omega : b \simeq b'$ to a group isomorphism $\mathcal{G}(\omega) : \mathcal{G}(b') \longrightarrow \mathcal{G}(b)$, in such a way that if $\omega_0 \simeq \omega_1$, then $\mathcal{G}(\omega_0) = \mathcal{G}(\omega_1)$.

4.1.10 EXAMPLE. Let $\pi : E \longrightarrow B$ be a Serre fibration. Define

$$\mathcal{G} : \Pi_1(B) \longrightarrow \mathcal{A}b \quad \text{by} \quad \mathcal{G}(b) = H^n(\pi^{-1}(b))\,,$$

and if $\omega : b \simeq b'$, then let $\mathcal{G}(\omega)$ be the composite

$$\begin{aligned} \omega^\star : H^n(\pi^{-1}(b')) \;\cong\; H^n(\widetilde{\pi}^{-1}(1)) \xrightarrow{(1)} H^n(E') \longleftarrow \\ \xleftarrow{(2)} H^n(\widetilde{\pi}^{-1}(0)) \;\cong\; H^n(\pi^{-1}(b))\,, \end{aligned}$$

where $\widetilde{\pi} : E' \longrightarrow I$ is the fibration induced by π over $\omega : I \longrightarrow B$. As in Definition 3.3.11, the homomorphisms (1) and (2) induced by the inclusions are isomorphisms.

We call this the *ordinary system of local coefficients induced by* the fibration $\pi : E \longrightarrow B$ on B.

4.1.11 DEFINITION. Let h^* be a generalized cohomology theory and $\pi : E \longrightarrow B$ a Hurewicz fibration, with a path lifting map

$$\Gamma : E \times_B B^I = \{(e, \omega) \in E \times B^I \mid \pi(e) = \omega(0)\} \longrightarrow E^I .$$

Given any path $\omega : b \simeq b'$, define a map

$$\alpha(\omega) : \pi^{-1}(b) \longrightarrow \pi^{-1}(b')$$

by $\alpha(\omega)(e) = \Gamma(e, \omega)(1)$.

4.1.12 EXERCISE. Prove the following facts:

(i) If $\omega_0 \simeq \omega_1 : b \simeq b'$, then $\alpha(\omega_0) \simeq \alpha(\omega_1) : \pi^{-1}(b) \longrightarrow \pi^{-1}(b')$.

(ii) If $\omega : b \simeq b'$ and $\omega' : b' \simeq b''$, then $\alpha(\omega\omega') \simeq \alpha(\omega') \circ \alpha(\omega) : \pi^{-1}(b) \longrightarrow \pi^{-1}(b'')$.

4.1.13 EXERCISE. Prove that there is a category of systems of local coefficients on a space B.

4.1.14 EXERCISE. Prove that a map $f : B \longrightarrow B'$ induces a covariant functor from the category of systems of local coefficients on B' to the category of systems of local coefficients on B. Prove that this correspondence is (contravariantly) functorial.

From Exercise 4.1.12 we conclude that there is a system of local coefficients as follows.

4.1.15 **Theorem**. *Let h^* be a generalized cohomology theory and $\pi : E \longrightarrow B$ be a Hurewicz fibration. Then the mapping*

$$[\omega] \longmapsto \alpha(\omega)^* : h^p(\pi^{-1}(b')) \longrightarrow h^p(\pi^{-1}(b))$$

determines a system of local coefficients. We call this the *h^p-system of local coefficients induced by* the fibration $\pi : E \longrightarrow B$ on B, and denote it by $h^p(\mathcal{F})$. □

4.1.16 EXERCISE. Prove that if h^* is ordinary cohomology, then the system of local coefficients $h^p(\mathcal{F})$ is the system of local coefficients $\mathcal{G}$ given in 4.1.10.

4.1.3 Singular Homology and Cohomology with Local Coefficients

4.1.17 Definition. Fix a system of local coefficients $\mathcal{G}$ on a space X, and denote by $\Delta_p(X)$ the set of *singular p-simplexes* on X, and by e_0 the leading vertex of Δ^p $\alpha : \Delta^p \longrightarrow X$. Define

$S_p(X;\mathcal{G})$
$= \{\text{functions } s : \Delta_p(X) \longrightarrow \bigcup_{x\in X} \mathcal{G}(x) \mid s(\alpha) \in \mathcal{G}(\alpha(e_0)),\ s(\alpha) \neq 0 \text{ for only finitely many singular maps } \alpha \in \Delta_p(X)\}$
$= \bigoplus_{\alpha\in\Delta^p(X)} \mathcal{G}(\alpha(e_0))$.

We call the elements of $S_p(X;\mathcal{G})$ the *singular p-chains on X with coefficients in $\mathcal{G}$*. A p-chain s is said to be *elementary* if $s(\alpha) \neq 0$ for only one p-simplex $\alpha \in \Delta_p(X)$. Thus a general p-chain s with coefficients in $\mathcal{G}$ can be written as a finite formal sum of elementary p-chains

$$s = \sum g_i\alpha_i\,, \qquad \text{where } g_i \in \mathcal{G}(\alpha_i(e_0))\,.$$

This explains the second equality.

Dually we define

$S^p(X;\mathcal{G})$
$= \{\text{functions } s : \Delta_p(X) \longrightarrow \bigcup_{x\in X} \mathcal{G}(x) \mid s(\alpha) \in \mathcal{G}(\alpha(e_0))\}$
$= \prod_{\alpha\in\Delta^p(X)} \mathcal{G}(\alpha(e_0))$.

We call the elements of $S^p(X;\mathcal{G})$ the *singular p-cochains on X with coefficients in $\mathcal{G}$*.

In order to describe a boundary operator on $S_*(X;\mathcal{G})$, we observe that the usual singular boundary operator behaves as follows with respect to the leading vertex e_0:

$$\partial_i\alpha = \begin{cases} \alpha(e_0) & \text{if } i \neq 0, \\ \alpha(e_1) & \text{if } i = 0. \end{cases}$$

In the case of local coefficients, the coefficients on certain simplex depend on the leading vertex, so we have to include a change of leading vertex. Let $\alpha : \Delta^p \longrightarrow X$ be a p-simplex and take the path

$$\omega_\alpha(t) = \alpha(te_0 + (1-t)e_1)$$

from $\alpha(e_0)$ to $\alpha(e_1)$. Define

$$(4.1.17) \qquad \partial s = \partial\left(\sum_i g_i\alpha_i\right) = \sum_i \left(\mathcal{G}(\omega_{\alpha_i})(g_i)\partial_0\alpha_i + \sum_{j=1}^{p}(-1)^j g_i\partial_j\alpha_i\right)\,.$$

The homomorphism ∂ is a differential; namely, we have the following.

4.1.18 **Lemma**. $\partial \circ \partial = 0$.

Proof: Just observe that $\omega_{\partial_i \alpha} = \omega_\alpha$ if $i > 1$ and $\omega_{\partial_1 \alpha} = \omega_{\partial_0 \alpha}\omega_\alpha$, and do the computations. $\square$

4.1.19 Definition. We define the (*singular*) *homology of* X *with local coefficients in* $\mathcal{G}$ to be

$$H_*(X;\mathcal{G}) = H(S_*(X;\mathcal{G}),\partial)\,.$$

This is a generalization of singular homology with regular coefficients as shown in the following.

4.1.20 **Proposition**. *If the system of local coefficients* $\mathcal{G}$ *is trivial or constant with value* G*, then* $H_*(X;\mathcal{G}) \cong H_*(X;G)$.

Proof: If $\mathcal{G}$ is *trivial*, then there exists a group isomorphism $\Phi_x : \mathcal{G}(x) \longrightarrow G$ for each x such that given any path ω in X, the diagram

$$\begin{tikzcd} \mathcal{G}(\omega(1)) \arrow[rr, "\mathcal{G}(\omega)"] \arrow[dr, "\Phi_{\omega(1)}"'] & & \mathcal{G}(\omega(0)) \arrow[dl, "\Phi_{\omega(1)}"] \\ & G & \end{tikzcd}$$

commutes. Thus, the isomorphisms Φ_x determine an isomorphism of chain complexes $S_*(X;\mathcal{G}) \longrightarrow S_*(X;G)$. If the system of local coefficients is *constant*, then $\mathcal{G}(x) = G$ for every $x \in X$, and $\mathcal{G}(\omega) = 1_G$ for every path ω in X. In this case, $S_*(X;\mathcal{G}) = S_*(X;G)$ and Formula 4.1.17 reduces to the regular boundary operator and so $H_*(X;\mathcal{G}) = H_*(X;G)$. $\square$

We now describe a boundary operator in $S^*(X;\mathcal{G})$ as follows.

$$(-1)^p \delta s(\alpha) = \mathcal{G}(\omega_\alpha)(s(\partial_0 \alpha)) + \sum_{i=1}^{p+1} (-1)^i s(\partial_i \alpha)\,, \tag{4.1.21}$$

for $\alpha \in \Delta_{p+1}(X)$ and $s \in S^p(X)$.

Similarly to 4.1.18, one can prove the following.

4.1.22 **Lemma**. $\delta \circ \delta = 0$. $\square$

4.1.23 Definition. We define the (*singular*) *cohomology of* X *with local coefficients in* $\mathcal{G}$ to be

$$H^*(X;\mathcal{G}) = H(S^*(X;\mathcal{G}),\delta)\,.$$

As in 4.1.20, we have the following.

4.1.24 **Proposition**. *If the system of local coefficients* $\mathcal{G}$ *is trivial or constant with value* G*, then* $H^*(X;\mathcal{G}) \cong H^*(X;G)$. □

If $A \subset X$ and $\mathcal{G}$ is a system of local coefficients on X, then we may consider the *restriction* $\mathcal{G}|_A$ of $\mathcal{G}$ to A by taking the composition of the functor $\mathcal{G}$ with the morphism of fundamental groupoids $\Pi_1(A) \longrightarrow \Pi_1(X)$. The inclusion $S_*(A;\mathcal{G}|_A) \longrightarrow S_*(X;\mathcal{G})$ has a cokernel that we denote by $S_*(X,A;\mathcal{G})$.

4.1.25 Definition. The (*singular*) *homology* of the pair (X,A) *with local coefficients in* $\mathcal{G}$ is given by

$$H_*(X,A;\mathcal{G}) = H_*(S_*(X,A;\mathcal{G}))\,,$$

and the short exact sequence of chain complexes

$$0 \longrightarrow S_*(A;\mathcal{G}|_A) \longrightarrow S_*(X;\mathcal{G}) \longrightarrow S_*(X,A;\mathcal{G}) \longrightarrow 0$$

provides the long exact sequence in homology of a pair

$$\cdots \to H_{p+1}(X,A;\mathcal{G}) \xrightarrow{\partial} H_p(A;\mathcal{G}|_A) \to H_p(X;\mathcal{G}) \to H_p(X,A;\mathcal{G}) \to \cdots\,.$$

This is the **Exactness** axiom for homology with local coefficients.

More generally than above, given any map $f: Y \longrightarrow X$ and a system of local coefficients $\mathcal{G}$ on X, we may induce a system of local coefficients $f^*\mathcal{G}$ on Y by composing the functor $\mathcal{G}$ with the groupoid morphism $f_*: \Pi_1(Y) \longrightarrow \Pi_1(X)$. Thus $f^*\mathcal{G}(y) = \mathcal{G}(f(y))$ for $y \in Y$, and $f^*\mathcal{G}(\beta) = \mathcal{G}(f \circ \beta)$ for any path β in Y. This induces a homomorphism

$$f_*: H_*(Y;f^*\mathcal{G}) \longrightarrow H_*(X;\mathcal{G})\,.$$

Similarly, a *morphism of systems of local coefficients* on X $\Phi : \mathcal{G} \longrightarrow \mathcal{H}$, namely a natural transformation of functors, or explicitely, a family of homomorphisms

$$\Phi_x : \mathcal{G}(x) \longrightarrow \mathcal{H}(x)$$

such that for any path $\alpha : x_0 \simeq x_1$ the diagram

$$\begin{tikzcd} \mathcal{G}(x_0) \arrow[r, "\mathcal{G}(\alpha)"] \arrow[d, "\Phi_{x_0}"'] & \mathcal{G}(x_1) \arrow[d, "\Phi_{x_1}"] \\ \mathcal{H}(x_0) \arrow[r, "\mathcal{H}(\alpha)"'] & \mathcal{H}(x_1) \end{tikzcd}$$

commutes, induces a homomorphism

$$\widehat{\Phi} : H_*(X; \mathcal{G}) \longrightarrow H_*(X; \mathcal{H}) .$$

4.1.26 Exercise. Let $\mathcal{G}$ be a system of local coefficients on X and let $g : Z \longrightarrow Y$, $f : Y \longrightarrow X$ be continuous maps. Prove that the induced systems of local coefficients $(f \circ g)^*\mathcal{G}$ and $g^* f^* \mathcal{G}$ are equal. Prove, moreover, that the diagram

$$\begin{tikzcd} H_*(Z; g^* f^* \mathcal{G}) \arrow[rr, "(f \circ g)_*"] \arrow[dr, "g_*"'] & & H_*(X; \mathcal{G}) \\ & H_*(Y; f^* \mathcal{G}) \arrow[ur, "f_*"'] & \end{tikzcd}$$

commutes. This is the **Functoriality** axiom for homology with local coefficients.

4.1.27 Exercise. Let $\mathcal{G}$ be a system of local coefficients on X, and let $f_0, f_1 : Y \longrightarrow X$ be homotopic maps. Prove that the induced systems of local coefficients $f_0^*\mathcal{G}$ and $f_1^*\mathcal{G}$ are isomorphic, say by an isomorphism of systems of local coefficients $\Phi : f_0^*\mathcal{G} \longrightarrow f_1^*\mathcal{G}$. Prove, moreover, that the homomorphisms induced by f_0 and f_1 in homology with local coefficients in $\mathcal{G}$ coincide up to the isomorphism, namely, that the diagram

$$\begin{tikzcd} H_*(Y; f_0^*\mathcal{G}) \arrow[dr, "f_{0*}"] \arrow[dd, "\widehat{\Phi}"'] & \\ & H_*(X; \mathcal{G}) \\ H_*(Y; f_1^*\mathcal{G}) \arrow[ur, "f_{1*}"'] & \end{tikzcd}$$

commutes. This is the **Homotopy** axiom for homology with local coefficients.

4.1.28 Exercise. Let $\mathcal{G}$ be a system of local coefficients on X and let $A \subset X$. Let moreover $\overline{U} \subset \overset{\circ}{A}$. Prove that the inclusion of pairs $(X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism

$$H_*(X - U, A - U; \mathcal{G}) \longrightarrow H_*(X, A; \mathcal{G}) .$$

(Hint: Compare with the proof of [14, 4.6.5].) This is the **Excision** axiom for homology with local coefficients.

4.1.29 Exercise. Observe that a system of local coefficients $\mathcal{G}$ on a singular space $*$ is nothing but an abelian group $G = \mathcal{G}(*)$. Prove that

$$H_p(*;\mathcal{G}) = \begin{cases} G & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

This is the **Dimension** axiom for homology with local coefficients.

The previous exercises show that homology with local coefficients satisfies axioms similar to the Eilenberg–Steenrod axioms (cf. Subsection 4.1.1 or see [1]). There is one more axiom that also plays an important role; namely, we have the following.

4.1.30 Exercise. Take pairs of spaces (X_α, A_α), with the indexes α varying in any set Φ. Let $i_\beta : (X_\beta, A_\beta) \in \coprod_{\alpha\in\Phi} (X_\alpha, A_\alpha)$, $\beta \in \Phi$, be the canonical inclusion of each of the pairs into their topological sum. If $\mathcal{G}$ is a system of local coefficients on the the topological sum and $\mathcal{G}_\alpha = i_\alpha^*\mathcal{G}$ is the induced system on each summand, then prove that the inclusions provide an isomorphism

$$\bigoplus_{\alpha\in\Phi} H_p\left(X_\alpha, A_\alpha; \mathcal{G}_\alpha\right) \xrightarrow{\cong} H_p\left(\coprod_{\alpha\in\Phi} (X_\alpha, A_\alpha)\,; \mathcal{G}\right).$$

This is the **Additivity** axiom for homology with local coefficients.

Take again $A \subset X$ and assume that $\mathcal{G}$ is a system of local coefficients on X. Let $\mathcal{G}|_A$ be the restriction of $\mathcal{G}$ to A. The projection $S^*(X;\mathcal{G}) \longrightarrow S^*(A;\mathcal{G}|_A)$ has a kernel that we denote by $S^*(X, A;\mathcal{G})$.

4.1.31 Definition. The (*singular*) *cohomology* of the pair (X, A) *with local coefficients in* $\mathcal{G}$ is given by

$$H^*(X, A;\mathcal{G}) = H^*(S^*(X, A;\mathcal{G}))\,,$$

and the short exact sequence of cochain complexes

$$0 \longrightarrow S^*(X, A;\mathcal{G}) \longrightarrow S^*(X;\mathcal{G}) \longrightarrow S^*(A;\mathcal{G}|_A) \longrightarrow 0$$

provides the long exact sequence in cohomology of a pair

$$\cdots \to H^p(X, A;\mathcal{G}) \to H^p(X;\mathcal{G}) \to H^p(A;\mathcal{G}|_A) \xrightarrow{\delta} H^{p+1}(X, A;\mathcal{G}) \to \cdots .$$

This is the **Exactness** axiom for cohomology with local coefficients.

Similarly to Exercise 4.1.30, one can solve the following.

4.1.32 Exercise. Under the same assumptions of Exercise 4.1.30, prove that the inclusions provide an isomorphism

$$H^p\left(\coprod_{\alpha\in\Phi}(X_\alpha, A_\alpha);\mathcal{G}\right) \xrightarrow{\cong} \prod_{\alpha\in\Phi} H^p(X_\alpha, A_\alpha;\mathcal{G}_\alpha)\,.$$

This is the **Additivity** axiom for cohomology with local coefficients.

4.1.33 Exercise. Give a proper formulation of the remaining axioms corresponding to the Eilenberg–Steenrod axioms for cohomology with local coefficients and prove them.

A slightly more general treatment of singular homology and cohomology with local coefficients can be read in [17].

4.1.4 Cellular Homology and Cohomology with Local Coefficients

Assume that (X,A) is a relative CW-complex and let

$$A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^p \subset X^{p+1} \subset \cdots$$

be its skeletal filtration, that is, for each $p \geq 0$ there are characteristic maps $\varphi : (\Delta^p, \dot{\Delta}^p) \longrightarrow (X^p, X^{p-1})$, such that the induced map

$$X^{p-1} \sqcup \coprod_{\varphi\in\Phi_p} \Delta^p \longrightarrow X^p$$

is an identification.

4.1.34 Definition. Suppose that (X,A) is a relative CW-complex and $\mathcal{G}$ a system of local coefficients on X. If we denote by X^p the p-skeleton of (X,A), $p \geq 0$, and $X^{-1} = A$, we define the *cellular complex* of (X,A) *with local coefficients in* $\mathcal{G}$ by

$$C_p(X,A;\mathcal{G}) = H_p(X^p, X^{p-1};\mathcal{G})\,,$$

and

$$\partial : C_p(X,A;\mathcal{G}) \longrightarrow C_{p-1}(X,A;\mathcal{G})$$

is the boundary operator of the triple (X^p, X^{p-1}, X^{p-2}).

4.1.35 **Proposition**. *Assume that* $\{\varphi_\alpha : (\Delta^p_\alpha, \dot{\Delta}^p_\alpha) \longrightarrow (X^p, X^{p-1})\}$ *is a collection of p-cells for* (X, A). *Then*

$$(\varphi_{\alpha *}) : \bigoplus_{\alpha \in \Phi_p(X,A)} H_p(\Delta^p_\alpha, \dot{\Delta}^p_\alpha; G_\alpha) \longrightarrow H_p(X^p, X^{p-1}; \mathcal{G}) = C_p(X, A; \mathcal{G})$$

determines a direct sum decomposition of $C_p(X, A; \mathcal{G})$, *where* Δ^p_α *is a copy of* Δ^p *and* G_α *denotes the local coefficient group* $\mathcal{G}(\varphi_\alpha(e_0))$.

Proof: The totality of the maps φ_α determine a relative homeomorphism

$$\varphi : \coprod_{\alpha \in \Phi_p(X,A)} \left(\Delta^p_\alpha, \dot{\Delta}^p_\alpha\right) \longrightarrow (X^p, X^{p-1}) \tag{4.1.36}$$

between CW-pairs. Thus it induces an isomorphism in homology. Since singular homology with local coefficients is additive (see Exercise 4.1.30), we have that the inclusions of the summands into the topological sum induce an isomorphism

$$\bigoplus_{\alpha \in \Phi_p(X,A)} H_p\left(\Delta^p_\alpha, \dot{\Delta}^p_\alpha; \varphi^*_\alpha \mathcal{G}\right) \cong H_p\left(\coprod_{\alpha \in \Phi_p(X,A)} \left(\Delta^p_\alpha, \dot{\Delta}^p_\alpha\right); \varphi^* \mathcal{G}\right). \tag{4.1.37}$$

On the other hand, since Δ^p is contractible, the system of local coefficients on Δ^p induced by $\mathcal{G}$ through φ_α is trivial. Hence, the result follows combining Equations (4.1.36) and (4.1.37), after applying 4.1.20. □

It is useful to describe $\partial : C_p(X, A; \mathcal{G}) \longrightarrow C_{p-1}(X, A; \mathcal{G})$ in terms of the direct sum decomposition given in the last result. In order to do it, we need the concept of *incidence isomorphism*, that can be defined as follows.

Suppose that Δ^{p-1}_i is the ith face of the simplex Δ^p, fact that we denote by $\Delta^{p-1}_i < \Delta^p$. The maps of pairs

$$\left(\Delta^{p-1}_i, \dot{\Delta}^{p-1}_i\right) \xrightarrow{j} \left(\dot{\Delta}^p, \dot{\Delta}^p - \left(\Delta^{p-1}_i - \dot{\Delta}^{p-1}_i\right)\right) \longrightarrow \left(\Delta^p, \dot{\Delta}^p\right) \tag{4.1.38}$$

induce isomorphisms

$$H_p\left(\Delta^p, \dot{\Delta}^p\right) \xrightarrow{\partial} H_{p-1}\left(\dot{\Delta}^p, \left(\Delta^{p-1}_i - \dot{\Delta}^{p-1}_i\right)\right) \xleftarrow{j_*} H_{p-1}\left(\Delta^{p-1}_i, \dot{\Delta}^{p-1}_i\right),$$

where the connecting homomorphism ∂ on the left-hand side is an isomorphism by the exact sequence of the triple $(\Delta^p, \dot{\Delta}^p, (\Delta^{p-1} - \dot{\Delta}^{p-1}_i))$, since both the first and the third spaces of it are contractible, while j_* on the right-hand side is an isomorphism by excision. We define

$$[\Delta^p, \Delta^{p-1}_i] = j_*^{-1} \circ \partial .$$

Let $\varphi : \Delta^p \longrightarrow X^p$ be the characteristic map for a p-cell of the CW-complex X, and Δ_i^{p-1} be the ith face of Δ^p. Then φ provides a map of pairs $\varphi : (\Delta^p, \Delta_i^{p-1}) \longrightarrow (X^p, X^{p-1})$. If $\varphi(e_0)$ is the image of the leading vertex of Δ^p, let e_0^i denote the image of the leading vertex of Δ_i^p. Since Δ^p is convex, we get a straight path $t \mapsto \varphi(te_0 + (1-t)e_0^i)$ in X which we denote by $\lambda(\Delta^p, \Delta_i^{p-1}, \varphi)$. On the other hand, we denote by $\langle \Delta^p, \Delta_i^{p-1}, \varphi \rangle$ the isomorphism $\lambda(\Delta^p, \Delta_i^{p-1}, \varphi)^\star : \mathcal{G}(\varphi(e_0)) \longrightarrow \mathcal{G}(\varphi(e_0^i))$. Then we have the following.

4.1.39 **Theorem**. *The boundary homomorphism of the cellular complex of a pair of spaces* (X, A)*,* $\partial : C_p(X, A; \mathcal{G}) \longrightarrow C_{p-1}(X, A; \mathcal{G})$*, can be expressed in terms of the direct sum decompositions given in Theorem 4.1.35*

$$\partial : \bigoplus_\alpha H_p(\Delta_\alpha^p, \dot{\Delta}_\alpha^p) \otimes G_\alpha \longrightarrow \bigoplus_\beta H_p(\Delta_\beta^{p-1}, \dot{\Delta}_\beta^{p-1}) \otimes G_\beta$$

by

$$\partial(u \otimes g) = \sum_{\Delta_\beta^{p-1} < \Delta_\alpha^p} [\Delta_\alpha^p, \Delta_\beta^{p-1}](u) \otimes \langle \Delta^p, \Delta_i^{p-1}, \varphi \rangle (g)\,.$$

Dually to the previous considerations we have the following.

4.1.40 Definition. Given a relative CW-complex (X, A) and a system of local coefficients $\mathcal{G}$ on X, we define the *cellular cocomplex* of (X, A) *with local coefficients in* $\mathcal{G}$ by

$$C^p(X, A; \mathcal{G}) = H^p(X^p, X^{p-1}; \mathcal{G})\,,$$

and take

$$\delta : C^{p-1}(X, A; \mathcal{G}) \longrightarrow C^p(X, A; \mathcal{G})$$

to be the coboundary operator of the triple (X^p, X^{p-1}, X^{p-2}).

Similarly to 4.1.35, we have the following.

4.1.41 **Proposition**. *Assume that* $\{\varphi_\alpha : (\Delta_\alpha^p, \dot{\Delta}_\alpha^p) \longrightarrow (X^p, X^{p-1})\}$ *is a collection of* p*-cells for* (X, A)*. Then*

$$(\varphi_\alpha^*) : C^p(X, A; \mathcal{G}) = H^p(X^p, X^{p-1}; \mathcal{G}) \longrightarrow \prod_{\alpha \in \Delta_p(X,A)} H^p(\Delta_\alpha^p, \dot{\Delta}_\alpha^p; G_\alpha)$$

determines a direct product decomposition of $C^p(X, A; \mathcal{G})$*, where* G_α *denotes the local coefficient group* $\mathcal{G}(\varphi_\alpha(e_0))$*.* $\square$

In order to describe $\delta : C^{p-1}(X,A;\mathcal{G}) \longrightarrow C^p(X,A;\mathcal{G})$ in terms of the direct product decomposition just given, we need the dual concept of *coincidence isomorphism* as follows.

Again the maps of pairs (4.1.38) give rise to isomorphisms

$$H^{p-1}\left(\Delta_i^{p-1},\dot\Delta_i^{p-1}\right) \xleftarrow{j^*} H^{p-1}\left(\dot\Delta^p,\dot\Delta^p-\left(\Delta_i^{p-1}-\dot\Delta_i^{p-1}\right)\right) \xrightarrow{\delta} H^p\left(\Delta^p,\dot\Delta^p\right)$$

and we define

$$[\Delta^p,\Delta_i^{p-1}]^* = \delta\circ j^{*-1}\,.$$

If we take $\langle\Delta^p,\Delta_i^{p-1},\varphi\rangle : \mathcal{G}(\varphi(e_0)) \longrightarrow \mathcal{G}(\varphi(e_0^i))$ as before, we have

4.1.42 **Theorem**. *The coboundary homomorphism of the cellular cocomplex of a pair of spaces (X,A), $\delta : C^{p-1}(X,A;\mathcal{G}) \longrightarrow C^p(X,A;\mathcal{G})$, can be expressed in terms of the direct product decompositions given in Theorem 4.1.41*

$$\delta : \prod_\beta H^{p-1}(\Delta_\beta^{p-1},\dot\Delta_\beta^{p-1})\otimes G_\beta \longrightarrow \prod_\alpha H^p(\Delta_\alpha^p,\dot\Delta_\alpha^p)\otimes G_\alpha$$

by

$$(-1)^{p-1}\delta_\alpha((u_\beta\otimes g_\beta)_\beta) = \sum_{\Delta_\beta^{p-1}<\Delta_\alpha^p} [\Delta_\alpha^p,\Delta_\beta^{p-1}]^*(u_\beta)\otimes\langle\Delta_\alpha^p,\Delta_\beta^{p-1},\varphi\rangle^{-1}(g_\beta)\,.$$

in each factor $H^p(\Delta_\alpha^p,\dot\Delta_\alpha^p)\otimes G_\alpha$. Observe that the sum on the right-hand side is always finite.

4.1.43 **Theorem**. *Let h^* be any cohomology theory and let F be any topological space. For fixed q, there are isomorphisms*

$$\xi^p : h^{p+q}\left(\left(\Delta^{p-1},\dot\Delta^{p-1}\right)\times F\right) \xrightarrow{\cong} H^p\left(\Delta^p,\dot\Delta^p\right)\otimes h^q(F)\,.$$

Proof: Recall the inclusions (4.1.38) and take the topological product with F to obtain inclusions

$$\left(\Delta_i^{p-1},\dot\Delta_i^{p-1}\right)\times F \xrightarrow{j_F} \left(\dot\Delta^p,\dot\Delta^p-\left(\Delta_i^{p-1}-\dot\Delta_i^{p-1}\right)\right)\times F \longrightarrow \left(\Delta^p,\dot\Delta^p\right)\times F\,.$$

We proceed inductively on p. Consider $p=1$ and the diagram

$$\begin{tikzcd} & h^{1+q}((\Delta^1,\dot\Delta^1)\times F) \arrow[d, "\xi^1"] \\ h^q(F) \arrow[ur, "\delta\circ j_F^{*-1}"] \arrow[r, "(\delta\circ j^{*-1})\otimes 1"'] & H^1(\Delta^1,\dot\Delta^1)\otimes h^q(F)\,, \end{tikzcd}$$

where δ on the top corresponds to the triple $(\Delta^1, \dot{\Delta}^1, \Delta^0) \times F$, while the one on the bottom corresponds to $(\Delta^1, \dot{\Delta}^1, \Delta^0)$ (observe that Δ^0 is a singular space consisting of the origin). Since the tilted arrows are isomorphisms, we may define ξ^1 just to make the diagram commutative. It is obviously an isomorphism.

Assume ξ^{p-1} already constructed, then take the diagram

$$\begin{tikzcd} h^{p-1+q}((\Delta_i^{p-1}, \dot{\Delta}_i^{p-1}) \times F) \arrow[r, "\delta\circ j_F^{*-1}"] \arrow[d, "\xi^{p-1}"'] & h^{p+q}((\Delta^p, \dot{\Delta}^p) \times F) \arrow[d, "\xi^p"] \\ H^{p-1}(\Delta_i^{p-1}, \dot{\Delta}_i^{p-1}) \otimes h^q(F) \arrow[r, "\delta\circ j^{*-1}\otimes 1"'] & H^p(\Delta^p, \dot{\Delta}^p) \otimes h^q(F)\,, \end{tikzcd}$$

where δ on the top corresponds to the triple $(\Delta^p, \dot{\Delta}^p, (\Delta^{p-1} - \dot{\Delta}_i^{p-1})) \times F$, while the one on the bottom to $(\Delta^p, \dot{\Delta}^p, (\Delta^{p-1} - \dot{\Delta}_i^{p-1}))$. Since the horizontal arrows are isomorphisms, so as also is the left arrow, we may define ξ^p to be an isomorphism such that the diagram commutes. $\square$

4.2 The Leray-Serre Spectral Sequence for Generalized Cohomology

We modify slightly the construction 3.3.1 given in Chapter 3. We assume that h^* is a generalized cohomology theory.

4.2.1 Construction. Let $\pi : E \longrightarrow B$ be a Hurewicz fibration over B, where (B, A) is a relative CW-complex. Denote by B^p the p-skeleton and by E^p its inverse image under π, $\pi^{-1}(B^p)$, $p \geq 0$. In particular, set $E^{-1} = \pi^{-1}A$ if $p < 0$. We have an exact couple (see 3.2.9).

(4.2.2)
$$\begin{tikzcd} A^{**} \arrow[rr, "i"] & & A^{**} \arrow[dl, "k"] \\ & C^{**}\,, \arrow[ul, "j"] & \end{tikzcd}$$

given by the definitions

$$\begin{aligned} A^{p,q} &= h^{p+q}(E^p)\,, \\ C^{p,q} &= h^{p+q}(E^p, E^q)\,, \\ i : h^{p+q}(E^{p+1}) &\longrightarrow h^{p+q}(E^p)\,, \\ j : h^{p+q}(E^p, E^{p-1}) &\longrightarrow h^{p+q}(E^p)\,, \end{aligned}$$

that are induced by the canonical inclusions, and

$$k : h^{p+q-1}(E^{p-1}) \longrightarrow h^{p+q}(E^p, E^{p-1})$$

given by the boundary homomorphism δ.

Dually to 3.3.1, the bidegrees of these homomorphisms clearly are:

$$\begin{aligned}
\mathrm{bideg}(i) &= (-1, 1)\,, \\
\mathrm{bideg}(j) &= (0, 0)\,, \\
\mathrm{bideg}(k) &= (1, 0)\,,
\end{aligned}$$

now with the opposite signs as in 3.3.1.

Dually as in 3.2.17, take $r \geq 1$, $r \in \mathbb{Z}$, $i^0 = \mathrm{id}_A$.

$$\begin{aligned}
d_r^{p,q} &= k(i^{r-1})^{-1} j : C^{p,q} \longrightarrow C^{p+r,q-r+1}\,, \\
Z_r^{p,q} &= \mathrm{Def}(d_r^{p,q})\,, \\
B_r^{p,q} &= \mathrm{Ind}(d_r^{p+r,q-r+1})\,, \\
Z_\infty^{p,q} &= \bigcap_{r=1}^{\infty} Z_r^{p,q}\,, \\
B_\infty^{p,q} &= \bigcup_{r=1}^{\infty} B_r^{p,q}\,, \\
E_r^{p,q} &= Z_r^{p,q}/B_r^{p,q}\,, \quad 1 \leq r \leq \infty\,.
\end{aligned}$$

Similarly to Section 3.2, we have that $d_r^{p,q} : C^{p,q} \longrightarrow C^{p+r,q-r+1}$ induces a homomorphism

$$\overline{d}_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}\,,$$

and one has the following result dual to 3.2.19.

4.2.3 **Theorem**. *The pair $(E_r, \overline{d}_r)$ is a cochain complex and its cohomology satisfies*

$$H^{p,q}(E_r, \overline{d}_r) \cong E_{r+1}^{p,q}\,;$$

that is, $(E_r, \overline{d}_r)$, $r = 1, 2, \ldots$, is a spectral sequence.

4.2.1 Computation of the E_1-term of the Spectral Sequence

Take a characteristic map $\varphi : \Delta^p \longrightarrow B^p$ and take the fibration induced by π through φ, namely, take the diagram

$$\begin{tikzcd} T_\varphi \arrow[r, "\widetilde{\varphi}"] \arrow[d] & E^p \arrow[d, "\pi"] \\ \Delta^p \arrow[r, "\varphi"'] & B^p . \end{tikzcd}$$

Since Δ^p is contractible, by 1.4.30, there is a (well-defined up to fiber homotopy) trivialization

$$\alpha_\varphi : \Delta^p \times F_{\varphi(e_0)} \longrightarrow T_\varphi ,$$

where as above $F_{\varphi(e_0)}$ represents the fiber $\pi^{-1}(\varphi(e_0))$. Consider the composite

$$\begin{aligned}\kappa_\varphi : h^{p+q}(E^p, E^{p-1}) \xrightarrow{\widetilde{\varphi}^*} h^{p+q}(T_\varphi, \dot{T}_\varphi) &\xrightarrow{\alpha_\varphi^*} h^{p+q}((\Delta^p, \dot{\Delta}^p) \times F_{\varphi(e_0)})) \\ &\xrightarrow{\xi^p} H^p(\Delta^p, \dot{\Delta}^p) \otimes h^q(F_{\varphi(e_0)})) ,\end{aligned}$$

where $\dot{T}_\varphi$ is the restriction of T_φ to the boundary $\dot{\Delta}^p$ of Δ^p, and ξ^p is the isomorphism given in Theorem 4.1.43. Since by 1.4.26 the map of pairs $\alpha_\varphi : (\Delta^p, \dot{\Delta}^p) \times F_{\varphi(e_0)}) \longrightarrow (T_\varphi, \dot{T}_\varphi)$ is a (fiber) homotopy equivalence, the homomorphism α_φ^* is also an isomorphism.

In what follows, we prove that the homomorphism

$$\kappa = (\kappa_\varphi) : h^{p+q}(E^p, E^{p-1}) \longrightarrow \prod_{\varphi \in \Phi_p} H^p(\Delta^p, \dot{\Delta}^p) \otimes h^q(F_{\varphi(e_0)}) = C^p(B, A; h^q(\mathcal{F}))$$

is an isomorphism. For that, it is enough to see that the maps $\widetilde{\varphi}^*$ determine an isomorphism

$$\kappa' = (\widetilde{\varphi}^*) : h^{p+q}(E^p, E^{p-1}) \longrightarrow \prod_{\varphi \in \Phi_p} h^{p+q}(T_\varphi, \dot{T}_\varphi) .$$

We have the following.

4.2.4 **Lemma**. *The homomorphism*

$$\kappa' = (\widetilde{\varphi}^*) : h^{p+q}(E^p, E^{p-1}) \longrightarrow \prod_{\varphi \in \Phi_p} h^{p+q}(T_\varphi, \dot{T}_\varphi)$$

is an isomorphism.

Proof: According to Definition 4.1.4, we have to prove that

$$\bar{\kappa}' = (\bar{\varphi}^*) : \bar{h}(B^p, B^{p-1}) \longrightarrow \prod_{\varphi\in\Phi_p} \bar{h}(\Delta^p, \dot{\Delta}^p)$$

is an isomorphism. The map

$$(\bar{\varphi}) : \coprod_{\varphi\in\Phi_p} (\Delta^p, \dot{\Delta}^p) \longrightarrow (B^p, B^{p-1})$$

is a relative homeomorphism, since it induces a homeomorphism

$$\coprod_{\varphi\in\Phi_p} \mathring{\Delta}^p \longrightarrow B^p - B^{p-1}\,.$$

Given that both, the inclusion $\coprod_{\varphi\in\Phi_p} \dot{\Delta}^p \hookrightarrow \coprod_{\varphi\in\Phi_p} \Delta^p$, and the inclusion $B^{p-1} \hookrightarrow B^p$ are cofibrations, by 4.1.8 we have an isomorphism

$$(\bar{\varphi})^* : \bar{h}^p(B^p, B^{p-1}) \longrightarrow \bar{h}^p\left(\coprod_{\varphi\in\Phi_p} (\Delta^p, \dot{\Delta}^p)\right)\,.$$

But since the cohomology theory h^* is additive, then so is also $\bar{h}^*$ (see 4.1.6); hence the homomorphisms induced by the inclusions $i_\varphi : (\Delta^p, \dot{\Delta}^p) \hookrightarrow \coprod_{\varphi\in\Phi_p} (\Delta^p, \dot{\Delta}^p)$ yield an isomorphism

$$\bar{h}^p\left(\coprod_{\varphi\in\Phi_p} (\Delta^p, \dot{\Delta}^p)\right) \cong \prod_{\varphi\in\Phi_p} \bar{h}(\Delta^p, \dot{\Delta}^p)\,.$$

Thus the homomorphism induced by $(\bar{\varphi})$ in cohomology, namely $\bar{\kappa}'$, is an isomorphism. □

4.2.5 **Theorem**. *Let $\pi : E \longrightarrow B$ be a Hurewicz fibration. If (B, A) is a relative* CW*-complex and $E_*^{p,q}$ is the spectral sequence associated to the filtration of E induced by the skeletal filtration of (B, A), then one has for the E_1-term an isomorphism*

$$\kappa : E_1^{p,q} \longrightarrow C^p(B, A; h^q(\mathcal{F}))\,,$$

where $C^p(B; h^q(\mathcal{F}))$ is the cellular cocomplex of B with local coefficients determined by $h^q(\pi^{-1}(b))$, $b \in B$. □

4.2.2 Computation of the E_2-term of the Spectral Sequence

In what follows, we prove that the isomorphism

$$\kappa = (\kappa_\varphi) : h^{p+q}(E^p, E^{p-1}) \longrightarrow \prod_{\varphi\in\Phi_p} H^p(\Delta^p, \dot\Delta^p)\otimes h^q(F_{\varphi(e_0)}) = C^p(B, A; h^q(\mathcal{F}))$$

commutes with the corresponding coboundary homomorphisms (see 4.1.40).

We have the following result.

4.2.6 **Lemma**. *The following is a commutative diagram:*

$$\begin{tikzcd} h^{p+q}(E^p, E^{p-1}) \arrow[r, "\kappa"] \arrow[d, "\delta"'] & C^p(B, A; h^q(\mathcal{F})) \arrow[d, "(-1)^p\delta"] \\ h^{p+q+1}(E^{p+1}, E^p) \arrow[r, "\kappa"] & C^{p+1}(B, A; h^q(\mathcal{F})) \,, \end{tikzcd}$$

where δ on the left-hand side is the connecting homomorphism for the triple (E^{p+1}, E^p, E^{p-1}) and δ on the right-hand side represents the coboundary operator of the cellular cochain complex of the pair (B, A) with local coefficients in $h^q(\mathcal{F})$ (see 4.1.40*).*

Proof: We have to prove the commutativity of the diagram

$$\begin{tikzcd} h^{p+q}(E^p, E^{p-1}) \arrow[r] \arrow[d, "\delta"'] & \prod_\beta H^p(\Delta^p_\beta, \dot\Delta^p_\beta)\otimes h^q(F_{\varphi_\beta(e_0^\beta)}) \arrow[d, "(-1)^p\delta"] \\ h^{p+q+1}(E^{p+1}, E^p) \arrow[r] & \prod_\alpha H^{p+1}(\Delta^{p+1}_\alpha, \dot\Delta^{p+1}_\alpha)\otimes h^q(F_{\varphi_\alpha(e_0^\alpha)}) \,, \end{tikzcd} \tag{4.2.7}$$

where the homomorphism δ on the right-hand side is as given in Theorem 4.1.42, while the horizontal arrows are given by composing κ with the isomorphism given in Proposition 4.1.41; $F_{\varphi(e_0)}$ denotes the fiber of π over the image of the leading vertex under the corresponding characteristic map.

We take the following diagrams:

$$\begin{tikzcd} h^{p+q}(E^p, E^{p-1}) \arrow[r, "\cong"] \arrow[d, "\delta"'] & h^{p+q}(\coprod_\beta(\Delta^p_\beta, \dot\Delta^p_\beta)\times F_{\varphi_\beta(e_0^\beta)}) \arrow[d, "\delta"] \\ h^{p+q+1}(E^{p+1}, E^p) \arrow[r, "\cong"] & h^{p+q+1}(\coprod_\alpha(\Delta^{p+1}_\alpha, \dot\Delta^{p+1}_\alpha)\times F_{\varphi_\alpha(e_0^\alpha)}) \,. \end{tikzcd}$$

Since both vertical arrows are given by connecting homomorphisms, that are natural, and inclusion maps, the commutativity of this diagram is quite clear.

$$\begin{array}{ccc}
h^{p+q}(\coprod_\beta(\Delta^p_\beta,\dot\Delta^p_\beta)\times F_{\varphi_\beta(e_0^\beta)}) & \xrightarrow{\cong} & \prod_\beta h^{p+q}((\Delta^p_\beta,\dot\Delta^p_\beta)\times F_{\varphi\beta(e_0^\beta)}) \\
\delta\downarrow & & \downarrow\delta \\
h^{p+q+1}(\coprod_\alpha(\Delta^{p+1}_\alpha,\dot\Delta^{p+1}_\alpha)\times F_{\varphi_\alpha(e_0^\alpha)}) & \xrightarrow{\cong} & \prod_\alpha h^{p+q+1}((\Delta^{p+1}_\alpha,\dot\Delta^{p+1}_\alpha)\times F_{\varphi_\alpha(e_0^\alpha)})\,.
\end{array}$$

This diagram commutes by naturality arguments, since both coboundary homomorphisms δ are given by the same formula.

$$\begin{array}{ccc}
\prod_\beta h^{p+q}((\Delta^p_\beta,\dot\Delta^p_\beta)\times F_{\varphi\beta(e_0^\beta)}) & \xrightarrow[\cong]{\xi^{p+q}} & \prod_\beta H^p(\Delta^p_\beta,\dot\Delta^p_\beta)\otimes h^q(F_{\varphi\beta(e_0^\beta)}) \\
\delta\downarrow & & \downarrow(-1)^p\delta \\
\prod_\alpha h^{p+q+1}((\Delta^{p+1}_\alpha,\dot\Delta^{p+1}_\alpha)\times F_{\varphi_\alpha(e_0^\alpha)}) & \xrightarrow[\xi^{p+q+1}]{\cong} & \prod_\alpha h^{p+1}(\Delta^{p+1}_\alpha,\dot\Delta^{p+1}_\alpha)\times h^q(F_{\varphi_\alpha(e_0^\alpha)})\,,
\end{array}$$

where the isomorphisms ξ are given in 4.1.43. This last diagram commutes because the definiton of the coboundary homomorphism on the right-hand side is given using the coincidence isomorphisms defined in page 166 previous to Theorem 4.1.42, that correspond precisely to the way that the coboundary homomorphism on the left-hand side is defined, and the sign comes from Theorem 4.1.42.

Putting these three diagrams together, we obtain the commutativity of Diagram (4.2.7), as desired. □

From Lemma 4.2.6, we obtain immediately the main result of this paragraph.

4.2.8 **Theorem**. *Let $\pi : E \longrightarrow B$ be a Hurewicz fibration. If (B,A) is a relative* CW*-complex and $E_*^{p,q}$ is the spectral sequence associated to the filtration of E induced by the skeletal filtration of B, then one has for the E_2-term an isomorphism*

$$\kappa^* : E_2^{p,q} \longrightarrow H^p(B,A;h^q(\mathcal{F}))\,,$$

where $H^p(B;h^q(\mathcal{F}))$ is the cellular cohomology of (B,A) with local coefficients determined by $h^q(\pi^{-1}(b))$, $b\in B$. □

References

[1] M. Aguilar, S. Gitler, C. Prieto, "Algebraic Topology from the Homotopical Viewpoint", Universitexts, Springer-Verlag, New York, Berlin, Heidelberg 2002

[2] C. Chevalley "Theory of Lie groups," Princeton University Press, Princeton, 1946

[3] A. Dold, *Partitions of unity in the theory of fibrations*, Annals of Math. 78 (1963), 223–255

[4] F. Hirzebruch, "Lectures on K-theory," mimeographed notes, Amer. Math. Soc. Summer Topology Institute, Seattle, 1963

[5] H. Hopf *Über die Abbildungen der* 3*-Sphäre auf die Kugelfläche* Math. Annalen 104 (1931), 637–665

[6] H. Hopf *Über die Abbildungen von Sphären auf Sphären niedrigierer Dimension* Fundam. Math. Annalen 25 (1935), 427–440

[7] S.-T. Hu, "Homotopy Theory," Academic Press, New York, 1959

[8] A. T. Lundell, S. Weingram, "The Topology of CW-Complexes," Van Nostrand Reinhold Co., New York, 1969

[9] S. Mac Lane, "Homology," Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Springer-Verlag, Berlin, Heidelberg, New York, 1975

[10] W. S. Massey, "A Basic Course in Algebraic Topology," Graduate Texts in Mathematics, Springer-Verlag, New York, Berlin, Heidelberg, 1991

[11] J. Milnor, "Topology from the Differentiable Viewpoint," The University of Virginia Press, Charlottesville, 1965

[12] H. Miyazaki, *The paracompactness of* CW*-complexes*, Tohoku Math. J. (2) 4 (1952), 309–313

[13] C. Prieto, "Topología Básica," 2001

[14] E. H. Spanier, "Algebraic Topology," McGraw-Hill, New York, 1966

[15] N. Steenrod, "The Topology of Fibre Bundles," (7th printing), Princeton Landmarks in Mathematics, Princeton University Press, Princeton, 1999

[16] N. Steenrod, *A convenient category of topological spaces*, Mich. Math. J. 14 (1967), 133–152

[17] G. W. Whitehead "Elements of Homotopy Theory," Graduate Texts in Mathematics, Springer-Verlag, New York, Heidelberg, Berlin, 1978

Index