Introduction

These notes, which are largely based on [11], are an introduction to the study of Indices of vector fields and Chern classes of singular varieties. We refer to [11] and the bibliography, for more information and details on the topics we explore here.

These notes have been prepared for the course I will deliver at the CIMPA School on Singularity Theory in Topology, Geometry and Foliations, organized by the Institute of Mathematics of Universidad Nacional Autónoma de México.

There are four sections, one for each lecture. The starting point is the celebrated theorem of Poincaré-Hopf about the index of vector fields, and an important concept arising from this, which are the Chern classes of complex manifolds and bundles. The aim of the course is discussing how these extend to singular varieties, and this is what we do in sections 2-4.

In Section 2 we speak of the so-called GSV-index of vector fields on isolated complex hypersurface singularities, introduced by X. Gómez Mont, J. Seade and A. Verjovsky in [40, 25]. The definition of this index actually extends easily to vector fields on isolated complete intersection singularities (ICIS for short). The GSV index was the first “index of vector fields” in the literature defined in general, though the Schwartz index and the local Euler obstruction for radial vector fields were already defined (for studying Chern classes of singular varieties). We refer to [11] for more on those subjects.

As noticed by M. Brunella in [12, 13] and by Khanedani and Suwa in [29], this index is important for the study of 1-dimensional holomorphic foliations
on complex surfaces. We briefly review in Section 2 some of Brunella’s work on the topic.

Just as the usual local index of Poincaré-Hopf, the GSV-index can be defined via Cher-Weil theory. This was done by D. Lehmann, M. Soares and T. Suwa for holomorphic vector fields (and extended by Suwa and myself to the general case). The corresponding index was called the virtual index; we refer to [11] for its definition and main properties. One of the interesting features of the virtual index is that this is defined for vector fields with arbitrary compact singular locus, not necessarily isolated points.

For holomorphic vector fields one also has the homological index, introduced by X. Gómez Mont. This is defined for vector fields on arbitrary normal isolated complex singularities, and in the case of ICIS, it coincides with the GSV index. We also review the homological index in section 2.

In Section 3 we study an extension for vector fields in general, of the Schwartz index for radial vector fields. This was first done by H. King and D. Trotman in a 1992 preprint which was finally published this year. In the late 1990s the same concept was studied from different viewpoints by Ebeling and Gusein-Zade in [18], and by Aguilar, Seade and Verjovsky in [1].

In Section 3 we also describe how these indices relate to the Milnor number of isolated complex singularities. This discussion leads to a global new invariant of compact varieties with isolated singularities. It would be interesting to know how this invariant relates to other invariants, such as the 0-degree Milnor class, whose definition is given in Section 4.

Last, in Section 4 we move from a local point of view to a global one: We give an introduction to the study of Chern classes for singular varieties, hoping that this will serve as an introduction to Jean Paul’s Brasselet lectures next week.

We start Section 4 by defining the Fulton-Johnson classes of complete intersections in compact complex manifolds. We then speak of stratifications and the radial extension process introduced by M. H. Schwartz to define what we call today the Schwartz classes: These provided the first extension of Chern classes for singular varieties. The definition is via obstruction theory, somehow following the classical definition of Chern classes, though several difficulties arise along the way.

The last part of section 4 discusses Milnor classes, which by definition are the difference between the Fulton-Johnson and the Schwartz classes. Milnor classes encode rich information about the geometry and topology of the singular variety as shown by several articles by various people, including J.
Schurman, L. Maxim, J. P. Brasselet, Sh. Yokura, P. Aluffi, and others. And I believe that my recent article [15] with R. Callejas-Bedregal and M. Morgado, throws new light on this, yet mysterious, subject, showing that for hypersurfaces in compact manifolds, the Milnor classes are closely related to the Lé-cycles, introduced by D. Massey for describing the local Milnor fibers at the singularities of $X$.

1 The Classical theory

1.1 The Poincaré-Hopf index of a vector field.

Let $v = \sum_{i=1}^{n} f_i \partial / \partial x_i$ be a vector field on an open set $U \subset \mathbb{R}^n$ with coordinates $\{(x_1, \ldots, x_n)\}$. The vector field is said to be continuous, smooth, analytic, etc., according as its components $\{f_1, \ldots, f_n\}$ are continuous, smooth, analytic, etc., respectively (here “smooth” means $C^\infty$, however in most cases $C^1$ is sufficient). A singularity $a$ of $v$ is a point where all of its components vanish, i.e., $f_i(a) = 0$ for all $i = 1, \ldots, n$. The singularity is isolated if at every point $x$ near $a$ there is at least one component of $v$ which is not zero.

The Poincaré-Hopf index of a vector field at an isolated singularity is its most basic invariant, and it has many interesting properties. To define it, let $v$ be a continuous vector field on $U$ with an isolated singularity at $a$, and let $S_\varepsilon$ be a small sphere in $U$ around $a$. Then the (local) Poincaré-Hopf index of $v$ at $a$, denoted by $\text{Ind}_{\text{PH}}(v, a)$ (if there is no fear of confusion, we will denote it simply by $\text{Ind}(v, a)$), is the degree of the Gauss map $\frac{v}{||v||}$ from $S_\varepsilon$ into the unit sphere in $\mathbb{R}^n$.

If $v$ and $v'$ are two such vector fields, then their local indices at $a$ coincide if and only if their Gauss maps are homotopic (special case of Hopf Theorem [32]). That is equivalent to saying that their restrictions to the sphere $S_\varepsilon$ are homotopic.

Let us consider now an $n$-dimensional smooth manifold $M$, then a vector field on $M$ is a section of its tangent bundle $TM$. Giving a local chart $(x_1, \ldots, x_n)$ on $M$, a vector field on $M$ is locally expressed as above and the definition of the local index at an isolated singularity extends in the obvious way. The index does not depend on the local chart.

**Definition 1.1.** The total index of $v$, denoted

$$\text{Ind}_{\text{PH}}(v, M),$$
is the sum of all its local indices at the singular points.

A fundamental property of the total index is the following classical theorem:

**Theorem 1.2. (Poincaré-Hopf)** Let $M$ be a closed, oriented manifold and $v$ a continuous vector field on $M$ with finitely many isolated singularities. Then one has

$$\text{Ind}_{\text{PH}}(v, M) = \chi(M),$$

independently of $v$, where $\chi(M)$ denotes the Euler-Poincaré characteristic of $M$.

The proof has two main steps. The first of these holds in great generality: Let $E$ be a fiber bundle over a finite CW-complex. Then “the primary obstruction” for constructing a section of $E$ is independent of the choice of section.

Then, in order to prove the previous theorem, it is enough to show that it holds for one specific choice of a vector field on $M$, for every $M$. This is achieved as follows. Take a triangulation of $M$. On each vertex, take a radial vector field, pointing outwards everywhere. Now extend it to the 1-skeleton, so that on each 1-simplex it is pointing towards its barycenter. Now, for each 2-face, we have it defined on its boundary, and we can extend it to the interior with one single point at the barycenter, where the vector field is attracting in all directions. We keep going like this till we get a vector field on $M$ with an isolated singularity at the vertices and at the barycenter of each simplex. By construction, the local index at each singularity is $\pm 1$ depending on the dimension. Thus we get that the total index is the alternate sum of the number of vertices, minus edges, plus faces and so on.

If $M$ is now an oriented manifold with boundary, one has a similar theorem:

**Theorem 1.3.** Let $M$ be a compact, oriented $n$-manifold with boundary $\partial M$, and let $v$ be a non-singular vector field on a neighborhood $U$ of $\partial M$. Then:

1. $v$ can be extended to the interior of $M$ with finitely many isolated singularities.

2. The total index of $v$ in $M$ is independent of the way we extend it to the interior of $M$. In other words, the total index of $v$ is fully determined by its behavior near the boundary.
If \( v \) is everywhere transverse to the boundary and pointing outwards from \( M \), then one has \( \text{Ind}_{PH}(v, M) = \chi(M) \). If \( v \) is everywhere transverse to \( \partial M \) and pointing inwards \( M \), then \( \text{Ind}_{PH}(v, M) = \chi(M) - \chi(\partial M) \).

**Remark 1.4.** It is worth saying that although \( \text{Ind}_{PH}(v, M) \) is determined by its behavior near the boundary, it does depend on the topology of the interior of \( M \).

We remark also that one of the basic properties of the index is its **stability under perturbations**. In other words, if \( v \) has an isolated singularity at a point \( a \) in a manifold \( M \) of index \( \text{Ind}(v, a) \) and we make a small perturbation of \( v \) to get a new vector field \( \hat{v} \) with isolated singularities, then \( \text{Ind}(v, a) \) will be the sum of the local indices of \( \hat{v} \) at its singular points near \( a \). In fact it is well-known that every vector field can be **morsified**, i.e., approximated by vector fields whose singularities are non-degenerate. Each such singularity has local index \( \pm 1 \) and the number of such points, counted with signs, equals the index of \( v \) at \( a \). In short, the local index of \( v \) at \( a \) is the number of singularities, counted with sign, into which \( a \) splits under a morsification of \( v \). We will see later that this basic property has its analogues in the case of vector fields on singular varieties.

This stability of the index is also preserved for vector fields with non-isolated singularities. To make this precise we need to introduce a few concepts, which will also be used later.

The following property of the local index is well-known and we leave the proof as an exercise:

**Proposition 1.5.** Let \( v \) be a vector field around \( 0 \in \mathbb{R}^n \) with an isolated singularity at \( 0 \) of index \( \text{Ind}(v, 0) \), and let \( w \) be a vector field around \( 0 \in \mathbb{R}^n \) with an isolated singularity at \( 0 \) of index \( \text{Ind}(w, 0) \). Then the direct product \( v \oplus w \) is a vector field in \( \mathbb{R}^{n+n} \) with an isolated singularity of index \( \text{Ind}(v, 0) \cdot \text{Ind}(w, 0) \).

A consequence of this result is the well-known fact that if \( M, N \) are closed, oriented manifolds, then \( \chi(M \times N) = \chi(M) \cdot \chi(N) \). Another consequence of 1.5 that will be used later is that if we have a vector field \( v \) in \( \mathbb{R}^n \) with an an isolated singularity at \( 0 \) of index \( \text{Ind}(v, 0) \), and if we extend it to \( \mathbb{R}^n \times \mathbb{R}^m \) by taking the vector field \( w = \sum_{i=1}^{m} y_i \partial/\partial y_i \) in \( \mathbb{R}^m \), then the index does not change, where \( (y_1, \ldots, y_m) \) are the coordinates on \( \mathbb{R}^m \). If we took the vector field \( -\sum_{i=1}^{r} y_i \partial/\partial y_i + \sum_{i=r+1}^{m} y_i \partial/\partial y_i \) in \( \mathbb{R}^m \), then the index in \( \mathbb{R}^n \times \mathbb{R}^m \)
would be $\pm \text{Ind}(v,0)$, depending on the parity of the number $r$ of negative signs.

Let us now introduce the concept of the difference which will be used in the sequel. For this we let $v$ and $v'$ be continuous vector fields on a neighborhood $U$ of a point $p$ in $M$, non-singular on $U \setminus p$. Let $\mathcal{T}$ and $\mathcal{T}'$ be discs around $p$ in $U$ such that interior of $\mathcal{T}$ contains the closure of $\mathcal{T}'$ and denote $X = \mathcal{T} \setminus \mathcal{T}'$. Let us consider $w$ a vector field on $X$ with isolated singularities which restricts to $v$ on $\partial \mathcal{T}$ and to $v'$ on $\partial \mathcal{T}'$; such a vector field $w$ always exists by Theorem 1.3. We may denote by $d(v,v') = \text{Ind}_{PH}(w,X)$ the difference between $v$ and $v'$. Then one has:

$$\text{Ind}_{PH}(v,S) = \text{Ind}_{PH}(v',S) + d(v,v'). \quad (1)$$

One can easily prove the following result that will be used later.

**Proposition 1.7.** Let $M_1$ and $M_2$ be compact oriented $n$-manifolds, $n > 1$, with the same boundary $N = \partial M_1 = \partial M_2$, and let $v$ be a non-singular vector field defined on a neighborhood of $N$. Then one has:

$$\text{Ind}_{PH}(v,M_1) - \text{Ind}_{PH}(v,M_2) = \chi(M_1) - \chi(M_2).$$

### 1.2 Collections of vector fields: The Chern classes of almost complex manifolds

Let us recall the definition of the Chern classes via obstruction [43, 11]. This can be done in full generality, however for simplicity we consider first the case of Chern classes of almost-complex manifolds, and later in this section we indicate how this generalizes to complex vector bundles in general.

Now we assume we are given an almost complex $n = 2m$-manifold $M$, so its tangent bundle $TM$ is endowed with the structure of a complex vector bundle of rank $m$.

**Definition 1.8.** An $r$-field on a subset $A$ of $M$ is a set $v^{(r)} = \{v_1, \ldots, v_r\}$ of $r$ continuous vector fields defined on $A$. A singular point of $v^{(r)}$ is a point where the vectors $(v_i)$ fail to be linearly independent. A non-singular $r$-field is also called an $r$-frame.

Let $W_{r,m}$ be the Stiefel manifold of complex $r$-frames in $\mathbb{C}^m$. Notice that we will use $r$-frames which are not necessarily orthonormal, but this does
not change the results, because every frame is homotopic to an orthonormal one. We know (see [43]) that \( W_{r,m} \) is \((2m - 2r)-\)connected and its first non-zero homotopy group is \( \pi_{2m-2r+1}(W_{r,m}) \simeq \mathbb{Z} \). The bundle of \( r \)-frames on \( M \), denoted by \( W_r(TM) \), is the bundle associated with the tangent bundle and whose fiber over \( x \in M \) is the set of \( r \)-frames in \( T_xM \) (diffeomorphic to \( W_{r,m} \)). In the following, we fix the notation \( q = m - r + 1 \).

The Chern class \( c^q(M) \in H^{2q}(M) \) is the first possibly non-zero obstruction to constructing a section of \( W_r(TM) \). Let us recall the standard obstruction theory process to construct this class. Let \( \sigma \) be a \( k \)-cell of the given cellular decomposition \((D)\), contained in an open subset \( U \subset M \) on which the bundle \( W_r(TM) \) is trivialized. If the section \( v^{(r)} \) of \( W_r(TM) \) is already defined over the boundary of \( \sigma \), it defines a map :

\[
\partial \sigma \simeq S^{k-1} \rightarrow W_r(TM)|_U \simeq U \times W_{r,m} \xrightarrow{pr_2} W_{r,m},
\]

thus an element of \( \pi_{k-1}(W_{r,m}) \).

If \( k \leq 2m - 2r + 1 \), this homotopy group is zero, so the section \( v^{(r)} \) can be extended to \( \sigma \) without singularity. It means that we can always construct a section \( v^{(r)} \) of \( W_r(TM) \) over the \((2q - 1)\)-skeleton of \((D)\).

If \( k = 2(m - r + 1) = 2q \), we meet an obstruction. The \( r \)-frame on the boundary of each cell \( \sigma \) defines an element, denoted by \( \text{Ind}(v^{(r)}, \sigma) \), in the homotopy group \( \pi_{2q-1}(W_{r,m}) \simeq \mathbb{Z} \).

**Definition 1.9.** The integer \( \text{Ind}(v^{(r)}, \sigma) \) is the (Poincaré-Hopf) index of the \( r \)-frame \( v^{(r)} \) on the cell \( \sigma \).

Notice that for this index, to be well defined, we need that the cell \( \sigma \) has the correct dimension. This will be essential for our considerations in the sequel.

The generators of \( \pi_{2q-1}(W_{r,m}) \) being consistent (see [43]), this defines a cochain

\[
\gamma \in C^{2q}(M; \pi_{2q-1}(W_{r,m})),
\]

by setting \( \gamma(\sigma) = \text{Ind}(v^{(r)}, \sigma) \), for each \( 2q \)-cell \( \sigma \), and then by extending it linearly. This cochain is actually a cocycle and the cohomology class that it represents is the \( q \)-th Chern class \( c^q(M) \) of \( M \) in \( H^{2q}(M) \).

The class one gets in this way is independent of the various choices involved in its definition. Note that \( c^m(M) \) coincides with the Euler class of the underlying real tangent bundle \( T \mathbb{R}M \), so these classes are natural generalization of the Euler class.
There is another useful definition of the index \( \operatorname{Ind}(v^{(r)}, \sigma) \): let us write the frame \( v^{(r)} \) as \((v^{(r-1)}, v_r)\), where the last vector is individualized, and suppose that \( v^{(r)} \) is already defined on \( \partial \sigma \). There is no obstruction to extending the \((r-1)\)-frame \( v^{(r-1)} \) from \( \partial \sigma \) to \( \sigma \) because the dimension of the obstruction for such an extension is \( 2(m - (r - 1) + 1) = \dim \sigma + 2 \). The \((r-1)\)-frame \( v^{(r-1)} \), defined on \( \sigma \), generates a complex subbundle \( G^{r-1} \) of rank \((r-1)\) of \( TM|_{\sigma} \) and one can write

\[
TM|_{\sigma} \simeq G^{r-1} \oplus Q^q,
\]

where \( Q^q \) is an orthogonal complement of (complex) rank \( q = m - (r - 1) \).

The obstruction to extending the last vector \( v_r \) inside a \( 2q \)-simplex \( \sigma \) as a non-vanishing section of \( Q^q \) is given by an element of \( \pi_{2q-1}(\mathbb{C}^q \setminus \{0\}) \simeq \mathbb{Z} \) corresponding to the composition of the map \( v_r : \partial \sigma \simeq S^{2q-1} \longrightarrow Q^q|_U \) with the projection on the fiber \( \mathbb{C}^q \setminus \{0\} \). Let us denote by \( \operatorname{Ind}_{Q^q}(v_r, \sigma) \) the integer so obtained. The obstruction to extending the \( r \)-frame \( v^{(r)}|_{\partial \sigma} \) inside \( \sigma \) as an \( r \)-frame tangent to \( M \) is the same as the obstruction to extending the last vector \( v_r \) inside \( \sigma \) as a non zero section of \( Q^q \). In fact there is a natural isomorphism \( \pi_{2q-1}(W_{r,m}) \simeq \pi_{2q-1}(\mathbb{C}^q \setminus \{0\}) \) (for compatible orientations) and by this isomorphism we have the equality of integers

\[
\operatorname{Ind}(v^{(r)}, \sigma) = \operatorname{Ind}_{Q^q}(v_r, \sigma).
\]

A different choice of \( v^{(r-1)} \) gives another choices of \( v_r \) and of \( Q^q \), however all such bundles \( Q^q \) are homotopic and the index we obtain is the same.

**Remark 1.10.** The Chern classes of complex vector bundles in general are defined in essentially the same way as above. If \( E \) is a complex vector bundle of rank \( k > 0 \) over a locally finite simplicial complex \( B \) of dimension \( n \geq k \), then one has \textit{Chern classes} \( c^i(E) \in H^{2i}(B; \mathbb{Z}), i = 1, \ldots, k \). The class \( c^i(E) \) is by definition the primary obstruction to constructing \((k - i + 1)\) linearly independent sections of \( E \).

The class \( c^0(E) \) is defined to be 1 and one has \textit{the Total Chern Class} of \( E \) defined by:

\[
c^*(E) = 1 + c^1(E) + \cdots + c^k(E)
\]

This can be regarded as an element in the cohomology ring \( H^*(B) \) and it is invertible in this ring.
1.3 Relative Chern classes

Suppose now that \((L)\) is a sub-complex of \((D)\) whose geometric realization \(|L|\) is also denoted by \(L\). Assume that we are already given an \(r\)-frame \(v^{(r)}\) on the \(2q\)-skeleton of \(L\), denoted by \(L^{(2q)}\). The same arguments as before say that we can always extend \(v^{(r)}\) without singularity to \(L^{(2q)} \cup D^{(2q-1)}\). If we wish to extend this frame to the \(2q\)-skeleton of \((D)\) we meet an obstruction for each corresponding cell which is not in \((L)\). This gives rise to a cochain which vanishes on \(L\) and is a cocycle in \(H^{2q}(M, L)\).

**Definition 1.11.** The relative Chern class

\[ c^q(M, L; v^{(r)}) \in H^{2q}(M, L), \]

is the class represented by the previous cocycle.

The image of \(c^q(M, L; v^{(r)})\) by the natural map in \(H^{2q}(M)\) is the usual Chern class but as a relative class it does depend on the choice of the frame \(v^{(r)}\) on \(L\). Let us discuss how the relative Chern class varies as we change the \(r\)-frame.

If we have two frames \(v_1^{(r)}\) and \(v_2^{(r)}\) on \(L^{(2q)}\) the difference between the corresponding classes is given by the difference cocycle of the frames on \(L\); in the product \(L \times I\), suppose \(v_1^{(r)}\) is defined at the level \(L \times \{0\}\) and \(v_2^{(r)}\) is defined at the level \(L \times \{1\}\), then the difference cocycle \(d(v_1^{(r)}, v_2^{(r)})\) is well defined in \(H^{2q}(L \times I, L \times \{0\} \cup L \times \{1\}) \simeq H^{2q-1}(L)\), as the obstruction to the extension of the given sections on the boundary of \(L \times I\) ([43] §33.3). As shown in [43], we have the following formula:

\[ c^q(M, L; v_2^{(r)}) = c^q(M, L; v_1^{(r)}) + \delta d(v_1^{(r)}, v_2^{(r)}), \]

where \(\delta : H^{2q-1}(L) \to H^{2q}(M, L)\) is the connecting homomorphism. Also, for three frames \(v_1^{(r)}, v_2^{(r)}\) and \(v_3^{(r)}\) as above, we have

\[ d(v_1^{(r)}, v_3^{(r)}) = d(v_1^{(r)}, v_2^{(r)}) + d(v_2^{(r)}, v_3^{(r)}) \]

For \(r = 1\) the frames consist of a single vector field and the difference above corresponds, via Poincaré duality, to the one previously defined for vector fields (cf 1).
In the sequel, we will show that the relative Chern class allows us to define Chern class in homology.

Let $S$ be a compact $(K)$-subcomplex of $M$, and $U$ a neighborhood of $S$. Let $T$ be a cellular tube in $U$ around $S$. Take an $r$-field $v^{(r)}$ defined on $D^{(2q)}$, possibly with singularities. We suppose that the only singularities inside $U$ are located in $S$. This implies that $v^{(r)}$ has no singularities on $(\partial T)^{(2q)}$ so there is a well defined relative Chern class (see 1.11)
\[ c^q(T, \partial T; v^{(r)}) \in H^{2q}(T, \partial T). \]

**Definition 1.13.** The Poincaré-Hopf class of $v^{(r)}$ at $S$, which is denoted by $\text{PH}(v^{(r)}, S)$, is the image of $c^q(T, \partial T; v^{(r)})$ by the isomorphism $H^{2q}(T, \partial T) \simeq H^{2q}(T, T \setminus S)$ followed by the Alexander duality (see [5])
\[ A_M : H^{2q}(T, T \setminus S) \xrightarrow{\sim} H_{2r-2}(S). \] (3)

For $r = 1$ the frame consist of a single vector field $v$ and the class $\text{PH}(v, S) \in H_0(S)$ is identified with the Poincaré-Hopf index of $v$ at $S$, $\text{Ind}_{\text{PH}}(v, S)$, previously defined (Definition ??)

Note that if $\dim S < 2r - 2$, then $\text{PH}(v^{(r)}, S) = 0$.

The relation between the Poincaré-Hopf class of $v^{(r)}$ and the index we defined above is the following :
\[ \text{PH}(v^{(r)}, S) = \sum \text{Ind}(v^{(r)}, d(\sigma)) \sigma, \]
where the sum runs over the $2(r-1)$-simplices $\sigma$ of the triangulation of $S$ and $d(\sigma)$ is the dual cell of $\sigma$ (of dimension $2q$).

Let us consider now the case of manifolds with boundary. Let $M$ be a compact almost complex $2m$-manifold, with non-empty boundary $\partial M$. Let $(K)$ be a triangulation of $M$ compatible with $\partial M$. The union of all “half-cells” dual to simplices in $\partial M$, denoted by $U$ is a regular neighborhood of $\partial M$. Its boundary is denoted by $\partial U$, which is a union of $(D)$-cells and is homeomorphic to $\partial M$. The pair $(M \setminus (\text{Int} U), \partial U)$ is homeomorphic to $(M, \partial M)$ and one can apply the previous construction.

Let $v^{(r)}$ be an $r$-field on the $(2q)$-skeleton of $(D)$, with singularities located on a compact subcomplex $S$ in $M \setminus (\text{Int} U)$. On the $(2q)$-skeleton of $U$, we have a well defined $r$-frame $v^{(r)}$. Let $\{S_\lambda\}$ be the connected components of $S$. Then, by setting $c_{r-1}(M; v^{(r)}) = c^q(M, \partial M; v^{(r)}) \cap [M, \partial M]$, we have
\[
\sum_{\lambda}(i_{\lambda})_{*}\text{PH}(v^{(r)}, S_{\lambda}) = c_{r-1}(M; v^{(r)}) \quad \text{in} \quad H_{2r-2}(M),
\]

where \(i_{\lambda} : S_{\lambda} \hookrightarrow M\) is the inclusion.

In particular, the sum of the Poincaré-Hopf classes is determined by the behavior of \(v^{(r)}\) near \(\partial M\) and does not depend on the extension to the interior of \(M\). Note that we may assume that \(v^{(r)}\) is non-singular on \(D^{(2q-1)}\).

If \(r = 1\) and \(v^{(1)} = \{v\}\), the relative Chern class is also called the \textit{Euler class of} \(M\) \textit{relative to} \(v\) and its evaluation on the orientation cycle of \((M, \partial M)\) gives the index of \(v\) on \(M\).

\section{Indices of Vector fields on hypersurfaces with isolated singularities}

\subsection{Vector fields on complex hypersurfaces}

We want to study vector fields which are “tangent to a singular variety” and define their local indices at the singular points of the vector field. The first point is discussing what does it mean for a vector field to be tangent to a singular variety.

For instance, consider the map \(f : \mathbb{R}^3 \to \mathbb{R}\) carrying \((x, y, z)\) into \(xyz\). Its zero-set \(V(xyz)\) consists of the three coordinate axis:

What does it mean that a vector field be tangent to this surface, for instance at the origin? In other words, who plays the role of the tangent bundle when there is no tangent bundle?

We will come back to this discussion later. For the moment, for simplicity we restrict the discussion to germs of varieties \((V, 0)\) in \(\mathbb{C}^m\) with an isolated singularity. In fact we may consider also real analytic germs. By a (continuous, smooth, holomorphic) vector field on the germ \((V, 0)\) we mean the restriction to \(V\) of a vector field in a neighborhood of 0 in \(\mathbb{C}^m\), which (is continuous, smooth, holomorphic, and) vanishes at 0 and is tangent to \(V\) at every other point.

For instance, let \((V, 0)\) be defined by a holomorphic function

\[f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0),\]

on a small ball \(\mathbb{B}_\varepsilon\), with a unique critical point at 0. Let \(v\) be a continuous section of the bundle \(T\mathbb{C}^{n+1}|_V\). We notice that for each \(x \in V^* = V \setminus \{0\},\)
the tangent space $T_x V^*$ consists of all vectors in $T_x \mathbb{C}^{n+1}$ which are mapped to 0 by the derivative of $f$:

$$T_x V^* = \{ \zeta \in T_x \mathbb{C}^{n+1} \mid df_x(\zeta) = 0 \}.$$ 

For example, if $f$ is the polynomial map in $\mathbb{C}^2$ defined by $f(z_1, z_2) = z_1^2 + z_2^3$, then the line tangent to $V = f^{-1}(0)$ at a point $(z_1, z_2)$, other than the origin, is spanned by the vector $\tilde{\zeta}(z_1, z_2) = (-3z_2^2, 2z_1)$. To see this notice one has

$$df_z = 2z_1 dz_1 + 3z_2^2 dz_2.$$ 

Hence:

$$df_z(\tilde{\zeta}) = df_z(-3z_2^2, 2z_1) = 0.$$ 

Now, a vector field $v$ on $V$ can be thought of as being a continuous map $(V, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$ which is non-zero on $V^*$ and whose image is contained in the linear space tangent to $V$ at each given point. Since $V$ is a closed subset of $\mathbb{B}_\varepsilon$, this map extends to a neighborhood of $V$ in $\mathbb{C}^{n+1}$. Geometrically this means that the vector field $v$ on $V$ can always be extended to the ambient
space, or equivalently that $v$ can always be considered as the restriction to $V$ of a vector field in the ambient space. However the extension of $v$ to $V$ is by no means unique. Furthermore, all these statements also hold in the holomorphic category:

**Theorem 2.1.** (\cite{3}) Let $V$ be a complex analytic variety in $\mathbb{C}^m$ with an isolated singularity at 0. Then:

1. There exist holomorphic vector fields on $V$ with an isolated singularity at 0. In fact the space of such vector fields is infinite-dimensional.

2. If $v$ is a holomorphic vector field on $V$ with an isolated singularity, then there are infinitely many extensions of $v$ to a neighborhood of 0 in the ambient space with an isolated singularity.

As an example, if $V$ is defined in $\mathbb{C}^2$ by a map $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, then the Hamiltonian vector field $\tilde{\zeta}(z_1, z_2) = (-\frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_1})$ is tangent to $V$ and it is zero only at the origin. Notice that this vector field is actually tangent to all the fibers $f^{-1}(t)$. Let $\zeta$ be the restriction of $\tilde{\zeta}$ to $V$. Notice that $\zeta$ can be extended to $\mathbb{C}^2$ in many other ways; for example, if $g$ is a holomorphic function on $\mathbb{C}^2$ that vanishes exactly on $V$ and represents a non-zero element in the local ring $\mathcal{O}_{(\mathbb{C}^2, 0)}$, then

$$\xi = \left(g - \frac{\partial f}{\partial z_2}, g + \frac{\partial f}{\partial z_1}\right),$$

coincides with $\zeta$ on $V$ and is no longer tangent to the fibers of $f$; choosing $g$ appropriately we can also assure that $\xi$ has an isolated singularity at 0.

In $\mathbb{C}^3$ one has the following example of \cite{22}. Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ have an isolated critical point at 0, set $V = f^{-1}(0)$ and choose the coordinates $(z_1, z_2, z_3)$ so that $V$ meets only at 0 the analytic set where the partial derivatives of $f$ with respect to $z_2$ and $z_3$ vanish:

$$V \cap \left\{ \frac{\partial f}{\partial z_2} = \frac{\partial f}{\partial z_3} = 0 \right\} = \{0\}.$$

Define a holomorphic vector field in $\mathbb{C}^3$ by

$$\tilde{\zeta} = \left(f, \frac{\partial f}{\partial z_3}, -\frac{\partial f}{\partial z_2}\right),$$
Notice $\tilde{\zeta}$ has an isolated singularity at 0 and

$$df(\tilde{\zeta}) = f \frac{\partial f}{\partial z_1},$$

hence $df(\tilde{\zeta})$ vanishes at the points where $f$ vanishes, so $\tilde{\zeta}$ is tangent to $V$. If we set $\zeta = \tilde{\zeta}|_V$, then we have a holomorphic vector field on $V$ with an isolated singularity at the origin, and an extension $\tilde{\zeta}$ of it to $\mathbb{C}^3$ which also has an isolated singularity. Notice however that, unlike the previous example, $\tilde{\zeta}$ is no longer tangent to the fibers of $f$. Yet, we may forget we are given $\tilde{\zeta}$ and just consider the vector field $\zeta$ on $V$. Since $f$ vanishes exactly on $V$, $\zeta$ takes the form $\zeta = (0, \frac{\partial f}{\partial z_3}, -\frac{\partial f}{\partial z_2})$ and we can extend it to a holomorphic vector field $\tilde{\xi}$ on $\mathbb{C}^3$ defined by:

$$\tilde{\xi} = (0, \frac{\partial f}{\partial z_3}, -\frac{\partial f}{\partial z_2}).$$

This is tangent to all the non-singular hypersurfaces $f^{-1}(t)$, $t \neq 0$. The singular set of $\xi$ is the complete intersection curve defined by the ideal $(\frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3})$, which meets each non-singular fiber $f^{-1}(t)$ at finitely many points, whose total sum (counting multiplicities) is constant (see Chapter 7). This constant is an index that depends only on $\zeta$ and the way $V$ is embedded in $\mathbb{C}^3$. This is the index that we study below.

Now, given a continuous, smooth or holomorphic vector field $v$ on an isolated singularity germ $(V, 0)$, what is the index of $v$ at 0?

The answer is not clear nor unique, since there is not a tangent space of $V$ at 0. It depends on what properties of the index we want to preserve.

### 2.2 The GSV index

One of the basic properties of the local Poincaré-Hopf index is stability under perturbations. In other words, if a vector field has an isolated singularity on an open set in $\mathbb{R}^n$ and if we perturb it slightly, then the singularity may split into several singular points, with the property that the sum of the indices of the perturbed vector field at these singular points equals the index of the original vector field at its singularity.

If we now consider an analytic variety $V$ defined by a holomorphic function $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ with an isolated critical point at 0, and if $v$ is a
vector field on $V$, with an isolated singularity at 0, then one may like “the index” of $v$ at 0 to be stable under small perturbations of both, the function $f$ and the vector field $v$. This leads naturally to another concept of index, called the GSV index, introduced by X. Gómez-Mont, J. Seade and A. Verjovskyy in [40, 25] for hypersurface germs, and extended in [41] to complete intersections.

This index was later interpreted by D. Lehmann, M. Soares and T. Suwa in [31] via Chern-Weil theory as the virtual index (and it was actually there where the terminology ”GSV-index” was first used). This point of view provides, amongst other things, an extension for vector fields on varieties with non-isolated singularities. When the vector field $v$ is holomorphic, one also has the homological index of Gómez-Mont [22], that we describe later. This is defined for holomorphic vector fields on arbitrary normal isolated complex singularities.

Let us recall the classical definition of the GSV-index for vector fields on isolated complex hypersurface singularities. The same definition extends easily to complete intersections.

Let $(V,0)$ be defined in $\mathbb{C}^{n+1}$ by a holomorphic map

$$ f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0), $$

with a unique critical point at 0. If $n = 1$ we further assume (for the moment, cf. Remark 2.6) that $V = f^{-1}(0)$ is irreducible.

Since 0 is an isolated singularity of $V$, it follows that the (complex conjugate) gradient vector field $\overline{\text{grad}} f(x)$ is normal to $V$ for the usual hermitian metric in $\mathbb{C}^{n+1}$ at all points $x \neq 0$ near the origin. Let $v$ be a germ of a continuous vector field on $V$ singular only at 0. The set $\{v(x), \overline{\text{grad}} f(x)\}$ is a (2)-frame at each point in $V^* := V \setminus \{0\}$, and up to homotopy, it can be assumed to be orthonormal, i.e., each vector has norm 1 and they are pairwise orthogonal. Thence these vector fields define a continuous map from $V^*$ into the Stiefel manifold of complex orthonormal (2)-frames in $\mathbb{C}^{n+1}$, denoted $W_2(n+1)$.

Let $K = V \cap S_\varepsilon$ be the link of 0 in $V$. It is an oriented, real manifold of dimension $(2n - 1)$ and the above frame defines a continuous map

$$ \phi_v = (v, \overline{\text{grad}} f) : K \rightarrow W_2(n+1). $$

It is an exercise in algebraic topology to show that the homotopy classes of maps from $K$ into $W_2(n+1)$ are classified by the integers, i.e., they have
a degree. In fact, it is well known (see [43]) that all the homology groups of $W_2(n + 1)$ vanish in dimension less than $2n - 1$ and

$$H_{2n-1}(W_2(n + 1); \mathbb{Z}) \cong \mathbb{Z} \cong H_{2n-1}(K; \mathbb{Z}).$$

So the degree of a map from $K$ into $W_2(n + 1)$ can be defined in the usual way using homology (see [11, Chapter 3] for details).

**Definition 2.2.** The GSV index of $v$ at $0 \in V$, $\text{Ind}_{GSV}(v, 0)$, is the degree of the above map $\phi_v$.

This index depends not only on the topology of $V$ near 0, but also on the way $V$ is embedded in the ambient space. For example, the singularities in $\mathbb{C}^3$ defined by

$$\{x^2 + y^7 + z^{14} = 0\} \text{ and } \{x^3 + y^4 + z^{12} = 0\},$$

are orientation preserving homeomorphic as abstract varieties, disregarding the embedding, and one can prove that the GSV index of the radial vector field is 79 in the first case and 67 in the latter; this follows from the fact (see 2.3 below) that for radial vector fields the GSV index is $1 + (-1)^{\dim V} \mu$, where $\mu$ is the Milnor number, which in the examples above is known to be 78 and 66 respectively, by [33, Theorem 9.1].

We recall that one has a Milnor fibration associated to the map $f$, see [33], and the Milnor fiber $F$ can be regarded as a compact $2n$-manifold with boundary $\partial F = K$. The fiber $F$ has the homotopy type of a bouquet of spheres of middle dimension:

$$F \simeq \wedge S^n.$$ 

The number of spheres in this wedge is the Milnor number of $f$, denoted $\mu$ of $\mu(f)$, and one has (see [33]):

$$\mu = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_{n+1}} )}.$$ 

By the Transversal Isotopy Lemma there is an ambient isotopy of the sphere $S_\varepsilon$ taking $K$ into $\partial F$, which can be extended to a collar of $K$, which goes into a collar of $\partial F$ in $F$. Hence $v$ can be regarded as a non-singular vector field on $\partial F$. 16
Theorem 2.3. The GSV index has the following properties:

(1) The GSV index of $v$ at 0 equals the Poincaré-Hopf index of $v$ in the Milnor fiber:

$$\text{Ind}_{GSV}(v, 0) = \text{Ind}_{PH}(v, F).$$

(2) If $v$ is everywhere transverse to $K$, then

$$\text{Ind}_{GSV}(v, 0) = 1 + (-1)^n \mu,$$

where $n$ is the complex dimension of $V$ and $\mu$ is the Milnor number of 0.

Proof. Since the conjugate gradient vector field $\{\text{grad} f\}$ is normal to $V^*$, the degree of $\phi_v$ can be identified with the obstruction to extending $v$ to a tangent vector field on $F$. Hence $\text{Ind}_{GSV}(v, 0) = \text{Ind}_{PH}(v, F)$ as claimed in (1). Statement (2) follows from statement (1) together with Theorem 1.3 above and the fact that, by [33], the Euler-Poincaré characteristic of $F$ is $1 + (-1)^n \mu$. 

Remark 2.4. Theorem 2.3 says that if we perturb the mapping $f$ by adding to it a small constant, then the index is preserved in the sense that the GSV
index of the vector field on the singular fiber becomes the sum of Poincaré-Hopf indices in the nearby fibers. One may of course look at more general deformations of the map-germ $f$. From the previous discussion we see that the way the GSV index changes when we perturb $f$ does not depend on the choice of vector field, but only on the way the topology of the Milnor fiber changes, i.e., on the behavior of the Milnor number under perturbations. This is an interesting subject that has been studied by several authors, including Lazzeri, Gabrielov, Lê and Massey, among many others.

One has a Poincaré-Hopf type theorem for this index:

**Theorem 2.5.** Let $V$ be a compact, complex analytic variety with isolated singularities $x_1, \ldots, x_r$, which are all isolated complete intersection germs. Let $v$ be a continuous vector field on $V$, singular at the $x_i$'s and possibly at some other smooth points $y_1, \ldots, y_s$ of $V$. Let $\text{Ind}_{\text{GSV}}(v, V)$ denote the total GSV index of $v$, i.e., the sum of the local GSV indices at the $x_i$'s and the usual Poincaré-Hopf indices at the $y_i$'s. Then one has:

$$\text{Ind}_{\text{GSV}}(v, V) = \chi(V) + (-1)^n \sum_{i=1}^{r} \mu(x_i),$$

where $\chi(V)$ is the Euler characteristic of $V$, and $\mu(x_i)$ is the Milnor number at the $x_i$'s.
where $\mu(x_i)$ is the Milnor number of $V$ at $x_i$.

**Proof.** The proof of this theorem is very similar to that of 3.3. One removes from $V$ conical neighborhoods $N_i$ of the $x_i$ and replaces these by copies $F_1, \ldots, F_r$ of the corresponding Milnor fibers. The new manifold $\hat{V}$ is smooth, and if $v$ is radial at each $x_i$ then its Euler characteristic is $\text{Ind}_{\text{GSV}}(v, V)$, essentially by definition. But $\chi(\hat{V})$ equals $\chi(V) + (-1)^n \sum_{i=1}^r \mu(x_i)$, proving the theorem when $v$ is radial at each $x_i$. The general case follows easily from this and Proposition 1.7.

**Remark 2.6.** In the above discussion we ruled out the case where the dimension of $V$ is 1 and $V$ has several branches. This case is of course interesting and it was first considered by Brunella [12, 13] and Khanedani-Suwa [29] in their study of holomorphic 1-dimensional foliations on complex surfaces (cf. Chapter 6). In this case the GSV index is defined as the Poincaré-Hopf index of an extension of $v$ to a Milnor fiber. If a curve $C$ has only one branch at a singular point $x_0$ this coincides with Definition 2.2. But if $C$ has several branches at $x_0$ one has an integer attached by 2.2 to each branch. If $C$ is a plane curve, the relation among all these indices is well understood and it is determined by the intersection number of the various branches. In fact, Milnor in [33, Theorem 10.5 and Remark 10.10] proved that if $C_1, \ldots, C_r$ are the irreducible components of $C$ then one has the formula:

$$
\mu = \sum_{i=1}^r \mu_i + 2I - r + 1,
$$

where $\mu$, respectively $\mu_i$, is the Milnor number of $C$, respectively $C_i$, at $x_0$ and $I$ is defined as $\sum_{i<j} C_i \cdot C_j$. This formula implies that if $v$ is a vector field on $C$ then we have

$$
\text{Ind}_{\text{GSV}}(v, x_0; C) := \sum_{i=1}^r \text{Ind}_{\text{GSV}}(v, x_0; C_i) = \text{Ind}_{\text{PH}}(v, F_t) - 2I,
$$

where $\text{Ind}_{\text{PH}}(v, F_t)$ is the Poincaré-Hopf index of an extension of $v$ to a Milnor fiber $F_t$ of $C$ at $x_0$, a formula proved independently in [12] and [29].

### 2.3 The homological index

For a holomorphic vector field $v$ in $\mathbb{C}^n$ with an isolated singularity at 0, the local Poincaré-Hopf index satisfies:

$$
\text{Ind}_{\text{PH}}(v, 0) = \dim \mathcal{O}_{\mathbb{C}^n, 0}/(a_1, \ldots, a_n),
$$

(5)
where \((a_1, \ldots, a_n)\) is the ideal generated by the components of \(v\).

This fact motivated the search for algebraic formulas for indices of vector fields on singular varieties. A major contribution in this direction was given by V. I. Arnold for gradient vector fields. There are also significant contributions by various authors, such as X. Gomez-Mont, S. Gusein-Zade, W. Ebeling and others.

An algebraic formula for the GSV-index of holomorphic vector fields on singular varieties was given in [25], inspired by (5), but that formula applies only under very stringent conditions: for holomorphic vector fields on a hypersurface germ \(V\) which are tangent to the fibers of a defining function \(f\) of \(V\). We remark that the right hand side in 5 is always positive, while the GSV of holomorphic vector fields can be negative. Hence an algebraic formula for this index must necessarily be more “sophisticated”.

Later, X. Gómez-Mont introduced the notion of a homological index for holomorphic vector fields. The homological index has the important property of being defined on arbitrary complex analytic isolated singularity germs \((V, 0)\). When the germ \((V, 0)\) is a complete intersection, the homological index coincides with the GSV-index, by [22, 4].

Let \((V, 0) \subset (\mathbb{C}^m, 0)\) be a germ of a complex analytic variety of pure dimension \(n\), which is regular on \(V \setminus \{0\}\). So \(V\) is either regular at 0 or else it has an isolated singular point at the origin. A vector field \(v\) on \((V, 0)\) can always be defined as the restriction to \(V\) of a vector field \(\tilde{v}\) in the ambient space which is tangent to \(V \setminus \{0\}\); \(v\) is holomorphic if \(\tilde{v}\) can be chosen to be holomorphic. So we may write \(v\) as \(v = (a_1, \ldots, a_m)\) where the \(a_i\) are restriction to \(V\) of holomorphic functions on a neighborhood of 0 in \((\mathbb{C}^m, 0)\).

It is worth noting that given any space \(V\) as above, there are always holomorphic vector fields on \(V\) with an isolated singularity at 0. This non-trivial fact is indeed a weak form of a stronger result ([3, p. 19]): in the space \(\Theta(V, 0)\) of germs of holomorphic vector fields on \(V\) at 0, those having an isolated singularity form a connected, dense open subset \(\Theta_0(V, 0)\). Essentially the same result implies also that every \(v \in \Theta_0(V, 0)\) can be extended to a germ of holomorphic vector field in \(\mathbb{C}^m\) with an isolated singularity, though it can also be extended with a singular locus of dimension more that 0, a fact that may be useful for explicit computations (c.f. the last part of the following section).

A (germ of) holomorphic \(j\)-form on \(V\) at 0 means the restriction to \(V\) of a holomorphic \(j\)-form on a neighborhood of 0 in \(\mathbb{C}^m\); two such forms in \(\mathbb{C}^m\) are equivalent if their restrictions to \(V\) coincide on a neighborhood of
We denote by $\Omega^j_{V,0}$ the space of all such forms (germs); these are the Kähler differential forms on $V$ at 0. So $\Omega^0_{V,0}$ is the local structure ring $\mathcal{O}_{(V,0)}$ of holomorphic functions on $V$ at 0 and each $\Omega^j_{V,0}$ is an $\Omega^0_{V,0}$-module. Notice that if the germ of $V$ at 0 is determined by $(f_1, \cdots, f_k)$ then one has:

$$\Omega^j_{V,0} := \Omega^j_{\mathbb{C}^m,0} + df_1 \wedge \Omega^{j-1}_{\mathbb{C}^m,0}, \cdots, f_k \Omega^j_{\mathbb{C}^m,0} + df_k \wedge \Omega^{j-1}_{\mathbb{C}^m,0}$$

where $d$ is the exterior derivative.

Now, given a holomorphic vector field $\hat{v}$ at 0 $\in \mathbb{C}^m$ with an isolated singularity at the origin, and a differential form $\omega \in \Omega^j_{\mathbb{C}^m,0}$, we can always contract $\omega$ by $v$ in the usual way, thus getting a differential form $i_v(\omega) \in \Omega^{j-1}_{\mathbb{C}^m,0}$. If $v = \hat{v}|_V$ is tangent to $V$, then contraction is well defined at the level of differential forms on $V$ at 0 and one gets a complex $(\Omega^*_{V,0}, v)$:

$$0 \to \Omega^n_{V,0} \to \Omega^{n-1}_{V,0} \to \cdots \to \mathcal{O}_{V,0} \to 0,$$

where the arrows are contraction by $v$ and $n$ is the dimension of $V$; of course one also has differential forms of degree $> n$, but those forms do not play a significant role here. We consider the homology groups of this complex:

$$H_j(\Omega^*_{V,0}, v) = \text{Ker} (\Omega^j_{V,0} \to \Omega^{j-1}_{V,0}) / \text{Im} (\Omega^{j+1}_{V,0} \to \Omega^j_{V,0})$$

The first observation in [22] is that if $V$ is regular at 0, so that its germ at 0 is that of $\mathbb{C}^n$ at the origin, and if $v = (a_1, \cdots, a_n)$ has an isolated singularity at 0, then this is the usual Koszul complex. In that case, its homology groups vanish for $j > 0$, while

$$H_0(\Omega^*_{V,0}, v) \cong \mathcal{O}_{\mathbb{C}^n,0}/(a_1, \cdots, a_n).$$

In particular the complex is exact if $v(0) \neq 0$. Since the contraction maps are $\mathcal{O}_{V,0}$-module maps, this implies that if $V$ has an isolated singularity at the origin, then the homology groups of this complex are concentrated at 0, and they are finite dimensional because the sheaves of Kähler differentials on $V$ are coherent. Hence it makes sense to define, for $V$ a complex analytic germ with an isolated singularity at 0 and $v$ a holomorphic vector field on $V$ with an isolated singularity at 0:

**Definition 2.10.** The **homological index** $\text{Ind}_{\text{hom}}(v, 0; V)$ of the holomorphic vector field $v$ on $(V, 0)$ is the Euler characteristic of the above complex:

$$\text{Ind}_{\text{hom}}(v, 0; V) = \sum_{i=0}^n (-1)^i h_i(\Omega^*_{V,0}, v),$$

21
where $h_i(\Omega^*_{V,0}, v)$ is the dimension of the corresponding homology group as a vector space over $\mathbb{C}$.

We recall that an important property of the Poincaré-Hopf local index is its stability under perturbations. This means that if we perturb $v$ slightly in a neighborhood of an isolated singularity, then this zero of $v$ may split into a number of isolated singularities of the new vector field $v'$, such that the sum of indices of $v'$ at these singular points equals the index of $v$. If the ambient space $V$ has an isolated singularity at 0, then every vector field on $V$ necessarily vanishes at 0, since in the ambient space the vector field defines a local flow with 0 as fixed point. Hence every perturbation of $v$ producing a vector field tangent to $V$ must also vanish at 0, but new singularities may arise with this perturbation. The homological index satisfies a stability under this type of perturbations (called the “Law of Conservation of Number” in [22]):

**Theorem 2.11.** (Gómez-Mont [22, Theorem 1.2]) For every holomorphic vector field $v'$ on $V$ sufficiently close to $v$ one has:

$$\text{Ind}_{\text{hom}}(v, 0; V) = \text{Ind}_{\text{hom}}(v', 0; V) + \sum \text{Ind}_{\text{PH}}(v'),$$

where the sum on the right runs over the singularities of $v'$ at regular points of $V$ near 0.

### 2.4 The case of 1-dimensional foliations

We now consider an interesting setting, which is specially relevant for this meeting.

Let $\mathcal{F}$ be a holomorphic, 1-dimensional foliation with isolated singularities on a complex surface $X$. Let $p \in X$ be a singular point of $\mathcal{F}$; near $p$ the foliation is given by a holomorphic vector field $v = F(z, w) \frac{\partial}{\partial z} + G(z, w) \frac{\partial}{\partial w}$, where $(z, w)$ are local coordinates centered at $p$ and $F, G$ are holomorphic functions with $F^{-1}(0) \cap G^{-1}(0) = \{(0, 0)\}$.

The Baum-Bott theory of residues says how to associate invariants to each singular point of the foliation, out from the Chern numbers of the normal sheaf. In the setting we now envisage, there are essentially two such invariants: One is obtained by localizing the 2nd Chern class and coincides with the local Poincaré-Hopf index; the other is the one that Brunella calls the Baum-Bott index; this is obtained by localizing de Chern number $c_1^2$ and
it coincides with the Camacho-Sad index, introduced in [16] for proving the separatrix theorem. Both of these invariants are determined by the normal sheaf of $\mathcal{F}$.

One also has the GSV-index of $\mathcal{F}$, which depends on the “tangential behavior” of the foliation: it equals the GSV-index of the vector field $v$ (and is easily seen to be independent of the choice of vector field, see for instance [25]).

In [12, 13] (see also [29]) Brunella studied relations amongst all these indices, throwing interesting light into the geometry and topology of 1-dimensional foliations from a global point of view.

In fact, it was already known from [16] that if $S \subset X$ is a compact holomorphic non-singular curve invariant by $\mathcal{F}$, one obtains that the sum of all the Camacho-Sad indices of points in $S$ equals the self-intersection number $S^2$. That is:

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap S} \text{Ind}_{CS}(\mathcal{F}, S, p) = S \cdot S.$$ 

In the same vein, Brunella proved:

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap S} \text{Ind}_{GSV}(\mathcal{F}, S, p) = c_1(N_{\mathcal{F}}) \cdot S - S \cdot S.$$ 

That is, the difference between the total sum of the Camacho-Sad and the GSV indices for points in $S$ is determined by the invariant $c_1(N_{\mathcal{F}}) \cdot S$. This latter invariant actually is the sum over the points in $S$ of another invariant called the variation, previously introduced in [29].

Brunella used these and other facts to study the geometry of 1-dimensional holomorphic foliations on compact complex surfaces. A key step for that was looking at nondicritical singularities: We recall that a singular point $p$ of $\mathcal{F}$ is nondicritical if $\mathcal{F}$ has only a finite number of separatrices at $p$. This is equivalent to saying that if $\pi : \hat{X} \to X$ is the resolution of $\mathcal{F}$ at $p$ and $\hat{\mathcal{F}}$ is the natural extension of the lifting of $\mathcal{F}$ to $\hat{X}$, then $\pi^{-1}(p)$ is $\hat{\mathcal{F}}$-invariant. And a separatrix $S$ of $\mathcal{F}$ at $p$ is nondicritical if there is a sequence of blow-ups $\pi : \hat{X} \to X$, based at $p$, such that:

i) $\pi$ is a resolution of $S$, i.e., $\pi^{-1}(S)$ is a curve with only normal crossing singularities; and,

ii) $\pi^{-1}(p)$ is $\hat{\mathcal{F}}$ invariant,
For instance, if $p$ is a nondicritical singular point of $\mathcal{F}$, then any separatrix of $\mathcal{F}$ at $p$ is nondicritical, with $\pi$ being the resolution map. On the other hand, if either $S$ is smooth at $p$ or if $S$ has a normal crossing singular point at $p$, then $S$ is nondicritical, independently on $\mathcal{F}$ (taking $\pi$ to be the identity in the above definition).

Brunella proved (Propositions 6 and 7 in [13]):

- If $S$ is a nondicritical separatrix at $p$, then
  \[ \text{Ind}_{GSV}(\mathcal{F}, S, p) \geq 0. \]

- If the singularity of $\mathcal{F}$ at $p$ actually is a generalized curve and $S$ is the union of all separatrices, then
  \[ \text{Ind}_{GSV}(\mathcal{F}, S, p) = 0. \]

Recall that the notion of a generalized curve was introduced in [17] and it means that the singularity of $\mathcal{F}$ at $p$ is nondicritical and there are no saddle-nodes in its resolution.

\section{Schwartz index and the Milnor number}

\subsection{The Schwartz or radial index}

In 1965 M. H. Schwartz published a seminal article with the first notion of Chern classes for singular varieties. She used for this a special class of vector fields, obtained by “radial extension”. Her work leads to another index of vector fields on singular varieties, that we now review. This was first done by H. King and D. Trotman (in a 1996 preprint, finally published in 2014), and then, at about the same time, by W. Ebeling and S. Gusein-Zade and by M. Aguilar, A. Verjovsky and myself.

Consider a (real or complex) analytic variety $V \subset \mathbb{R}^m$ of dimension $n > 1$ with an isolated singularity at 0. Let $U$ be an open ball around $0 \in \mathbb{R}^m$, small enough so that every sphere in $U$ centered at 0 meets $V$ transversally (see [32]). For simplicity we restrict the discussion to $U$.

\textbf{Definition 3.1.} A continuous vector field $v_{\text{rad}}$ tangent to $V$ is \textit{radial} at $0 \in V$ if it transverse (outwards-pointing) to all spheres $S_\varepsilon$ around 0 for $\varepsilon$ small enough. By definition \textit{the Schwartz, or radial, index} of $v_{\text{rad}}$ is 1.
We will say later more about why this index is 1 for radial vector fields. For the moment we content ourselves by saying that 1 is the Euler characteristic of the point $0 \in V$, which is the singular set of $v_{\text{rad}}$.

Consider now an arbitrary continuous vector field $v$ on $V$ with an isolated singularity at 0. By this we mean a continuous section $v$ of $T\mathbb{C}^m|_V$ which is tangent to $V^* = V \setminus \{0\}$. We want to define its Schwartz index. For this we consider the difference between $v$ and $v_{\text{rad}}$ at 0, just as we did in the previous section: consider small spheres $S_\varepsilon, S_{\varepsilon'}; \varepsilon > \varepsilon' > 0$, and let $w$ be a vector field on the cylinder $X$ in $V$ bounded by the links $K_\varepsilon = S_\varepsilon \cap V$ and $K_{\varepsilon'} = S_{\varepsilon'} \cap V$, such that $w$ has finitely many singularities in the interior of $X$, it restricts to $v$ on $K_\varepsilon$ and to $v_{\text{rad}}$ on $K_{\varepsilon'}$. The difference of $v$ and $v_{\text{rad}}$ is defined as

$$d(v, v_{\text{rad}}) = \text{Ind}_{PH}(w, X),$$

the Poincaré-Hopf index of $w$ on $X$.

**Definition 3.2.** The Schwartz, or radial, index of $v$ at 0 $\in V$ is:

$$\text{Ind}_{\text{Sch}}(v, 0; V) = 1 + d(v, v_{\text{rad}}).$$

The following result is well-known (see [41, 18, 1]). For vector fields with radial singularities, this is a special case of the work of M.-H. Schwartz; the general case follows easily from this.

**Theorem 3.3.** Let $V$ be a compact complex analytic variety with isolated singularities $q_1, \cdots, q_r$ in a complex manifold $M$, and let $v$ be a continuous vector field on $V$, singular at the $q_i$ and possibly at some other isolated points in $V$. Let $\text{Ind}_{\text{Sch}}(v, V)$ be the sum of the Schwartz indices of $v$ at the $q_i$ plus its Poincaré-Hopf index at the singularities of $v$ in the regular part of $V$. Then:

$$\text{Ind}_{\text{Sch}}(v, V) = \chi(V)$$

The proof is very simple. Assume first the vector field $v$ is radial at each $q_1, \cdots, q_r$, so its local Schwartz index at each $q_i$ is 1. Now take small discs $D_i$ in $M$ around each $q_i$ and remove from $V$ the interior of each $V \cap D_i$; we get a manifold $V^*$ which is compact with boundary. The vector field is transverse to the boundary everywhere. Hence its total Poincaré-Hopf index there equals $\chi(V^*)$. The result then follows from the Poincaré-Hopf index theorem because one has:

$$\chi(V) = r + \chi(V^*).$$
Now, if \( v \) is non-radial at (some) \( q_i \) we do a simple trick: for each \( \varepsilon > 0 \) sufficiently small, denote by \( B_{i,\varepsilon} \) the ball in \( M \) of radius \( \varepsilon \) around \( q_i \) (for some metric) and set \( K_{i,\varepsilon} = V \cap B_{i,\varepsilon} \) and \( V^* = V \setminus \bigcup_i (V \cap B_{i,\varepsilon}) \) for some fixed \( \varepsilon > 0 \) sufficiently small. By [33] each boundary component \( K_{i,\varepsilon} \) of \( V^* \) has a neighborhood diffeomorphic to a cylinder \( K_{i,\varepsilon} \times [0,1] \). Choose \( \varepsilon_1, \varepsilon_2 > 0 \) such that \( \varepsilon > \varepsilon_1 > \varepsilon_2 \), and let \( X_{\varepsilon,\varepsilon_1} \) and \( X_{\varepsilon_1,\varepsilon_2} \) be the cylinders in \( M \) bounded by \( \{ K_{i,\varepsilon}, K_{i,\varepsilon_1} \} \) and \( \{ K_{i,\varepsilon_1}, K_{i,\varepsilon_2} \} \) respectively. Put the vector field \( v \) on each \( K_{i,\varepsilon_1} \) and on each \( K_{i,\varepsilon} \) and \( K_{i,\varepsilon_2} \) we put a radial vector field \( v_{\text{rad}} \). Then the local Schwartz index of \( v \) at each \( q_i \) is 1 plus the difference \( d(v,v_{\text{rad}}) \) between \( v \) on \( K_{i,\varepsilon_1} \) and \( v_{\text{rad}} \) on \( K_{i,\varepsilon_2} \), which equals \(-d(v_{\text{rad}},v)\), the difference between \( v_{\text{rad}} \) on \( K_{i,\varepsilon} \) and \( v \) on \( K_{i,\varepsilon_1} \). Hence this case reduces to the previous one of radial vector fields, proving theorem 3.3.

The Schwartz index has the advantage of being defined always, for continuous vector fields on real or complex varieties with arbitrary singularities, and it always gives the right number, i.e., the total index on compact varieties equals the Euler characteristic.

### 3.2 On the Milnor number

The Milnor number of hypersurface singularities is a key-invariant, which plays an important role in singularity theory and in several other branches of mathematics, such as foliations, open-book decompositions, knots theory, etc.

This invariant was extended by H. Hamm in the early 1970s for isolated complete interaction singularities (ICIS for short). Hamm proved that every ICIS has a Milnor fibration and its Milnor fibre has the homotopy type of a bouquet of spheres of middle dimension. Just as for hypersurfaces, the Milnor number is defined to be precisely the number of spheres in that bouquet, i.e., the middle Betti number of the Milnor fiber.

The question I want to address now is very easy to state:

What is (or rather, what ought to be) the Milnor number of a complex analytic singularity \((V,P)\) in general?

Another way of formulating this question is: Who plays the role of the Milnor number when there is not a Milnor fiber?

In fact these questions are by no means original: There are several interesting papers in the literature with different viewpoints. Let us recall some
of the well-known results.

The most basic case is considering curve singularities which are not complete intersection germs. This was done by Buchweitz and Greuel in [14], following previous work by Bassein. That was later extended for functions on curves by D. Mond and D. Van Straten (2001).

Buchweitz-Greuel defined the Milnor number of a reduced complex curve singularity \((X, P)\) as:

\[
\mu = \mu_{BG} := \dim \frac{\omega_X}{d\mathcal{O}_X}
\]

where \(\omega_X\) is Grothendieck’s dualizing module at \(P\).

When the germ is smoothable, \(\mu_{BG}\) equals the first Betti number of the smoothing, hence for ICIS germs it coincides with the usual Milnor number.

The next natural step is considering complex dimension 2. This was done by G.-M. Greuel and J. Steenbrink, proving (Arcata’s volume, 1980):

**Theorem:** Every smoothable, Gorenstein normal surface singularity \((V, P)\) has a well-defined Milnor number \(\mu_{GS}\): The 2nd Betti-number of a smoothing.

Furthermore Steenbrink generalized to this setting a formula of H. Laufer for ICIS of dimension 2, thus proving that for smoothable Gorenstein normal surface singularities, this invariant satisfies:

\[
\mu_{GS} + 1 = \chi(\tilde{V}) + K^2 + 12\rho_g(V),
\]

where \(\chi\) is the usual Euler characteristic of a good resolution \(\tilde{V}\), \(K^2\) is the self-intersection number of the canonical class and \(\rho_g\) is the geometric genus \(\dim H^1(\tilde{V}, \mathcal{O})\).

A variant of this formula was proved by myself in [39] using cobordism:

\[
\mu_{GS} + 1 \equiv \chi(\tilde{V}) + K^2 + 12 \text{Arf} K \mod (24),
\]

where \(\text{Arf} K\) is always 0 or 1 and it is the Arf invariant of a certain quadratic form.

Yet, H. Pinkham (Asterisque 1974) proved that there are normal surface singularities with different smoothings (with distinct Euler characteristic). Hence: there is no hope of generalizing the concept of the Milnor number in such a way that when there are smoothings, the Milnor number is determined only by the topology of the smoothing. There has to be something else.
For instance, consider again the Laufer-Steenbrink formula:

\[ \mu_{GS} + 1 = \chi(\tilde{V}) + K^2 + 12\rho_g(V) \]

Observe that right hand side is defined always (whether the singularity is smoothable or not), and if the singularity is Gorenstein, this is always an integer, independent of the choice of resolution. It is thus natural to ask:

What should there be on the LHS of this equation when the singularity is non-smoothable? This is an open question.

One approach to define a Milnor number arises from my work [20] with Ebeling and Gusein-Zade in 2004. In fact, from Theorem 2.3 one easily gets that for all hypersurface germs one has:

\[ \mu = (-1)^n \left( \text{Ind}_{GSV}(v) - \text{Ind}_{Sch}(v) \right), \]

independently of the choice of the vector field \( v \).

Now, the Schwartz index is defined in full generality, for all continuous vector fields on singular varieties, while the GSV-index is defined only for vector fields on ICIS. Yet we know from [22, 4] that for holomorphic vector fields on ICIS, the GSV index coincides with the homological index, and the latter is defined for holomorphic vector fields on arbitrary normal isolated singularities.

Therefore one may take the following as definition for a Milnor number:

\[ \mu = (-1)^n \left( \text{Ind}_{Hom}(v) - \text{Ind}_{Sch}(v) \right), \]

where \( v \) is a holomorphic vector field on \( V \). This is essentially what we did in [20], replacing vector fields by 1-forms, since one also has for these all the corresponding indices, and 1-forms are sometimes easier to handle than vector fields. We proved:

- For curves, the Milnor number so defined equals \( \dim \frac{\Omega_1(V, p)}{\Omega_0(V, p)} \). Recall that the Milnor number of Buchweitz-Greuel is \( \dim \frac{\omega(V, p)}{\Omega_0(V, p)} \), where \( \omega(V, p) \) is the dualizing module.

- For the examples of surface singularities I know the value of this Milnor number, it satisfies the Laufer-Steenbrink equality. (I do not know whether or not these are particular cases of a general theorem waiting to be proved).
3.3 A global point of view. The 0-degree Milnor class

Now consider a “global point of view”:

Let \( X \hookrightarrow M \) be a compact subvariety in some complex \( m \)-manifold and assume all its singularities are normal and isolated. Then \( X \) has a well defined invariant, that we may call its homological index, \( \text{Ind}_{\text{Hom}}(X; M) \).

To define it, we choose a holomorphic vector field on a neighborhood of each singularity of \( X \), and extend these to a global, continuous vector field \( v \) on all of \( X \), possibly with other isolated singularities on the regular part of \( X \).

We may define the total homological index associated to the vector field \( v \) on \( X \): It is the sum of the local homological indices at the singular points of \( X \) and the usual Poincaré-Hopf indices and the singular points of \( v \) contained in the regular set of \( X \). It is an exercise to show that this integer actually is independent of the choices of vector fields and depends only on \( X \). Thus we denote it \( \text{Ind}_{\text{Hom}}(X; M) \).

**Problem:** determine what the invariant \( \text{Ind}_{\text{Hom}}(X; M) \) is (for instance in terms of other better-known invariants).

When \( X \) is a complete intersection in \( M \) the answer to this question comes from work by Tatsuo Suwa and myself in 1998 (see [41]), together with the fact that, by [22, 4], for ICIS the homological and the GSV indices coincide, and by [31], the latter equals the virtual index. One has:

**Theorem 3.4.** If \( X \) is a complete intersection in \( M \), then:

\[
\text{Ind}_{\text{Hom}}(X; M) = c^n_FJ(X)[X]
\]

\[
= \chi(X) + (-1)^n \mathcal{M}_0(X),
\]

where \( c^n_FJ(X)[X] \) is the top dimensional Fulton-Johnson class of \( X \) evaluated on the fundamental cycle \([X]\), \( \chi \) is the usual Euler characteristic and \( \mathcal{M}_0(X) \) is the Milnor class of \( X \) in degree 0.

We remark that all the invariants in this theorem are defined for all compact \( X \) in \( M \), regardless of whether or not it is a complete intersection. It is thus natural to ask whether this theorem holds in a more general setting. In the following section we shall define and discuss all these invariants.
4 Varieties with non-isolated singularities

In the first section we defined the Poincaré-Hopf local index of a vector field and we described how this extends to an index for $r$-frames, thus giving rise to the concept of Chern classes. Next we described several extensions of the Poincaré-Hopf local index to the case of singular varieties. We now discuss how these relate to different notions of Chern classes for singular varieties.

This is very much related with deciding who plays the role of the tangent bundle when considering singular varieties. For instance:

a) M. H. Schwartz looked at varieties embedded in some complex manifold, equipped these with an appropriate partition (a stratification), and considered collections of “stratified” vector fields. This leads to what we know as the Schwartz classes.

b) R. MacPherson looked at the Nash bundle $\tilde{T}$ over the Nash blow up $\tilde{X}$ of a complex variety $X$, which is obtained by replacing the singular set by all limits of spaces tangent to the regular part. He used these to get the so-called MacPherson classes, solving a conjecture of Deligne and Grothendieck on Chern classes for singular varieties. Later Brasselet and Schwartz proved that these classes essentially coincide with those defined by Schwartz.

c) W. Fulton and K. Johnson noticed that a complete intersection in a compact complex manifold has a natural “virtual tangent bundle” that can be used to define characteristic classes. These are called the Fulton-Johnson classes. The definition of these classes actually extends to all algebraic varieties.

In this section we briefly review the Schwartz classes, the Fulton-Johnson classes, and the difference amongst these, which are called Milnor classes.

4.1 Fulton-Johnson classes

For simplicity we restrict the discussion to the case when $X$ is a compact variety of pure codimension 1 in a complex manifold $M$ of dimension $n + 1$. In this case we say that $X$ is a hypersurface in $X$.

We recall that if $E$ is a complex vector bundle over $X$ with fiber $\mathbb{C}^k$, $k \leq n$, the Chern class $c^i(E) \in H^{2i}(E; \mathbb{Z})$, $i = 1, \cdots, k$, is the obstruction
for constructing $k - i + 1$ linearly independent sections of $E$ over the skeleton of dimension $2i$ of some triangulation of $X$. We can phrase this in the converse way: If we want to construct an $r$-frame of $E$, meaning by this a set of $r$ linearly independent sections. Then the “obstruction dimension” is $2k - 2r + 2$, and that is where the corresponding Chern class appears.

The total Chern class of $E$ is a polynomial in the cohomology of $X$:

$$c(E) = 1 + c^1(E) + \cdots + c^k(E) \in H^*(X; \mathbb{Z}).$$

Since $X$ has codimension 1 in $M$, it is a divisor for some holomorphic line bundle $L$.

If $X$ were smooth, the restriction of $L$ to $X$ essentially coincides with normal bundle $\nu(X)$ and we have a $C^\infty$ splitting:

$$TM|_X = TX \oplus \nu(X).$$

One of the basic properties of Chern classes is their nice behavior (called Whitney decomposition) with respect to direct sums. One gets:

$$c(T(X)) = c(TM|_X) \cdot (1 + c^1(L)|_X)^{-1} \in H^*(X),$$

where $(1 + c^1(L)|_X)^{-1}$ is the inverse in $H^*(X)$ of the element $(1 + c^1(L)|_X)$.

When $X$ is singular, it does not have a tangent bundle, but even so the right hand side of the above equation still makes sense and we can use it to define “Chern classes” for $X$:

**Definition 4.1.** Let $X$ be a possibly singular hypersurface in $M$. The total Fulton-Johnson class of $X$ is:

$$c_{FJ}(X) := c(TM|_X) \cdot (1 + c^1(L)|_X)^{-1} \in H^*(X),$$

where $(1 + c^1(L)|_X)^{-1}$ is the inverse of $c(L|_X)$ in $H^*(X)$ and $c(TM|_X)$ is the total Chern class of this bundle.

Notice $c_{FJ}(X)$ is a polynomial in $c^1(L|_X)$ and the Chern classes of $TM|_X$. Let us write

$$c_{FJ}(X) = 1 + c_{FJ}^1(X) + \cdots + c_{FJ}^n(X),$$

where $c_{FJ}^i(X)$ denotes the sum of all terms of dimension $2i$.

**Definition.** The term $c_{FJ}^i(X)$ is the $i^{th}$ Fulton-Johnson class of $X$. 

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Notice one has the Poincaré morphism
\[ P : H^*(X) \to H_{2n-*}(X), \]
cap product with the fundamental cycle \([X]\). So we can look at each class \(c^F_j(X)\) regarded in the homology of \(X\). This will be used later.

## 4.2 Whitney stratifications

Let us make a short summary of what we need in the sequel about stratifications. We refer to [26] for more on the subject. A stratification of a space \(X\) is a particularly nice decomposition of this space into pieces, all of which are smooth manifolds called the strata.

**Definition 4.2.** Let \(V\) be a complex analytic variety of dimension \(n\) in some complex manifold \(M\). An *analytic stratification* of \(V\) means a locally finite family \((V_\alpha)_{\alpha \in A}\) of non-singular analytic subspaces of \(V\) (i.e., each \(V_\alpha\) is a complex manifold) such that:

1. The family is a partition of \(V\), i.e., they are pairwise disjoint and their union covers \(V\).
2. For each \(V_\alpha\), the closures in \(V\) of both \(\overline{V_\alpha}\) and \(\overline{V_\alpha} \setminus V_\alpha\) are analytic in \(V\).
3. For each pair \((V_\alpha, V_\beta)\) such that \(V_\alpha \cap \overline{V_\beta} \neq \emptyset\) one has \(V_\alpha \subset \overline{V_\beta}\).

The highest dimensional stratum, which may not be connected, is called the regular stratum and usually denoted by \(V_0\) or \(V_{\text{reg}}\).

**Definition 4.3.** A stratification \((V_\alpha)_{\alpha \in A}\) of \(V\) is said to be Whitney if it further satisfies the following two conditions, known as the Whitney conditions (a) and (b), for every pair \((V_\alpha, V_\beta)\) such that \(V_\alpha \subset \overline{V_\beta}\).

Let \(x_i \in V_\beta\) be an arbitrary sequence converging to some point \(y \in V_\alpha\) and \(y_i \in V_\alpha\) a sequence that also converges to \(y \in V_\alpha\). Suppose these sequences are such that (in the appropriate Grassmannian) the sequence of secant lines \(l_i = \overline{xy_i}\) also converges to some limiting line \(l\), and the tangent planes \(T_{x_i}V_\beta\) converges to some limiting plane \(\tau\). The Whitney conditions (a) and (b) are the following. For all sequences as above we have:

1. The limit space \(\tau\) contains the tangent space of the stratum \(V_\alpha\) at \(y\), i.e., \(T_yV_\alpha \subset \tau\).
2. The limit space \(\tau\) contains all the limits of secants, i.e., \(l \subset \tau\).
One knows that condition (b) implies condition (a), but it is useful to
have both conditions stated explicitly.

**Remark 4.4.** Whitney stratifications are very important for several reasons,
some of which will become apparent along this text. Some important facts
about these stratifications are:

1. Every closed (sub)analytic subset of an analytic manifold admits a Whit-
ney stratification. Furthermore, every stratification of such a set can be
refined to become Whitney.

2. Whitney stratified spaces can be triangulated compatibly with the strat-
ification.

3. The transversal intersection of two Whitney stratified spaces is a Whitney
stratified space, whose strata are the intersections of the strata of the two
spaces.

4. Whitney stratifications are locally topologically trivial along the strata.
That is, given a complex (or real) analytic space \( V \) with a Whitney strat-
ification \((V_a)_{a \in A}\), a point \( x \in V_a \) and a local embedding of \((V, x)\) in \( \mathbb{C}^m \),
there is a neighborhood \( W \) of \( x \) in \( \mathbb{C}^m \), diffeomorphic to \( \Delta \times U_a \), where \( U_a \)
is a ball, neighborhood of \( x \) in \( V_a \) and \( \Delta \) is a small closed disk through \( x \) of
complex dimension \( m - \dim_{\mathbb{C}} V_a \), transverse to all the strata of \( V \), and such
that \( W \cap V_\beta = (\Delta \cap V_\beta) \times U_a \) for each stratum \( V_\beta \) with \( x \in V_\beta \). This is
essentially a consequence of the Thom first Isotopy Lemma.

### 4.3 Radial extension of vector fields

Let us describe briefly the radial extension technique developed by M.-H.
Schwartz. The idea is simple though there are technical difficulties that we
shall omit. A detailed exposition of this construction can be found in Section
7 of [10].

We consider a complex analytic \( n \)-variety, i.e., a reduced complex analytic
space \( V \) of (complex) dimension \( n \), embedded in a complex manifold \( M \)
of dimension \( m \). We endow \( M \) with a Whitney stratification adapted to \( V \); i.e.,
\( V \) is union of strata. Since each stratum \( V_a \) is itself a complex manifold we
have its tangent bundle \( TV_a \). The singular set of \( V \) is denoted by \( \text{Sing}(V) \)
and the regular one \( V_{\text{reg}} = V \setminus \text{Sing}(V) \). If \( V \) is reducible, we assume it is
pure dimensional.
Definition 4.5. A stratified vector field on $V$ means a (continuous, smooth) section $v$ of the tangent bundle $TM|_V$ such that for each $x \in V$ the vector $v(x)$ is contained in the subspace tangent to the stratum $V_\alpha$ that contains $x$.

We now describe the local radial extension process. This is based on the local topological triviality of Whitney stratifications explained in 4) of Remark 4.4, and it has two main steps.

Let $v_\alpha$ be a vector field in a neighborhood of a point $x \in V_\alpha$ with possibly a singularity at $x$. According to 4) of Remark 4.4, there is a product neighborhood $W \cong \Delta \times U_\alpha$ of $x$ in the ambient space. We may assume that $x$ is the only possible singularity of $v_\alpha$ in $U_\alpha$.

Denoting by $p_1 : W \to \Delta$ and $p_2 : W \to U_\alpha$ the projections on the two factors of the product, we have a decomposition

$$TW = p_1^*T\Delta \oplus p_2^*TU_\alpha.$$ 

On one hand, the pull-back $p_2^*v_\alpha$ is a vector field on $W$, which is “parallel” to $v_\alpha$. It is stratified, since it is tangent to the fibers of $p_1$. On the other hand, let $\Delta$ be equipped with the induced stratification and let $v_\Delta$ be a stratified vector field on $\Delta$, which is radial in the usual sense and singular at $x$. Then $p_1^*v_\Delta$ is a stratified vector field on $W$ since it is tangent to the fibers of $p_2$ and $v_\Delta$ is stratified. It is thus radial in each slice $\Delta \times \{q\}$ for $q$ in $U_\alpha$. The local radial extension of $v_\alpha$ in $W$ is the following:

Definition 4.6. The local radial extension of $v_\alpha$, denoted by $v$, is the stratified vector field defined on the neighborhood $W$ as the sum:

$$v = p_1^*v_\Delta + p_2^*v_\alpha.$$ 

The fundamental property of the local radial extension is the following:

Lemma 4.7. The local radial extension $v$ of $v_\alpha$ has no singularity along the boundary of $W$ and is pointing outwards $W$ along its boundary. If $v_\alpha$ has a singularity at $x$ with index $\text{Ind}_{PH}(v_\alpha, x; V_\alpha)$, then the local radial extension $v$ of $v_\alpha$ has $x$ as unique singular point in $W$, and one has

$$\text{Ind}_{PH}(v, x; W) = \text{Ind}_{PH}(v_\alpha, x; V_\alpha).$$

In other words, if we start with a vector field $v$ tangent to a given stratum $V_\alpha$ with an isolated singularity at some point $x \in V_\alpha$, the local radial extension technique, just described, gives an extension $\tilde{v}$ of $v$ to a neighborhood of $x$ in the ambient space, such that:
1. $\tilde{v}$ is stratified;

2. Away from $V_\alpha$ it is “escaping away” from every tubular neighborhood of $V_\alpha$ (since it is radial in each slice $\Delta \times \{q\}$ for $q$ in $U_\alpha$).

3. The local Poincaré Hopf index of $v$ in the stratum $V_\alpha$ equals the local Poincaré Hopf index of $\tilde{v}$ in the ambient space.

In particular one has:

**Remark 4.8.** At an isolated singularity of $X$, the only vector fields obtained by radial extension are those which actually are radial.

**Definition 4.9.** Let $v$ be a stratified vector field obtained as in Definition 4.6. Then the Schwartz index of $v$ at $x$ on $V$ is defined to be the Poincaré-Hopf index of $v$ on $W$:

$$\text{Ind}_{\text{Sch}}(v, x; V) = \text{Ind}_{\text{PH}}(v, x; W).$$

The local radial extension allows to define the global radial extension. For this one starts by choosing an arbitrary vector field with isolated singularities on the stratum of lowest dimension. Then the extension is done step by step on the dimension of the strata. This process yields to the following theorem of M.-H. Schwartz (see [10] for details):

**Theorem 4.10.** ([36, 38]) Let $V$ be a complex analytic variety in a complex manifold $M$, and let $(V_\alpha)_{\alpha \in \Lambda}$ be a Whitney stratification of $M$ adapted to $V$. Then there exists stratified vector fields on a neighborhood of $V$ in $M$ constructed by radial extension as above, such that every such vector field $v$ satisfies:

1. Given any stratum $(V_\alpha)$, the total Poincaré-Hopf index of $v$ on $T(\overline{V_\alpha})$ is $\chi(\overline{V_\alpha})$, where $T(\overline{V_\alpha})$ is a regular neighborhood of the closure of $(V_\alpha)$.

2. $v$ is transverse, outwards pointing, to the boundary of every small regular neighborhood of $V$ in $M$.

3. The Poincaré-Hopf index of $v$ at each singularity $x$ is the same if we regard $v$ as a vector field on the stratum that contains $x$ or as a vector field in a neighborhood of $x$ in $M$. Hence the total Schwartz index of $v$ on $V$ is $\chi(V)$. 

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4.4 The Schwartz classes

We consider now an arbitrary $n$-dimensional complex subvariety of an $m$-dimensional compact complex manifold $M$.

We consider a triangulation $(K)$ of $M$ compatible with a Whitney stratification for which $X$ is union of strata. Following M. H. Schwartz, we consider the dual cell decomposition $(D)$: Given a simplex $\sigma \in (K)$, to define its dual we take the first barycentric subdivision $(K')$. Then the dual cell of $\sigma$ is the union of all simplices in $(K')$ whose closure meets $\sigma$ at its barycenter. The cells of $(D)$ are transversal to $X$. Thence, if $\gamma \in (D)$ has dimension $p = 2m - 2r + 2$, which is the obstruction dimension for an $r$-frame in $M$, then $\gamma \cap X$ has dimension $q = 2n - 2r + 2$, since $X$ has real codimension $2m - 2n$, and $q$ is the obstruction dimension for a stratified $r$-frame on $X$.

Using the radial extension process M. H. Schwartz proved:

**Theorem 4.11.** For each $r = 1, \ldots, n$, there exists an $r$-frame $F_r = (v_1, \ldots, v_r)$ on the $(2m - 2r + 1)$-skeleton of $(D)$ in a regular neighborhood $U$ of $X$ in $M$, such that:

1. Each vector field $v_i$ is stratified and obtained by radial extension.

2. For each $(2m - 2r + 2)$-cell $\gamma$ we have that the $(r-1)$-frame $(v_2, \ldots, v_r)$ extends with no singularity to the interior of $\gamma$, and the first vector $v_1$ extends with at most an isolated singularity at the barycenter of $\gamma$.

3. The singularities of $F_r$ in $U$ are all contained in $X$.

Notice this defines a cochain in the usual way: To each cell $\gamma$ of dimension $(2m - 2r + 2)$ we associate the local Poincaré-Hopf index of $v_1$ at the barycenter of $\gamma$, and we extend these to $(2m - 2r + 2)$-chains by linearity.

These cochains actually are cocycles that represent the Chern classes of $U$, by definition. But we notice that by construction, these cocycles vanish away from $X$, because all the singularities of $v_1$ are in $X$. Hence we actually get cocycles in

$$H^{2m-2r+2}(U, U \setminus X) \cong H_{2r-2}(X),$$

where the isomorphism with the homology of $X$ is determined by Alexander duality.

**Definition 4.12.** The classes $c^\delta_{\text{Sch}} \in H^*(U, U \setminus X) \cong H^*(M, M \setminus X)$ that we get in this way are, by definition, the Schwartz classes of $X$. 

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When we look at them in homology, we get the MacPherson classes, by a theorem in [10]. Therefore in the modern literature these are called the Schwartz-MacPherson classes of \( X \) and regarded in homology.

We remark:

1. The top dimensional Schwartz class corresponds to the obstruction for constructing a stratified vector field on \( X \), and its image in \( H_0(X) \) is \( \chi(X) \).

2. One has the usual exact sequence:

\[
\cdots \rightarrow H^{i-1}(U \setminus X) \rightarrow H^i(U, U \setminus X) \rightarrow H^i(U) \rightarrow \cdots ,
\]

and the image in \( H^*(U) \) of the Schwartz classes are the usual Chern classes of the complex manifold \( U \), which is a regular neighborhood of \( X \) in \( M \). In other words one has:

**Proposition 4.13.** The Schwartz classes of \( X \) can be thought of as being a specific choice of a lifting of the Chern classes of \( U \) to the relative cohomology \( H^*(U, U \setminus X) \). These liftings are defined, in each dimension, by the choice of a stratified frame on \( X \) obtained by radial extension (which are non-singular on \( U \setminus X \)).

### 4.5 Milnor classes

Let \( X \) be now a hypersurface in a compact complex manifold \( M \) of dimension \( n+1 \). We know from the previous sections that \( X \) has Fulton-Johnson classes \( c_{FJ}^i(X) \in H^{2i}(X) \), and Schwartz classes \( c_{Sch}^j(X) \in H^{2j}(M, M \setminus X) \). It is an exercise to show these all coincide with the usual Chern classes when \( X \) is non-singular.

The Fulton-Johnson classes can be mapped into \( H_*(X) \) by the Poincaré morphism, which is given by taking cap product with the fundamental cycle; we set:

\[
c_i^{FJ}(X) = c_{FJ}^i(X)[X] \quad \forall i \geq 1 .
\]

Similarly, Alexander’s duality gives an isomorphism between \( H^*(M, M \setminus X) \) and \( H_*(X) \), so we can look at the Schwartz classes regarded in homology, and we know from [10] that these actually coincide with the MacPherson classes. Thus we call them Schwartz-MacPherson classes and denote them by \( c_i^{SM}(X) \in H_*(X) \).
Definition 4.14. For each $i = 0, \cdots, n$, the $i^{th}$ Milnor class of $X$ is

$$M_i := (-1)^i \left( c_i^{FJ} - c_i^{SM}(X) \right).$$

Milnor classes were originally defined as elements in the singular homology group of the hypersurface (or a complete intersection). It was observed in the work of W. Fulton and K. Johnson ([21]) and G. Kennedy ([28]) that the Fulton-Johnson classes and the Schwartz-MacPherson classes, and hence the Milnor classes, can actually be considered as cycles in the corresponding Chow group. Milnor classes have support in the singular set of the variety, and there is a Milnor class in each complex dimension, from 0 up to the dimension of the singular set.

The concept of Milnor classes appeared first in P. Aluffi’s work [2] under the name of $\mu$-class. Milnor classes also appear implicitly in A. Parusinski and P. Pragacz’ article [35]. The actual name of Milnor classes was coined later by various authors at about the same time (see [7, 8, 44, 34]); the name comes from aforementioned theorem, that when the singularities of $Z$ are all isolated, the Milnor class in dimension 0, which is an integer, is the sum of the local Milnor numbers (by [41]).

There are interesting recent articles about Milnor classes by various authors, as for instance P. Aluffi, J.-P. Brasselet, J. Schürmann, M. Tibăr, S. Yokura, T. Ohmoto, L. Maxim and others. There are also important generalisations of this concept to different settings and I believe Jean-Paul Brasselet will speak about that in his lectures next week.

Even so, Milnor classes are still somehow mysterious objects, even “esoteric”, and there is still a lot to be said for decoding the rich information about the variety $X$ that these classes encode. In particular, in my recent paper [15] with R. Callejas-Bedregal and M. Morgado, we show that there are deep relations amongst Milnor classes and Lê cycles, which are analytic cycles introduced by D. Massey that describe fully the topology of the local Milnor fibers of the hypersurface $X$ at its singular points.

References


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